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which experience plastic deformation. It is of the utmost importance, however, that the material used in the determination of the characteristic expansion-stress and length-stress curves have the same history as a test specimen for which the unknown superposed stress history is to be determined. This method of stress analysis may be especially useful in the analysis of failure in metals and perhaps in the limit design of structures.

As stated in the introduction, this research was undertaken in an attempt to anticipate how brittle minerals or rocks might react if subjected to compressive stresses under confining pressures. Griggs[5] has already demonstrated that some rocks, such as limestone and marble, behave like metals under high confining pressures. They display an elastic limit and deform by plastic flow when stressed above this point. It is anticipated, therefore, that some minerals and rocks which have been subjected to high, differential pressures will give characteristic curves somewhat similar to those of steel and magnesium. If this proves to be the case, a method will be available for achieving a better understanding of the range of pressures which have been responsible for the great deformational effects produced during metamorphism.

[5] D. Griggs, J. Geol. 44, 541 (1936).

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Thermal Noise at High Frequencies

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Spenke's discussion of thermal noise from the point of view of the electron theory leads to a result which differs from Nyquist's original formula at frequencies ν such that $h\nu/kT \geqq 1$. This discrepancy is due to the assumptions which Spenke made in his analysis.

1. INTRODUCTION

About 20 years ago Nyquist[1] published his well-known thermodynamical discussion of thermal noise; since then it has been shown that Nyquist's results could be derived with the help of the electron theory of conduction (Bernamont,[2] Bakker and Heller,[3] and Spenke[4].) Unfortunately Spenke's and Nyquist's results differ at frequencies ν such that $h\nu/kT \geqq 1$. It will be shown that the source of this discrepancy is in Spenke's treatment of the problem.

For our discussion we have to define "available power." A signal generator has an e.m.f. e and an internal impedance $Z=R+jX$; its available power P_a is defined as the maximum power which can be dissipated by an external impedance Z'. This maximum occurs if $Z'=R-jX$, so that:

$$P_a=\tfrac{1}{4}e^2/R. \tag{1}$$

2. THERMODYNAMICAL THEORY OF THERMAL NOISE

Nyquist's[1] analysis consists of three steps:

(a) The available noise power P_a in a small frequency interval $\Delta\nu$ of *any* thermal noise generator is a *universal* function of the frequency ν and the absolute temperature T of the generator. He proved this with the help of the second law of thermodynamics.

(b) The available noise power P_a in a frequency interval $\Delta\nu$ around a central frequency ν is:

$$P_a=\bar{E}\Delta\nu, \tag{2}$$

where $\bar{E}$ is the average energy of a harmonic oscillator of frequency ν and absolute temperature T (a system with two degrees of freedom). Nyquist proved this in a somewhat artificial way but his result cannot be doubted. A more detailed proof of (2) was given by Schremp.[5]

(c) $\bar{E}$ is given by the equipartition law. The average energy of an harmonic oscillator of frequency ν and temperature T is:

$$\bar{E}=kT\cdot f(\nu);\quad f(\nu)=h\nu/kT[\exp(h\nu/kT)-1]^{-1}, \tag{3}$$

and one obtains:

$$P_a=kT\Delta\nu\cdot f(\nu). \tag{4}$$

$f(\nu)=1$ if $h\nu/kT \ll 1$; at those frequencies:

$$P_a=kT\Delta\nu. \tag{5}$$

Equation (5) has been verified experimentally; the deviation between (4) and (5) becomes important in the

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[1] H. Nyquist, Phys. Rev. 32, 110 (1928).
[2] J. Bernamont, Ann. d. Physik. Paris 7, 71 (1937).
[3] C. J. Bakker and G. Heller, Physica 6, 262 (1939).
[4] E. Spenke, Wiss. Veroeff. Siemenswerke 18, 54 (1939).
[5] E. J. Schremp, in *Valley and Wallman*, "Vacuum Tube Amplifiers," M.I.T. series (McGraw-Hill Book Company, Inc., New York, 1948), Vol. 18, p. 529.

infra-red (at least at normal room temperature) and no experimental verification of (4) has been published up to now.*

In order to prove (4) in a less artificial way consider a hollow sphere at a uniform temperature T containing an harmonic oscillator of frequency ν_0 and a circuit consisting of a self-induction L, a capacity C and a resistance R and being tuned at the frequency ν_0. The hollow sphere is filled with black-body radiation of temperature T; the harmonic oscillator and the tuned circuit both interact with this radiation and will assume the temperature T. Their average energy E will be the same and the equipartition law will hold. Half of the average energy of the circuit is stored in the condenser, hence:

$$\tfrac{1}{2}C\langle v^2\rangle=\tfrac{1}{2}\bar{E}, \tag{6}$$

where v is the fluctuating voltage across C. A Fourier analysis shows that P_a is proportional to $\Delta\nu$:

$$P_a=\alpha\Delta\nu. \tag{7}$$

We now have to prove: $\alpha=\bar{E}$. Let the thermal noise of the circuit for the frequency interval $\Delta\nu$ be described by an e.m.f. e in series with R, then after (1):

$$\langle e^2\rangle=4P_aR=4\alpha R\Delta\nu. \tag{7a}$$

The contribution of e to the mean square value of the noise voltage v across C in the frequency interval $\Delta\nu$ is:

$$\langle e^2\rangle\left|1+\left(\frac{1}{j\omega L}+j\omega C\right)R\right|^{-2}=4\alpha R[1+4(\omega-\omega_0)^2C^2R^2]^{-1},$$

if $(\omega-\omega_0)\ll\omega_0$ and $\omega_0{}^2LC=1$. Integrating over all intervals $\Delta\nu$:

$$\langle v^2\rangle=\alpha/C \quad\text{or}\quad \tfrac{1}{2}C\langle v^2\rangle=\tfrac{1}{2}\alpha \quad\text{or}\quad \alpha=\bar{E}.$$

3. THE CORPUSCULAR THEORY OF THERMAL NOISE

The following model of a conductor is used here: Electrons in the conductor move in all possible directions, they have a Fermi velocity distribution at a temperature T. The electrons which have a velocity close to the top of the Fermi distribution can gain energy when an electric field is applied and thus give rise to conduction. Those electrons can collide with the positive ions of the lattice after having traveled an average "free path length" Ω; as they all have practically the same velocity v, the average time of flight τ_0 along the free path length Ω is Ω/v. It is assumed that the collisions with the positive ions of the lattice occur without an exchange of energy and that the electrons are scattered in random directions after such a collision.

The motion of electons along the free path length Ω gives rise to current pulses of length τ_0 in the conductor (thermal noise). Carrying out a Fourier analysis of the resultant noise current $I(t)$, Bakker and Heller[3] and also Spenke[4] found for its Fourier components i:

$$\langle i^2\rangle=\langle i_0{}^2\rangle(1+\omega^2\tau_0{}^2)^{-1}, \tag{10}$$

where $\langle i_0{}^2\rangle$ is the l.f. value of $\langle i^2\rangle$.

If a d.c. field is applied to the conductor, the electrons gain energy in the field; due to the collisions with the lattice a state of equilibrium is established in which the electrons have an average velocity in the direction of the field. This drift velocity, which depends upon τ_0, constitutes a current and so the collisions with the lattice give rise to a finite d.c. conductivity G_0. If an a.c. field is applied, the conductivity G remains independent of frequency as long as its period is small in comparison to τ_0. If it is comparable to τ_0, the electrons will on the average gain less energy in the field, as its phase changes appreciably during the motion along the free path length; at those frequencies $G<G_0$. Spenke[4] found:

$$G=G_0(1+\omega^2\tau_0{}^2)^{-1}; \tag{11}$$

this result was first obtained by Kronig.[6] Hence:

$$P_a\equiv\langle i^2\rangle/4G\equiv\langle i_0{}^2\rangle/4G_0. \tag{12}$$

Evaluation of $\langle i_0{}^2\rangle$ and G_0 then yields (5). This holds for all frequencies.

Spenke's result is not a very surprising one, for in his model no exchange of energy occurs when the electrons collide with the lattice. The average energy stored in the condenser C of an LCR circuit tuned at the frequency ν_0 will then be in equilibrium with the thermal agitation of the electrons. The average energy for translatory motion as given by the equipartition law does not contain the Planck factor $f(\nu)$ and hence one would expect that, independent of the frequency ν_0:

$$\tfrac{1}{2}C\langle v^2\rangle=\tfrac{1}{2}kT$$

so that (5) should be valid for all frequencies. Nowhere in this model is any room left for an expression containing $h\nu/kT$ and this cannot be improved by using another velocity distribution either.

But in a genuine thermodynamical discussion it is not sufficient to state that the electrons have a velocity distribution corresponding to the temperature T; one has to immerse the conductor in a heat reservoir of large heat capacity and one has to investigate how the spontaneous current fluctuations in the conductor are coupled to this heat reservoir. Undoubtedly this coupling occurs as follows: The heat reservoir is in thermal equilibrium with the lattice vibrations and the electrons couple the current fluctuations to those vibrations. This coupling is due to the fact that in each collision an energy exchange $\pm h\nu$ occurs between the colliding electron and the lattice (ν is one of the frequencies of the lattice vibrations). The average energy stored in the condenser C of an LCR circuit tuned at the frequency ν_0

* It would be worthwhile to test its validity in the centimeter wave-length region at low temperatures. If $T=1.2°K$ then $h\nu/kT=1$ at 1 cm wave-length ($\nu=3\times10^{10}$); the deviation between (4) and (5) is then about 40 percent, which might be detectable with present techniques.

[6] R. de L. Kronig, Proc. Roy. Soc. **A124**, 409 (1929); **133**, 255 (1931).

will therefore be in equilibrium with the average energy $\bar{E}$ of the *lattice vibrations* of frequency ν_0, and $\bar{E}$ is equal to the average energy of an harmonic oscillator given by (3). Hence in the correct theory:

$$\tfrac{1}{2}C\langle v^2\rangle = \tfrac{1}{2}\bar{E}$$

so that we obtain (4) instead of (5).

4. THE SOURCE OF THE ERROR

As it is now clear that Spenke's analysis is wrong, it has to be investigated whether the error is in (10) or in (11). According to the Fourier analysis, fluctuating quantities $X(t)$ have Fourier components x_n of frequency ω_n such that:

$$\langle x_n{}^2\rangle = 4\Delta\nu \int_0^\infty \langle X(t)X(t+w)\rangle \cos\omega_n w dw. \quad (13)$$

The correlation function $\langle X(t)X(t+w)\rangle$ is zero if $|w| \ll \tau$, where τ is the *correlation time* of the fluctuating quantity. If $(\omega_n\tau) \ll 1$:

$$\langle x_n{}^2\rangle = 4\Delta\nu \int_0^\infty \langle X(t)X(t+w)\rangle dw.$$

This integral is independent of the frequency and is completely determined by the equipartition law; any theory using this law will give the right formula at low frequencies even though it gives wrong values for τ and for $\langle X(t)X(t+w)\rangle$. In the latter case the noise spectrum will have the wrong shape at high frequencies.

Lameris[7] found from his reflection measurements in the infra-red that (11) was correct. He deduced from his measurements that at room temperature $\omega\tau_0$ was equal to unity at the frequencies $\nu(=\omega/2\pi)$: 2.36×10^{13} for Ag, 2.39×10^{13} for Au and 5.23×10^{13} for Pt. But after (4) the available noise power shows a marked decrease if $h\nu/kT=1$, this corresponds to $\nu=0.6\times10^{13}$ at room temperature. Hence the correlation time τ of the fluctuations is about 5–10 times as large as the correlation time τ_0 of the resistance. But in Spenke's model $\tau=\tau_0$ so that the error is definitely in (10).

Spenke[4] made the following assumptions:

(a) After collisions the electrons are scattered in random directions. In fact the scattering over small angles is favored.

(b) No energy is exchanged in the collision process. In fact each collision corresponds to an exchange of energy with the lattice. It is therefore likely that an exact calculation of $\langle i^2\rangle$, in which the above facts are taken into account will give (4) instead of (5). The calculation is very difficult, whereas the thermodynamical analysis is very simple.

In order to show that the two above facts work in the right direction, consider a number of electrons dN having a velocity between v and $v+dv$. In calculating the correlation time of the resistance we have the problem: "What is the probability that the electrons will obtain a different velocity v' when traveling along the distance dx?" This will give a correlation time equal to the time τ_0 between two collisions, if on the average the electrons lose the energy gained by traveling in the electric field after each collision (for a more detailed discussion compare Spenke's[4] paper). In calculating the correlation time τ of the current fluctuations the problem is: "How long does it take before there is no longer any correlation between the current due to the motion of these dN electrons and the original current?" If the scattering over small angles is favored, it will take a number of collisions before this is achieved so that τ is much longer than τ_0, just as was required.

5. AVAILABLE NOISE POWER OF AN ANTENNA

Burgess[8] calculated the available noise power P_a in the frequency interval $\Delta\nu$ of an antenna receiving black body radiation of uniform temperature and showed that it was given by (4). That this leads to (4) and not to (5) can be shown by a simple argument.

At low frequencies the energy density of black body radiation is:

$$U_\nu \equiv (8\pi\nu^2/c^3)\cdot kT\Delta\nu \text{ (Rayleigh-Jeans' law)}. \quad (14)$$

But at those frequencies $P_a = kT\Delta\nu$, hence:

$$U_\nu = (8\pi\nu^2/c^3)\cdot P_a. \quad (15)$$

Equation (15) describes a property of the antenna which does not depend upon quantum phenomena so that it will hold for all frequencies. But at high frequencies (14) has to be replaced by Planck's law. As this means that (14) has to be multiplied by the Planck factor $f(\nu)$, P_a has to be multiplied by the same factor, so that (4) holds.

The author is indebted to Dr. A. J. Dekker and Dr. F. A. Kaempffer for many discussions on this subject.

[7] P. F. Lameris, Ph.D. Thesis, Groningen, 1936.

[8] R. Burgess, Proc. Phys. Soc. 53, 293 (1941).