2/25/14

MARKOV NETS / MARKOV RANDOM FIELDS

(1) Recall BN: MN

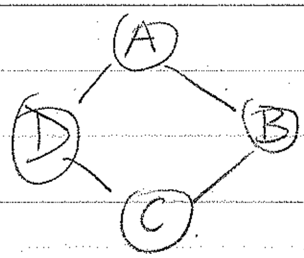
→ Vars $\mathcal{X} = \{X_1 \ldots X_n\}$ → Same

→ DAG $G = (V, E)$ → Arbitrary undirected graph

→ CPTs $P(X_i \mid Pa_{X_i})$ → "Factors" $\phi_C(D_C)$

$D_C \subseteq \{X_1, \ldots, X_n\}$

Lets start with a "Pairwise MRF"



Factors: $\underbrace{\phi_1(A,B)\ \phi_2(B,C)\ \phi_3(C,D)\ \phi_4(D,A)}_{\text{all "pairwise" terms}}$

$|\text{Scope}(\phi_i)| = 2$

Sometimes pairwise MRFs also have "unary" terms/factors

e.g. $\phi(A), \phi(B), \phi(C), \phi(D)$

but not strictly neccesary ∵ can be "absorbed" into pairwise terms

ie $\tilde{\phi}_1(A,B) = \phi_1(A,B)\,\phi(A)$

$\hat{\phi}_2(B,C) = \phi_1(B,C)\,\phi(B)$

⋮

etc.

Now $P(X_1, \ldots, X_n) = \dfrac{1}{Z} \prod_{(i,j) \in E} \phi_{ij}(X_i, X_j)$

normalization constant / "Partition Function"

$Z = \sum_{x_1 \ldots x_n} \prod_{ij} \phi_{ij}(x_i, x_j)$

Intuition #1: $\phi(A, B)$ denotes "local happiness"
or "affinity"
or "compatibility"
of variables in scope of ϕ

Note:
$\phi(A, B) \not\propto P(A, B)$

$\not\propto P(A \mid B)$
$\not\propto P(A, B \mid C, D)$

$P(A, B)$ is a "global" measure
requires knowing about all factors
(to marginalizing)

Compare w\ BN where CPTs where $P(X_i \mid Pa_{X_i})$

(2) General MRF "Gibbs Distribution"

→ Arbitrary sized factors $\phi_c(D_c)$

e.g $\phi(x_1, x_3, x_4, x_5)$

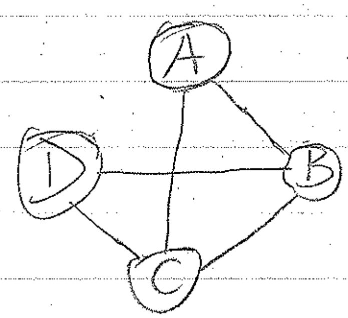
$\phi(A, B, C)$

etc

→ $\boxed{\text{Note}}$: $G = (V, E)$ is now an "induced graph"

⇒ ∃ an edge (i,j) iff X_i & $X_j \in$ Scope(ϕ_c)

e.g



Factors: $\phi_1(A, B, C)$, $\phi_2(B, C, D)$

$$P(A, B, C, D) = \frac{1}{Z} \phi_1(A, B, C) \phi_2(B, C, D)$$

in general $P(X_1 \ldots, X_n) = \frac{1}{Z} \prod_c \phi_c(D_c)$

→ Note #2: Looking at graph G, we can't tell which factors exist

e.g above graph could be

→ $\phi_1(A, B, C)\, \phi(B, C, D)$

or

→ $\phi_1(A, B)\, \phi_2(B, C)$, $\phi_3(C, D)$ $\phi_4(C, A)\, \phi_5(B, D)$

or → $\phi_1(A, B, C)\, \phi_2(B, C)\, \phi_3(C, D)$ $\phi_4(B, D)$

③ BN Factorization Theorem Vs MNs

Recall In BN

G is an I-map $\rightleftarrows$ P factorizes according to G
ie $I(G) \subseteq I(P)$

In MNs. (definition)
→ P factorizes over G if
→ $\exists$ subset of vars $D_1 \subseteq \vec{X}$ ..., $D_m \subseteq \vec{X}$
s.t vars D_c are cliques in G
→ and P can be written as:

$$P(X_1 \ldots X_n) = \frac{1}{Z} \prod_c \phi_c(D_c)$$

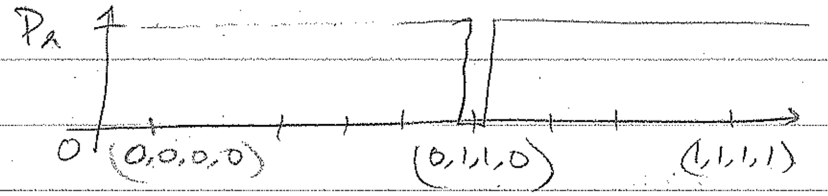
Now, for MNs one direction above is easy
P factorizes to G $\Longrightarrow$ G is an I-map of P
[so any indep $X \perp Y \mid Z \in I(G)$ is actually true in P]

but other dir is tricky!
G is an I-map of P $\underbrace{\Longrightarrow}_{\text{Not necessary}}$ P factorizes to G

Need P to be non-zero everywhere.

Why?

Example $P(A, B, C, D)$ all binary vars

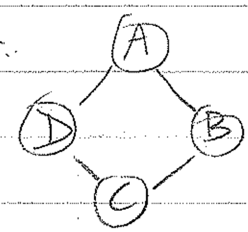


This needs a factor with all 4-vars

$$\phi(A, B, C, D) = \begin{cases} 0 & \text{if } (A=0, B=1, C=1, D=0) \\ 1 & \text{else} \end{cases}$$

Now consider $P(A,B,C,D)$ places "support" only on 8 (out of 16) states: $(0,0,0,0)$ $(1,0,0,0)$ $(1,1,0,0)$ $(1,1,1,0)$
$(0,0,0,1)$ $(0,0,1,1)$ $(0,1,1,1)$ $(1,1,1,1)$

I claim:



is an I-map

Consider $A \perp C \mid B, D$

Does P satisfy this?

Need to show $P(A=\mid B, D, C) = P(A \mid B, D)$

e.g

$$P(A=0 \mid B=0, D=0) = 1$$

so $P(A=0 \mid B=0, D=0, C=c) = 1 \quad \forall c$

so this works.

Can verify $\forall$ other settings

So P satisfies $A \perp C \mid B, D$

So G is an I-map BUT P doesn't factorize to G (need 4-clique)