2

Components

Resistors, capacitors, and inductors are the basic components that make up electrical circuits. For each, we will give a formula that relates voltage and current that is based on an underlying physical law. We will see what happens when we apply a voltage or current source to circuits with these components. In addition, we consider circuits with a semiconductor device, the diode. Analyzing circuits requires a lot of algebra. The expressions get so complicated that it is difficult to see what is going on. For this reason, it is important to express the results in a simple form that can be understood, and it is worthwhile to try different approaches to find the simplest path to a solution and to make sure that you understand it.

2.1 Resistors

In a resistor, the voltage is proportional to the current. This is Ohm's law. We call the ratio of voltage and current the *resistance* and write it as

$$R = V/I. \tag{2.1}$$

The units of resistance are ohms. The abbreviation for ohms is Ω (the Greek capital letter *omega*). Figure 2.1a gives the circuit symbol for a resistor. In a series connection of resistors (Figure 2.1b), the voltage across the entire combination is the sum of the voltages across the individual resistors. Kirchhoff's current law tells us that the current in each resistor is the same. This means that the resistance of a series connection is the sum of the individual resistances:

$$R = \sum_i R_i, \tag{2.2}$$

where i is an index for the resistors.

The inverse of the resistance is called the *conductance*, with units of siemens (S). The conductance G is written as

$$G = I/V. \tag{2.3}$$

In a parallel connection of resistors (Figure 2.1c), the total current is the sum of the individual currents. Kirchhoff's voltage law tells us that voltage across each

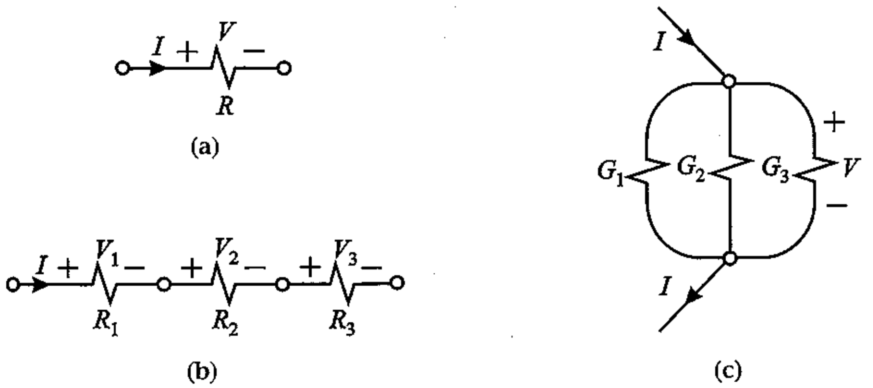


Figure 2.1. Circuit symbol for a resistor (a), series connection (b), and parallel connection (c).

resistor is the same. This means that we can write the conductance of a parallel connection as the sum of the individual conductances:

$$G = \sum_i G_i. \tag{2.4}$$

Since most people think in terms of resistances rather than conductances, we can substitute to find the equivalent formula and get

$$R = \frac{1}{\sum_i 1/R_i}. \tag{2.5}$$

This formula is easy to compute on a calculator, but it is a little complicated to write, and we will use the shorthand $\|$ to indicate a parallel connection. For example, for three resistors connected in parallel, we write

$$R_1 \parallel R_2 \parallel R_3 = \frac{1}{1/R_1 + 1/R_2 + 1/R_3} \tag{2.6}$$

to avoid writing out the formula. Circuits can often be simplified by repeatedly applying the parallel and series formulas. In more complicated cases, Kirchhoff's laws can be used to set up matrix equations for a solution.

The power dissipated in a resistor can be written in several ways. We can write

$$P(t) = V(t)I(t) = V^2(t)/R = I^2(t)R. \tag{2.7}$$

In terms of conductance, we can write

$$P(t) = V^2(t)G = I^2(t)/G. \tag{2.8}$$

These formulas mean that power is proportional to the square of a voltage or current. For cosine signals, we can rewrite Equation 1.11 to find the average power P_a as

$$P_a = \frac{V_p I_p}{2} = \frac{V_p^2}{2R} = \frac{I_p^2 R}{2}, \tag{2.9}$$

(a)

Color	Digit	Multiplier
Silver		0.01
Gold		0.1
Black	0	1
Brown	1	10
Red	2	100
Orange	3	1,000
Yellow	4	10,000
Green	5	100,000
Blue	6	1,000,000
Violet	7	
Grey	8	
White	9	

(b)

Figure 2.2. Axial lead resistor (a), and the resistor color code (b). Silver and gold bands are not used for numbers, only for multipliers, while violet, grey, and white are used only for numbers. The color sequence from red to violet is the same as that of the visible spectrum, and this may help you remember it.

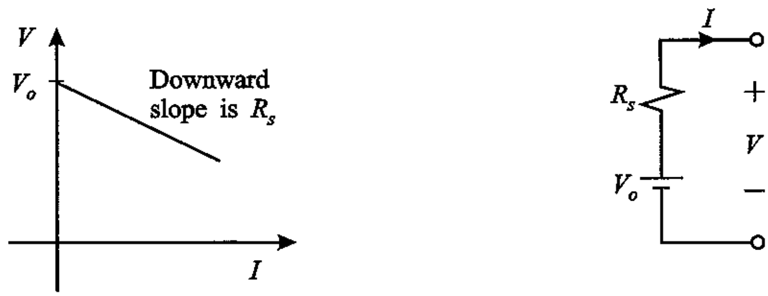
where V_p and I_p are peak values. In terms of the peak-to-peak values that we measure in the lab, we have

$$P_a = \frac{V_{pp}I_{pp}}{8} = \frac{V_{pp}^2}{8R} = \frac{I_{pp}^2 R}{8}. \tag{2.10}$$

Resistors come in a variety of sizes and configurations. The ones we use in the NorCal 40A are called axial-lead resistors (Figure 2.2a), because the leads extend along the axis of the resistor. It is traditional to indicate the resistance with three color bands. The first two bands give the first two digits of the resistance, while the third band indicates a multiplier. The colors and their digits and multipliers are given in Figure 2.2b. For example, a 100-Ω resistor would have the following bands: brown, black, brown. There is a fourth band that indicates the tolerance. We use 5% resistors with a gold band. The actual paint colors you see vary somewhat, so that it is not always easy to read the code, and you may have to check the resistance with a meter.

2.2 Sources

To provide the power for electronic circuits, we can connect a battery or a plug-in power supply. These *sources* are specified by the voltage that they deliver, such as 1.5 V for AA batteries or 12 V for an adapter. However, one should think of this as a nominal voltage, because the actual voltage depends on the current, generally dropping as more current is drawn (Figure 2.3a). It is as if there is a resistor inside



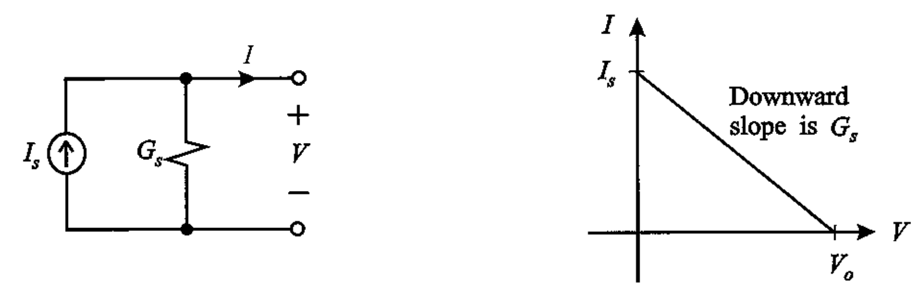
(a) (b)

Figure 2.3. Voltage versus current for a source (a), and circuit model with an ideal voltage source V_o and a resistor R_s (b).

the battery or adapter. As the current increases, the voltage drops because the voltage across the resistor increases. Often, this is really what is happening, and you may feel a battery or adapter getting hot if you draw a lot of current. We will represent our source by a circuit *model* that has two parts (Figure 2.3b). The first is an *ideal voltage source* V_0. An ideal voltage source maintains the same voltage regardless of the current. The second is a *source resistance* R_S that provides the drop in voltage as the current increases.

We can relate the circuit model to our voltage plot to find the values of V_0 and R_S. When the circuit has no components attached, no current is drawn. This is the *open-circuit* condition. Because there is no current, there is no voltage drop across R_S, and the output voltage is the same as that of the ideal source. We call V_0 the open-circuit voltage, and it is just the y intercept of the voltage plot. The downward slope of the plot is the voltage drop per unit current, which is just the resistance R_S. With these values, the circuit model has the same relation between voltage and current as the real source. This circuit model with an ideal voltage source and a series resistor is called a *Thevenin* equivalent circuit.

Alternatively, we could use a different circuit model called the *Norton* equivalent circuit (Figure 2.4a). The Norton circuit also has two parts: an *ideal current source* I_S that maintains the same current regardless of what is connected to it



(a) (b)

Figure 2.4. Norton equivalent circuit consisting of an ideal current source and a parallel resistor (a). Finding I_s and G_s from a plot of the current (b).

and a parallel resistor with a conductance G_S. We can find the components of the Norton equivalent circuit from a plot of the current (Figure 2.4b). We get I_S by considering the current when the voltage is zero. We call this the *short-circuit* condition, because it corresponds to putting a short circuit across the output. If the voltage is zero, then the current in the source conductance G_S is zero and the entire current I_S is delivered to the load. This means that I_S is the y intercept on our current plot. Next we find G_S by letting the output voltage of the circuit model increase. As the output voltage increases, the output current drops because some of the current begins to flow through G_S. The slope is just the negative of the conductance G_S. Because the Norton equivalent circuit also produces the same voltage and current as the real source, we could use it in place of the Thevenin. Usually we will choose the one that gives the simpler algebra.

If we compare the Norton circuit with the Thevenin circuit, the only difference in using the slope to calculate R_S and G_S is that we swap the current and voltage axes. This means that

$$R_S = 1/G_S \tag{2.11}$$

and that the resistors are really one and the same. We could just as well use R_S in the Norton equivalent as G_S, or we could use G_S instead of R_S in the Thevenin equivalent. From the graph, we can also calculate R_S in terms of the open-circuit voltage V_o and the short-circuit current I_S from Figure 2.4b as

$$R_S = V_o/I_S. \tag{2.12}$$

Thus if we know the Thevenin components, we can calculate the Norton components and vice versa. As a practical matter, many batteries and power adapters will not tolerate a short circuit, so that you may only be able to measure the voltage and current over a small range. To find I_S you may have to extrapolate out to the axes.

Next consider what happens if we connect an ideal voltage source V to a Thevenin source (Figure 2.5a). We use Kirchhoff's laws to write the voltage V

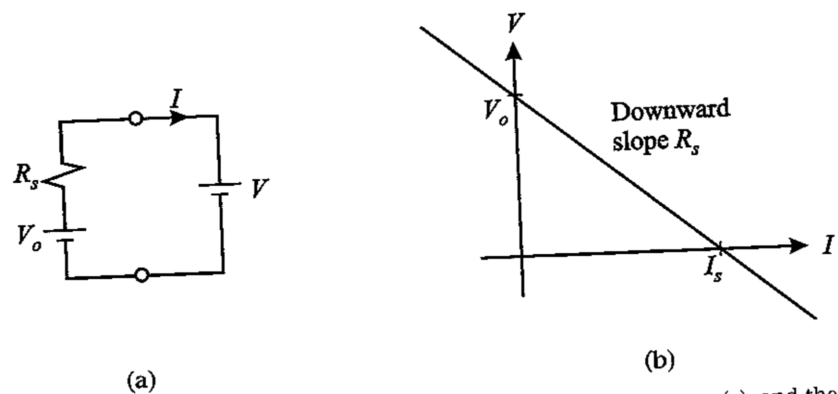


Figure 2.5. Connecting an ideal voltage source to a Thevenin source (a), and the extended voltage–current plot (b).

as

$$V = V_o - R_s I. \tag{2.13}$$

This is the equation of a straight line, with voltage intercept V_o and downward slope R_s (Figure 2.5b). If the load voltage V is negative, then the current is greater than the short-circuit current. However, if V is greater than V_o, then the current turns negative. This happens when we charge a battery. A battery charger has a larger voltage than the open-circuit voltage of the battery. For example, for "12-volt" lead-acid batteries, the open-circuit voltage for a battery that is 70% discharged is 12 V. We charge the batteries at 13.8 V. Initially, a large current flows into the battery, but this decreases as the battery charges. The open-circuit voltage when the battery is fully charged is 12.8 V.

2.3 Dividers

Now we are ready to analyze a circuit with a Thevenin source and a resistor R_l, which is called a *load* (Figure 2.6a). First we redraw the circuit in Figure 2.6b to emphasize that the two resistors are in series. We can write the output voltage V in terms of the current I as

$$V = IR_l. \tag{2.14}$$

We use Kirchhoff's laws to write the supply voltage V_o in a similar fashion as

$$V_o = I(R_l + R_s). \tag{2.15}$$

We can divide these two equations to find the output voltage:

$$\frac{V}{V_o} = \frac{R_l}{R_l + R_s} = \frac{1}{1 + R_s/R_l}. \tag{2.16}$$

The input voltage V_o divides proportionally between the load resistor R_l and the source resistor R_s, and for this reason we call this circuit a *voltage divider*. The larger the load resistor, the larger the output voltage. I have written two equivalent expressions, $R_l/(R_l + R_s)$ and $1/(1 + R_s/R_l)$. The first expression is easier to

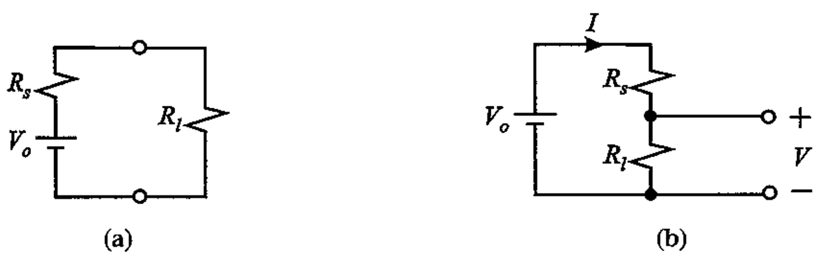


Figure 2.6. Voltage-divider circuit with a Thevenin source and a load resistor (a). Redrawn circuit for analysis (b).

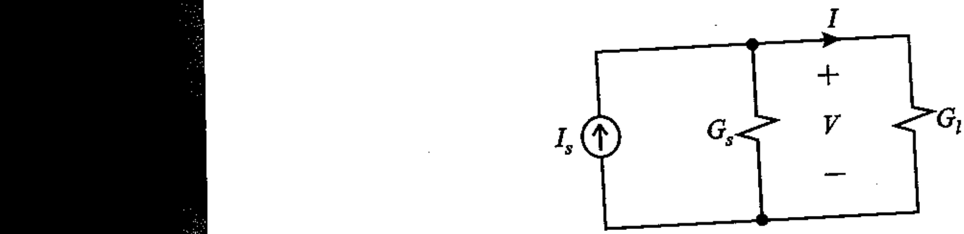


Figure 2.7. Current-divider circuit with a Norton source and a load resistor.

understand, because it can be interpreted as a ratio of load resistance to total circuit resistance. The second expression is convenient for a calculator because each resistance value need only be entered once.

Now consider the *current-divider* circuit, which has a Norton source and a load resistor with a conductance G_l (Figure 2.7). We can write the load current I in terms of the voltage V as

$$I = VG_l. \tag{2.17}$$

Using Kirchhoff's laws, we can write the supply current I_s as

$$I_s = V(G_l + G_s). \tag{2.18}$$

We divide these equations to find the output current:

$$\frac{I}{I_s} = \frac{G_l}{G_l + G_s} = \frac{R_s}{R_s + R_l}. \tag{2.19}$$

For the current divider, the larger the load conductance, the larger the output current. In terms of resistances, the smaller the load resistance, the larger the output current. Because voltage dividers and current dividers will arise repeatedly, it is a good idea to memorize these formulas.

2.4 Look-Back Resistance

So far we have considered a Thevenin or Norton equivalent circuit as a model for a real source. We make a plot of voltage and current to find the parameters of the model. We can also use a Thevenin or Norton source to simplify a section of a circuit diagram. The idea is to replace a complicated source circuit with a simpler equivalent. If the equivalent circuit has the same relationship between voltage and current, then the equivalent circuit will produce the same voltage and current in a load as the original source circuit, regardless of what the load is. This result is called *Thevenin's theorem*. To make the substitution, we need to find the components of the Thevenin equivalent circuit. One approach is to calculate the open-circuit voltage V_o and the short-circuit current I_s. Then we can take the ratio to find R_s.

There is another way to find R_s from the circuit diagram. If there is a voltage or current source in the circuit, then V_o and I_s are both proportional to the value of the source. This means that the slope of the voltage–current plot does not change

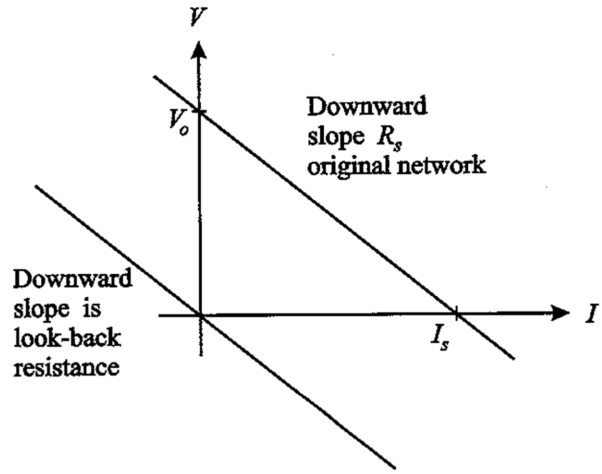


Figure 2.8. Voltage and current for a source network. The slope of the plot is the source resistance R_s. Also shown is a plot of voltage and current when the internal source is turned off. This slope is the look-back resistance. The slope of each plot is the same, so that the look-back resistance gives us R_s.

as we reduce the source to zero (Figure 2.8). We call the resistance when the internal source is zero the *look-back resistance*, because people talk about looking back into the circuit to find its resistance. The look-back resistance is equal to R_s.

As an example, we find the Thevenin equivalent circuit for a voltage-divider circuit (Figure 2.9a). We can write the open-circuit voltage V_o using the voltage-divider

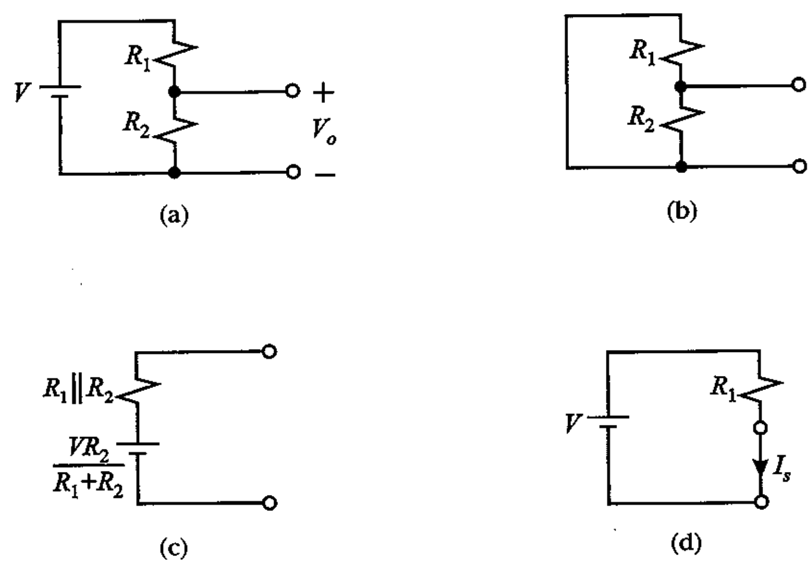


Figure 2.9. Finding the Thevenin equivalent circuit for a voltage divider (a). Circuit with source turned off for calculating the look-back resistance R_s (b), and the components of the Thevenin equivalent circuit (c). Calculating the short-circuit current I_s (d).

formula (Equation 2.16):

$$V_o = \frac{VR_2}{R_1 + R_2}. \tag{2.20}$$

To find the look-back resistance R_s, we set the ideal voltage source V to zero (Figure 2.9b). An ideal voltage source set to zero is no different from a short circuit, because its voltage is zero, regardless of the current. If we replace V by a short circuit, we have only the two resistors in parallel, which gives a look-back resistance of

$$R_s = R_1 \parallel R_2. \tag{2.21}$$

This gives us the Thevenin equivalent circuit shown in Figure 2.9c. This circuit is equivalent in that it will produce the same voltage and current in a load that the divider would, but it is simpler. As a check we can calculate the short-circuit current I_s by shorting out R_2 (Figure 2.9d). We can write

$$I_s = V/R_1. \tag{2.22}$$

If we divide V_o by I_s, we get R_s again.

However, the Thevenin equivalent is not identical to the original divider because the voltages and currents *inside* the divider and those of the Thevenin equivalent circuit are not the same. They could not be, because the Thevenin equivalent has fewer elements. It only produces the same voltage and current in an *outside* load. As an exercise, you should find the Thevenin and Norton equivalent for a current-divider circuit. For this you will need to set a current source to zero to find the look-back resistance. A current source set to zero acts just like an open circuit, since no current flows, regardless of the voltage. This means that the current source should be replaced by an open circuit for the calculation.

2.5 Capacitors

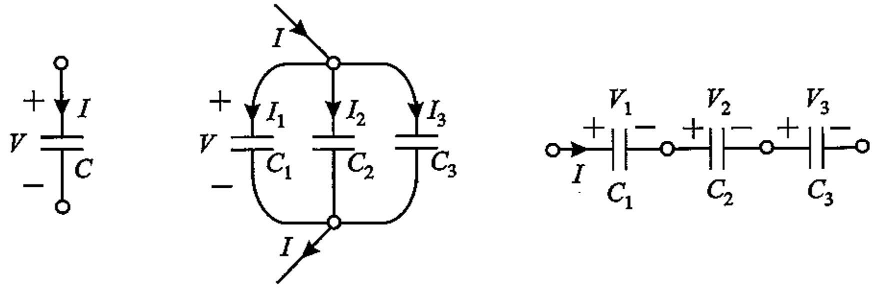
In a capacitor, voltage is proportional to charge. Fundamentally, this arises from the relation between charge and electric field expressed in Gauss's law. This is in contrast to a resistor, where voltage is proportional to current. We write

$$C = \frac{Q}{V}, \tag{2.23}$$

where C is the capacitance, with units of farads (F). We can think of the charge as the time integral of the current:

$$Q(t) = \int_0^t I(t)\, dt. \tag{2.24}$$

We have to be careful how we interpret this expression, because t is doing double duty mathematically. It is the argument of Q, and in this role, it appears in $Q(t)$



(a) (b) (c)

Figure 2.10. Circuit symbol for a capacitor (a), parallel connection (b), and series connection (c).

and as the limit of integration $\int_0^t$. It also appears in the integrand as the argument of I and the differential dt. Often people are disturbed by this, and for this reason they change the variable in the integrand to something else, like u. This makes the mathematical grammar correct but obscures the physics, because time really is the variable in both cases. We will say "ain't" in order to keep the physics. We can rewrite Equation 2.23 as

$$C = \frac{\int_0^t I(t)\,dt}{V}. \tag{2.25}$$

The circuit symbol for a capacitor represents a pair of plates (Figure 2.10a). If we have capacitors in parallel (Figure 2.10b), Kirchhoff's laws tell us that the voltage for each capacitor is the same, and the total current is the sum of the individual capacitor currents. The integral of the total current is also the sum of the integrals. This means that the capacitance of a parallel connection is the sum of the individual capacitances:

$$C = \sum_i C_i. \tag{2.26}$$

The series connection shown in Figure 2.10c is trickier, but we can follow the same logic we used for conductances. If we invert Equation 2.25, we have

$$\frac{1}{C} = \frac{V}{\int_0^t I\,dt}. \tag{2.27}$$

In a series circuit, the current in each capacitor is the same, and the current integral for each capacitor is the same. However, the total voltage is the sum of the individual voltages. Thus we can write

$$\frac{1}{C} = \sum_i \frac{1}{C_i}. \tag{2.28}$$

This is similar to the expression for the resistance of parallel resistors.

Earlier we studied resistor divider circuits. We can also make divider circuits with capacitors. Figure 2.11 shows a capacitive voltage divider. We can write the

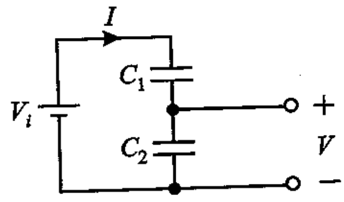


Figure 2.11. Capacitive voltage-divider circuit.

output voltage V as

$$V = \frac{\int_0^t I\,dt}{C_2} \tag{2.29}$$

and the input voltage V_i as

$$V_i = \frac{\int_0^t I\,dt}{C_1} + \frac{\int_0^t I\,dt}{C_2}. \tag{2.30}$$

If we divide these two expressions and simplify, we get

$$\frac{V}{V_i} = \frac{C_1}{C_1 + C_2}. \tag{2.31}$$

This formula says that the output voltage becomes larger if C_1 is made larger. This is the opposite of the resistive potential divider, where the voltage becomes larger if the load resistor is made larger.

2.6 Energy Storage in Capacitors

Capacitors store energy rather than dissipate it as heat like resistors. We can calculate the stored energy, starting with a capacitor with no charge or voltage on it at time $t = 0$. When a current flows into the capacitor, the voltage across it increases. The power going into the capacitor is given by $P(t) = V(t)I(t)$. We can write the energy $E(t)$ stored in the capacitor as an integral of the power $P(t)$. We write the stored energy $E(t)$ as

$$E(t) = \int_0^t P(t)\,dt = \int_0^t V(t)I(t)\,dt. \tag{2.32}$$

Now we rewrite Equation 2.25 in differential form as

$$I = CV', \tag{2.33}$$

where the prime denotes a time derivative. We substitute for current in the previous formula to get

$$E = \int_0^t VCV'\,dt. \tag{2.34}$$

If we use V as the variable of integration instead of t, we can write

$$dV = V'dt \tag{2.35}$$

and we get

$$E = C\int_0^V V\,dV = \frac{CV^2}{2}. \tag{2.36}$$

The fact that capacitors store energy rather than dissipate it means that ordinarily capacitors do not get hot. In practice, capacitors have some resistance, and large currents can cause them to heat up. The energy stored in a capacitor can discharge dangerously quickly if the output terminals are shorted together. It is important to remember this when working with large high-voltage capacitors. Even with the circuit off, the capacitors may be charged to a high voltage and can deliver a lethal shock.

2.7 RC Circuits

If we connect a resistor to a charged capacitor (Figure 2.12a), the capacitor will discharge through the resistor. This dissipates the energy stored in the capacitor as heat in the resistor. This is often used in high-voltage circuits to reduce the capacitor voltage to a safe level when the circuit is turned off. We call the resistor a *bleeder resistor*. We write the current I in two ways:

$$I = V/R = -CV'. \tag{2.37}$$

There is a minus sign because the current arrow points *out* from the capacitor. We could change the direction of the arrow and make it positive, but this would make the resistor current negative. We are stuck with one minus sign, no matter how we point the arrow. We can rewrite this as a differential equation

$$RCV' + V = 0. \tag{2.38}$$

The quantity RC has dimensions of time and is called the time constant. We will write it as τ (the Greek letter *tau*). We write

$$\tau = RC. \tag{2.39}$$

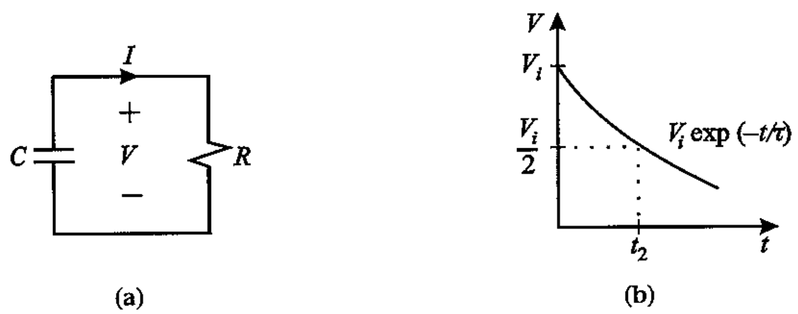


Figure 2.12. Charged capacitor with a bleeder resistor (a) and the decaying voltage (b).

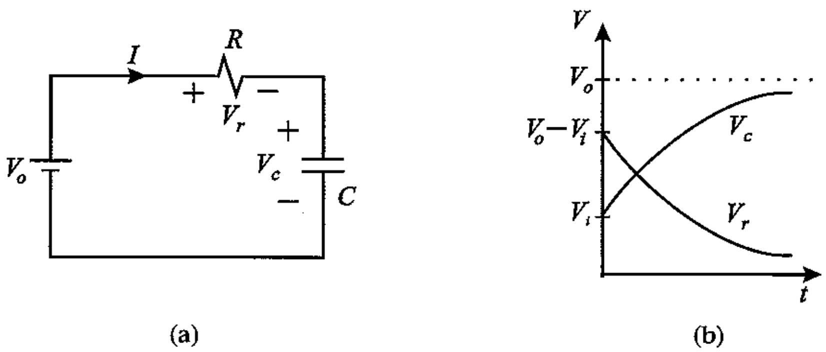


Figure 2.13. RC network with an added voltage source V_o (a). Resistor and capacitor voltages (b).

The solution is given by

$$V(t) = V_i \exp(-t/\tau), \tag{2.40}$$

where V_i is the initial voltage, and $\exp(x)$ is the exponential function e^x. The voltage decays exponentially with time (Figure 2.12b). It reaches half its initial level at time t_2, given by

$$t_2 = \tau \ln 2 = 0.69\tau. \tag{2.41}$$

In the lab, t_2 is convenient to measure, and this formula lets us work backward to find τ.

In the lab we study an RC network that is repeatedly charged and discharged by a function generator. We can understand the behavior if we add a voltage source V_o to our RC circuit (Figure 2.13a). The math is easier if we solve for the resistor voltage, which decays to zero. We use Kirchhoff's voltage law to write

$$V_o = V_c + V_r, \tag{2.42}$$

where V_c is the capacitor voltage and V_r is the resistor voltage. This formula lets us find V_c if we know V_r. The current I is written as

$$I = V_r/R = CV_c'. \tag{2.43}$$

Since V_o is fixed, Equation 2.42 tells us that $V_c' = -V_r'$, and we write

$$I = V_r/R = -CV_r'. \tag{2.44}$$

This gives us

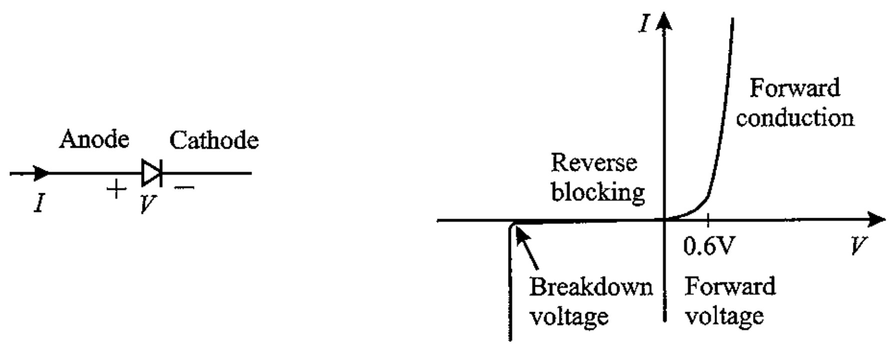
$$\tau V_r' + V_r = 0. \tag{2.45}$$

The solution is a voltage that decays from its initial value with the same time constant τ as before. The initial value of V_r is given by $V_o - V_i$, where V_i is the initial

voltage on the capacitor. We plot the capacitor voltage as $V_o - V_r$ (Figure 2.13b). This means that the capacitor voltage charges exponentially from V_i to V_o with the same time constant τ.

2.8 Diodes

Diodes are devices that let current pass more easily in one direction than the other. They do not obey a simple linear relation between voltage and current like resistors, or voltage and charge like capacitors, and so we say diodes are *nonlinear*. Figure 2.14 shows the schematic symbol for a diode and a representative plot of the current as a function of voltage. We call these plots I–V curves. For positive voltages, the diode conducts well (we say the diode is "on") if the voltage exceeds a small threshold that we call the *forward voltage*. For a silicon diode such as the 1N4148 in our transceiver, the forward voltage is 0.6 V. The power is usually low when the diode is conducting in the forward direction, because the voltage is low. For negative voltages the current is quite small (we say the diode is "off"). The reverse current for the 1N4148 is only a few nanoamps. Since the current is small, the power is also small. When we get to a sufficiently negative large voltage, 75 V for the 1N4148, the diode breaks down, and the current increases rapidly. In the breakdown region, both the voltage and current are large, and thus the power dissipated is large, and we have to be careful not to destroy the diode. Usually we avoid the breakdown region and operate with the diode either on or off. One way to think of a diode is that it limits a positive voltage and blocks a negative current.



(a) (b)

Figure 2.14. Circuit symbol and terminal names for a diode (a). We need names for each terminal, because unlike a resistor or capacitor, they are distinct. The names cathode and anode come from the days of vacuum-tube diodes. In a vacuum-tube diode the cathode emits electrons, and the anode collects them. Because electrons have a negative charge, however, we will usually say that current flows from the anode to the cathode, even though the electron flow is in the opposite direction. The diodes we use are made of silicon, but the names for the terminals are the same. Usually diode manufacturers mark the cathode end with a black stripe. In (b), a current plot for a diode is shown.

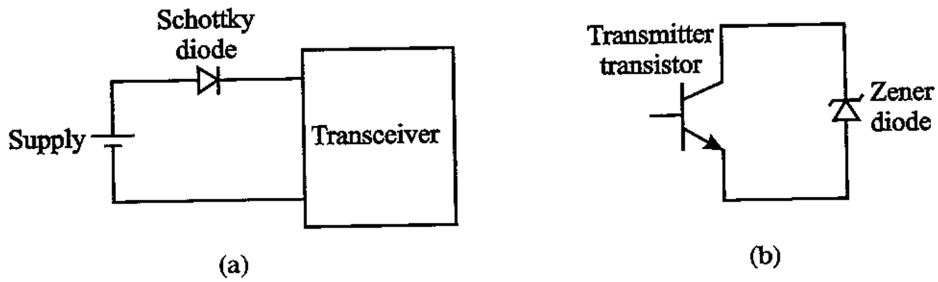


Figure 2.15. Protecting a transceiver from a negative voltage with a Schottky diode (a), and protecting a transistor from a high positive voltage with a Zener diode (b). The Zener symbol has short tags on the cathode bar that indicate that the diode conducts in both directions.

We can consider a diode as a kind of self-activated switch. When the voltage or current is positive, the switch is on. When the voltage or current is negative, the switch is off.

You should take note of the diode part number: 1N4148. There are thousands of different kinds of diodes and transistors, and there are standard registration and numbering systems to help us keep them straight. "1N" denotes a diode, and "2N" a transistor. The 1N and 2N designations are not trademarks, and they do not belong to a single manufacturer. Many different manufacturers make a 1N4148. For the complete specifications for this diode and other parts that appear in the transceiver, see Appendix D.

We use four different kinds of diodes in the NorCal 40A. We will only discuss how they act in a circuit, leaving the details of the operation of the diodes to a book on solid-state devices. In addition to the 1N4148 silicon diode, we use a Schottky diode, the 1N5817, that is made out of a contact between metal and silicon. Schottky diodes have a low forward voltage, only about 0.2 V, and this reduces the power dissipation if the current is large. We use the Schottky diode to prevent a negative power-supply voltage from being applied to the radio, which could damage some of the circuits (Figure 2.15a). If the voltage is positive, the current passes through the diode with only a small forward voltage drop. However, if the voltage is negative, the diode turns off, and current does not flow to the radio. In addition, we use a Zener diode that is fabricated to have a controlled breakdown voltage and to allow a reasonable amount of breakdown current to flow safely. A 1N4753A, 36-V, Zener appears across the output of our transmitter transistor (Figure 2.15b). This restricts the peak transistor voltage to 36 V to prevent damage to the transistor. The fourth type of diode we use is a varactor diode. A varactor operates with negative voltages, where the diode acts like a capacitor with a capacitance that is controlled by the voltage. We use Motorola's MVAM108 varactor to control the frequency of the transceiver.

Probably the single most important application of diodes is in changing the sinusoidal AC wall voltage to the steady DC voltage that is required in most circuits. AC is short for *alternating current*, and DC stands for *direct current*. This is called *rectification*. A rectifier circuit is shown in Figure 2.16a. The diode turns off whenever the voltage is negative, so that the negative parts of the waveform are removed (Figure 2.16b). This is called a *half-wave* rectifier.

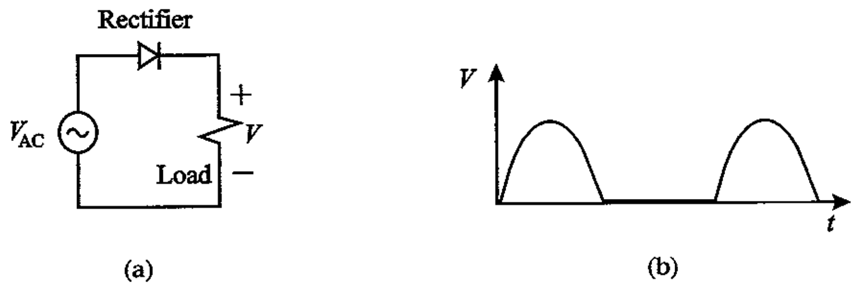


Figure 2.16. Rectifying an AC supply voltage with a diode to produce DC (a), and the voltage waveform (b).

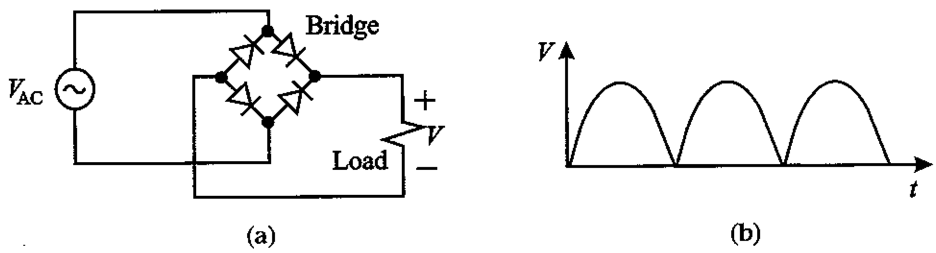


Figure 2.17. Bridge-rectifier circuit (a), and the full-wave rectified waveform (b).

Figure 2.17a shows a circuit that flips the negative parts of the AC waveform, rather than removes them. It has four diodes in a ring (Figure 2.17b). This arrangement is called a *bridge*. You should notice that the DC waveform is still pretty bumpy, and in most circuits we would have to smooth the voltage out with a capacitor before we could use it.

2.9 Inductors

In an inductor, the voltage is proportional to the time derivative of the current. The circuit symbol for an inductor is a coil (Figure 2.18a). The proportionality constant is called the inductance, with units of henries (H). The inductance is traditionally written as L, and we write

$$V = LI'. \tag{2.46}$$

The voltage arises from the fact that currents produce magnetic fields, and time-varying magnetic fields produce voltages through Faraday's law. We consider inductors in series (Figure 2.18b), and rewrite the inductance formula to get

$$L = \frac{V}{I'}. \tag{2.47}$$

In a series connection the current for each inductor is the same. This means that the denominator, I', is the same for each inductor. The total voltage is the sum of the individual inductor voltages. Therefore the total inductance is the sum of the

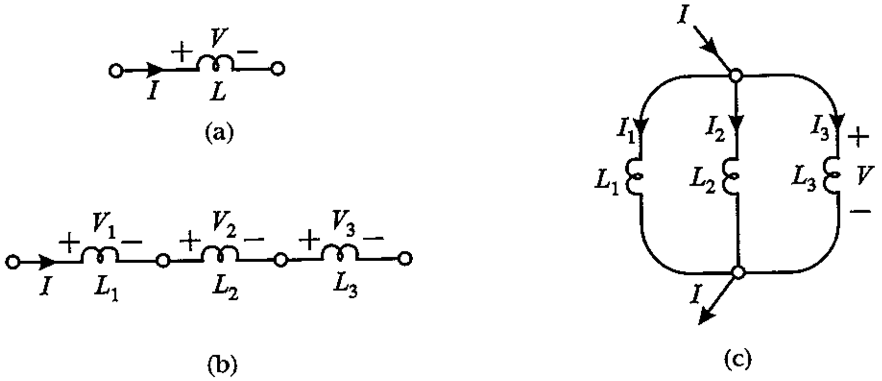


Figure 2.18. Circuit symbol for an inductor (a), series connection (b), and parallel connection (c).

inductances:

$$L = \sum_i L_i. \tag{2.48}$$

This is similar to the expression for series resistors. For the parallel connection (Figure 2.18c), we can start by inverting Equation 2.47 to get

$$\frac{1}{L} = \frac{I'}{V}. \tag{2.49}$$

In a parallel circuit, the voltage for each inductor is the same, and so the denominator does not change. The total current is the sum of the individual currents, and the derivative of the total current is the sum of the individual derivatives. This means that we can write

$$\frac{1}{L} = \sum_i \frac{1}{L_i}. \tag{2.50}$$

This is similar to the expression for parallel resistors.

2.10 Energy Storage in Inductors

Like capacitors, inductors store energy rather than dissipate it as heat. As in capacitors, there is some resistance, and if there are large currents, inductors will heat up. Again we calculate the energy $E(t)$ stored in the inductor as the integral of the power $P(t) = V(t)I(t)$. We write the stored energy $E(t)$ as

$$E(t) = \int_0^t P(t)\,dt = \int_0^t V(t)I(t)\,dt. \tag{2.51}$$

If we substitute for V from Equation 2.46, we get

$$E = \int_0^t LI'I\,dt. \tag{2.52}$$

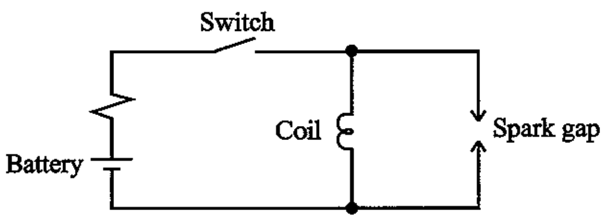


Figure 2.19. Inductor ignition system for cars. This circuit is simplified. In addition there would be a transformer to make sure that the spark voltage is bigger than the switch voltage. Otherwise the switch itself will arc.

We use I as the integration variable rather than t and get

$$E = L \int_0^I I\,dI = \frac{LI^2}{2}. \tag{2.53}$$

Notice that energy storage in an inductor is associated with the current rather than the voltage as in a capacitor. We said that for capacitors there is a danger in shorting the terminals because the current can be very large and the energy will be dissipated quickly in the short. In contrast, in an inductor, the danger is in trying to stop the current. For example, when a switch is opened in a circuit with an inductor that is carrying current, there will be a large voltage that can make an arc.

This feature is used in car ignition systems to make a spark inside the cylinders to ignite the gasoline. It is impressive because it takes 12 V from a battery and produces 10 kV to make the spark. Figure 2.19 shows a simplified ignition circuit. First a battery and resistor are connected by a switch to establish a current in the inductor. Then the switch is opened. When this happens, the current in the inductor drops quickly, causing a large voltage. When the voltage is high enough, there will be an arc across the spark gap, firing the spark plug.

2.11 RL Circuits

If we connect a resistor to an inductor carrying current (Figure 2.20a), the current will decay as the energy stored in the inductor is dissipated as heat in the resistor.

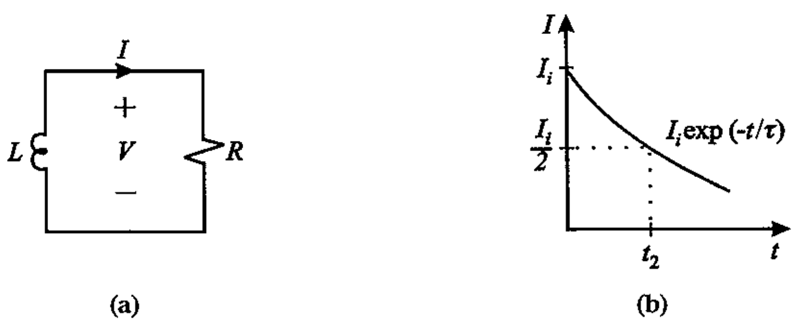


Figure 2.20. Inductor carrying current with a resistor (a), and the decaying current (b).

We can analyze the circuit by writing two expressions for the voltage:

$$V = IR = -LI'. \tag{2.54}$$

There is a minus sign because the current arrow points out from the inductor:

$$(L/R)I' + I = 0. \tag{2.55}$$

The time constant is given by

$$\tau = L/R. \tag{2.56}$$

We write the solution as

$$I(t) = I_i \exp(-t/\tau). \tag{2.57}$$

The current decays in an inductor just as voltage decays in a capacitor (Figure 2.20b). However, in an inductor, the current decays quickly if the resistor is large. In a capacitor, it is the other way around – the discharge is fast if the resistor is small. Since an oscilloscope measures voltage rather than current, it is not as convenient to measure this decay in an inductor as it is in a capacitor. Usually we will end up measuring the voltage across a series resistor and divide by the resistance to get the current.

In the lab, you will drive an RL circuit with a function generator. To understand the behavior, we will analyze an RL circuit with a current source I_s (Figure 2.21a). The approach is to solve for a current that decays to zero, the resistor current, and then to use the resistor current to find the inductor current that we are really interested in. Kirchhoff's current law gives us an expression for I_s:

$$I_s = I_l + I_r, \tag{2.58}$$

where I_l is the inductor current and I_r is the resistor current. We can write the voltage V as

$$V = RI_r = LI_l'. \tag{2.59}$$

For the derivative I_l' we can substitute $-I_r'$. This gives us

$$V = RI_r = -LI_r' \tag{2.60}$$

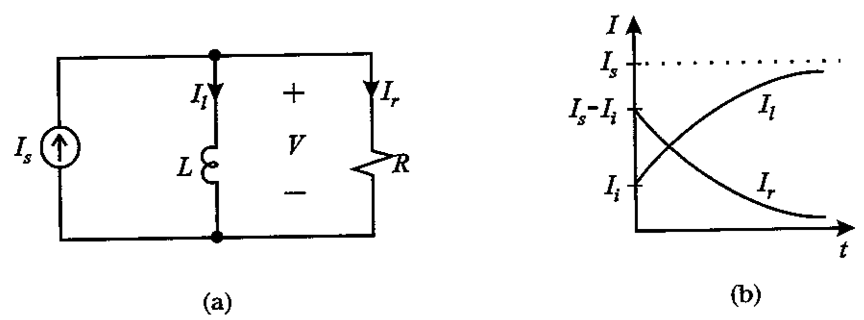


Figure 2.21. RL circuit with a current source I_s added (a), and the resistor and inductor currents (b).

or

$$(L/R)I_r' + I_r = 0, \tag{2.61}$$

which gives us a decaying exponential with the same time constant L/R as before. The initial value of I_r is given by $I_s - I_i$, where I_i is the initial inductor current. We find the inductor current I_l as the difference between I_s and I_r (Figure 2.21b). The inductor current builds up exponentially from the initial current I_i to I_s, with the same L/R time constant.

FURTHER READING

The classic book on electronics is the encyclopedic *The Art of Electronics*, by Horowitz and Hill, published by Cambridge University Press. This is a brilliant book, and it is more likely to be on the bookshelf of working engineers than any other book that I know of. I recommend consulting this book continuously as you learn about electronics. For a serious discussion of Thevenin's theorem, see Desoer and Kuh's *Basic Circuit Theory*, published by McGraw-Hill. This is also an excellent reference for matrix solutions of circuits. A good book for information on diodes is *Device Electronics for Integrated Circuits*, by Muller and Kamins, published by Wiley.

PROBLEM 1 - RESISTORS

A. Figure 2.22 shows a Thevenin source with a load resistor R_l. Find the formula for the power in the load. Find the load resistance R_l that gives the maximum power. What is the maximum load power? As a check, it is a good idea to find the formula for the maximum available power for a Norton source with a source conductance G_s and a load conductance G_l. Your result should be equivalent to the Thevenin result.

B. Figure 2.23 shows two resistive circuits that appear often in *attenuators*, which are circuits that reduce the power of a signal. Attenuators can prevent radios from

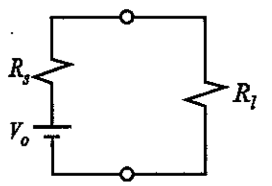


Figure 2.22. Source with a load.

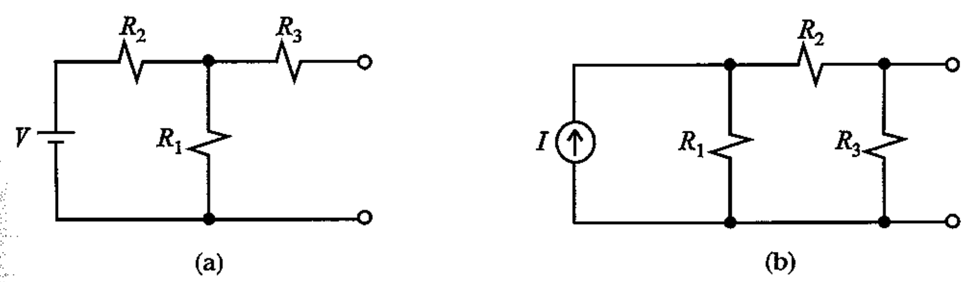


Figure 2.23. T resistor circuit (a), and π resistor circuit (b).

overloading and sensitive instruments from burning out. Figure 2.23a is called a *T* network, and Figure 2.23b is a π network. They get their names because the outlines resemble these letters. For each circuit, find the parameters needed to make Thevenin and Norton equivalent circuits: V_o, I_s, and R_s. Make sure that you express each quantity in the simplest form.

PROBLEM 2 - SOURCES*

For these measurements, we use a 12-V, 0.8-A-hr battery that is manufactured by Yuasa. You should start with four 510-Ω, quarter-watt resistors. Use a multimeter to measure the open-circuit voltage. Be careful not to attach the leads to the current jack on the multimeter. This jack is only for measuring current. Its resistance is quite low, and several amps will flow, blowing the fuse in the multimeter.

A. Connect the positive lead from the battery to the top row of holes on the breadboard and the negative lead to the bottom row (Figure 2.24). It is a good idea to follow the tradition of using black for low voltage and red for high voltage. This will save you from blowing out circuits later. Now you can add the resistors in parallel by plugging them into any of the holes in the top and bottom rows. You should wait two minutes after adding each resistor before you take a measurement for the battery voltage to stabilize. Plot the voltage you measure on the y axis versus the current on the x axis as the number of resistors increases from 0 to 4. To calculate the current, assume the nominal resistance value of 510 Ω. Use linear graph paper and choose scales carefully to show what is happening in the plot. Label your axes with units. Draw a smooth curve through the data points.

B. Find an equivalent circuit for the battery with an ideal voltage source V_o and a resistor R_s when the current is the neighborhood of 75 mA. You should notice that this circuit will not be accurate at currents that are much lower or higher than this.

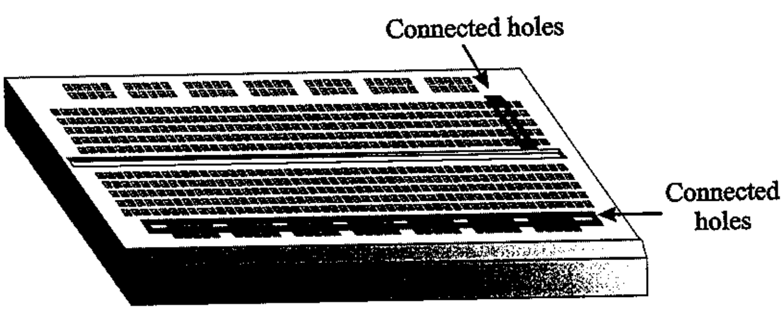


Figure 2.24. Hole pattern for a solderless breadboard. On the top and bottom are four rows of connected holes. In the center are columns of five connected holes. If you are confused about the connections, you can measure the resistance between holes with a multimeter.

* *Note:* Please see Appendix A for a complete list of the supplies and equipment that are used in each problem.

C. When the NorCal 40A is receiving, it draws 20 mA. What voltage would you expect from the battery? If the battery has an amp-hour rating of 0.8 A-hr, how long would you expect to be able to operate the radio as a receiver?

PROBLEM 3 - CAPACITORS

A combination of a series resistor and parallel capacitor is used in many circuits to give a time delay of about RC. In our transceiver, this delay is used to make sure the receiver is muted while the transmitter is turning on and off – otherwise there would be a loud pop, because the transmitter voltages are much larger than the ordinary signals that are received. In other circuits, delays may be unintentional. A major factor limiting the speed of computers is the resistance and capacitance of the metal patterns that connect different parts of a circuit.

Connect a function generator and an oscilloscope, using test hooks attached directly to the resistor and capacitor leads (Figure 2.25). Make sure that the red leads are across the resistor and that the black leads from the scope and the function generator are connected together. The black leads are connected to the ground through the AC outlets, but this is not a reliable connection. You should use a sync cable from the function generator to trigger the scope. Do not use a breadboard, because it adds capacitance that confuses the measurements.

A. The function-generator settings should be for a 20-Hz, 1-Vpp (peak-to-peak) square wave. For a function generator with a 50-Ω source resistance, this amplitude, 1 V peak-to-peak, is the voltage that we *would* see if the load were 50 Ω. For an

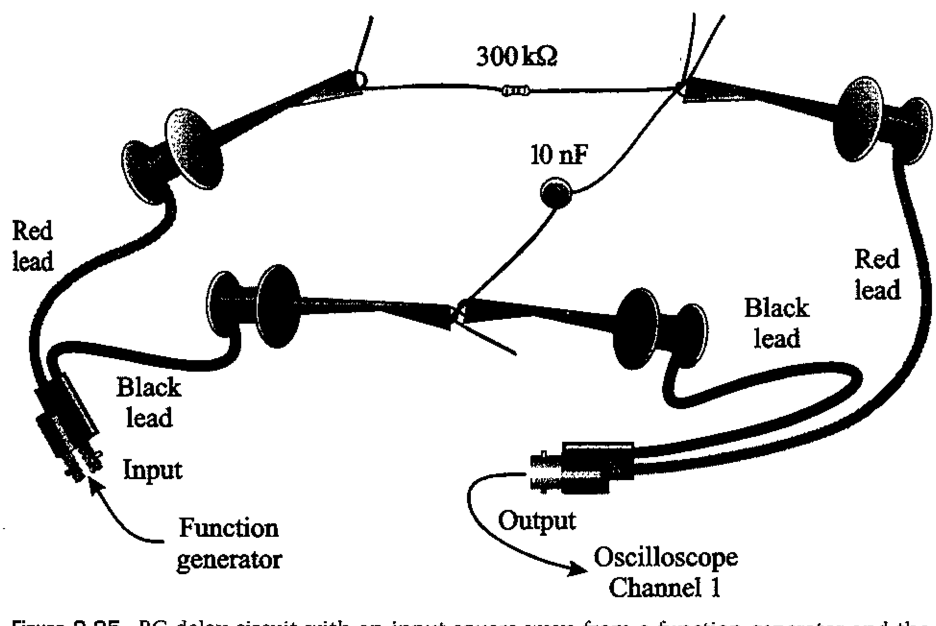


Figure 2.25. RC delay circuit with an input square wave from a function generator and the output to an oscilloscope. The output contains sections of exponential waveforms with a time constant RC.

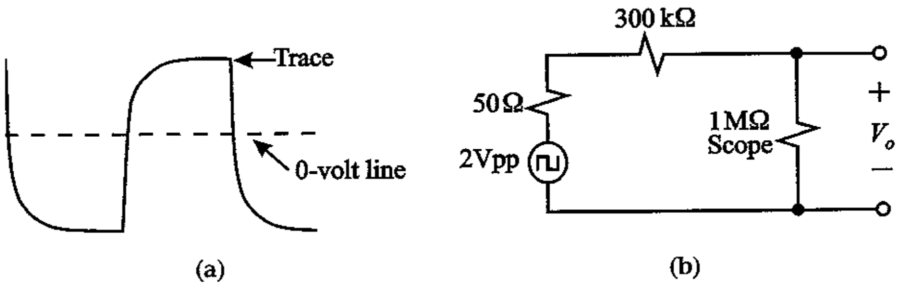


Figure 2.26. Oscilloscope waveform at output of the delay network (a), and simplified circuit diagram without capacitors (b).

open-circuit load, the amplitude is twice this, or 2 Vpp. The frequency, 20 Hz, is low enough that it allows the capacitor to charge fully each time the voltage rises and to discharge fully each time the voltage falls. With the scope voltage and time scales properly set, you should see the waveform in Figure 2.26a. It is a square wave with rounded corners. Measure the peak-to-peak output voltage on the oscilloscope.

B. Now calculate what the voltage should be. A good way to start is to consider the circuit without capacitors. Figure 2.26b shows the open-circuit voltage of the function generator (2 Vpp), the function-generator resistance, 50 Ω, the 300-kΩ load resistance, and a 1-MΩ resistance for the oscilloscope. This is a divider circuit. The function-generator resistance is much smaller than the others, and you can ignore it. Make a Thevenin equivalent for this circuit by finding V_o and the look-back resistance R_S. The open-circuit voltage V_o should be close to the value you measured, and you will need R_S later to calculate the delays.

C. The delays come from the time required to charge the capacitor when the voltage rises and to discharge the capacitor when the voltage falls. Expand the time scale on the oscilloscope so that the falling part of the waveform occupies the entire screen (Figure 2.27a). When the voltage drops to zero volts, the capacitor has discharged halfway. The time for it to reach 0 V is the time t_2 that we related to the time constant τ. Measure t_2.

D. Now calculate what t_2 should be, from the Thevenin source resistance R_S and the load capacitance 10 nF (Figure 2.27b).

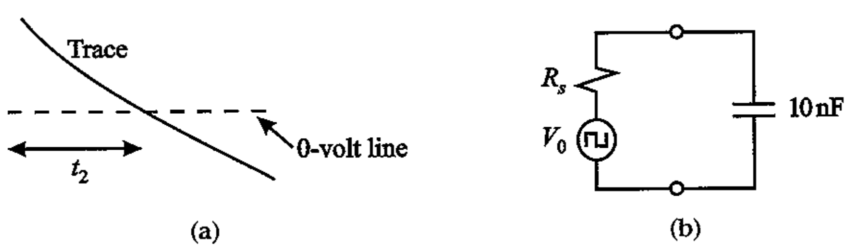


Figure 2.27. Output waveform with time scale expanded to show detail of discharging capacitor with the measuring time t_2 (a), and Thevenin circuit with 10-nF load (b).

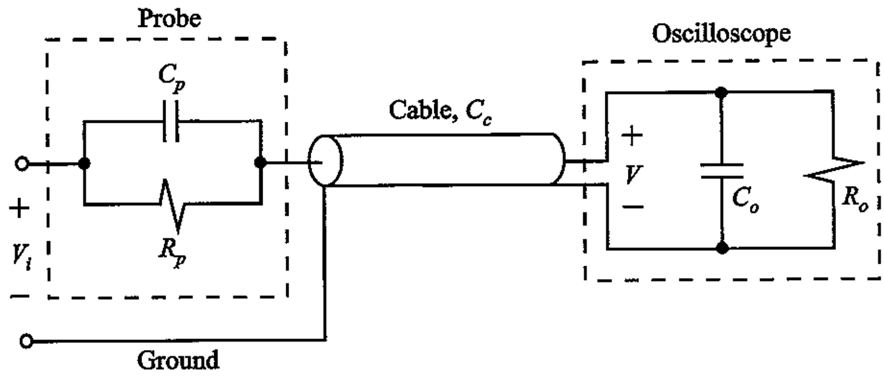


Figure 2.28. Construction of a scope probe.

E. We can try to eliminate the delay by removing the capacitor from the circuit. This reduces the delay considerably, but not as much as you might expect, because there is capacitance in the scope and the cable. Measure t_2 again. To make the measurement accurate, you should expand the time scale so that the delay is several large divisions long. Use this delay measurement to figure out the total scope and cable capacitance C.

F. The oscilloscope capacitance C_o is usually written by the scope input jack. Subtract it from C to find the cable capacitance C_c. We will study cable capacitance when we discuss transmission lines later, but for now you should know that it is proportional to the length. This can be a good reason to make cables short. Divide the cable capacitance by the length of the scope cable to get the capacitance per unit length. This is called the *distributed capacitance*. The cable from the function generator does *not* contribute to this capacitance, and we will see why when we study transmission lines.

G. We can reduce the delay further with a high-impedance probe. Figure 2.28 shows a simplified view of the construction of a high-impedance scope probe. We will calculate the values for R_p, C_p, and C_c. Start by considering only the resistances. What value must the probe resistance R_p have to make the resistance marked on the probes correct, given the resistance R_o marked on the scope? You may want to check the probe resistance R_p with a multimeter to verify your answer. For these values of R_p and R_o, what is the ratio of the input voltage V_i to the output voltage V?

H. Now consider only the capacitances in the circuit. You should find the values that the series probe capacitance C_p and parallel cable capacitance C_c must have to make the capacitance marked on the probe correct. You should use the capacitance C_o marked on the scope and the same ratio of input voltage to output voltage that you calculated for the resistors.

I. Replace the scope cable with a high-impedance probe and measure t_2 again.

J. Now calculate what t_2 should be, using the capacitance marked on the probe. You should take into account that R_s has changed because the probe resistance is larger than the scope resistance.

K. Measure the peak-to-peak output voltage again, and calculate what it should be for comparison.

PROBLEM 4 – DIODE DETECTORS

In Chapter 1, we discussed amplitude modulation, which is used by AM radio stations. We wrote the voltage as

$$V(t) = V_c \cos(2\pi f t) + a(t)\cos(2\pi f t). \tag{2.62}$$

The first term, $V_c \cos(2\pi f t)$, is the carrier, and $a(t)$ is the audio modulating signal. We characterize the modulation by the *modulation depth*, which is written as

$$m = a_p / V_c, \tag{2.63}$$

where a_p is the peak value of $a(t)$. Usually m is expressed as a percentage. Commercial broadcasters monitor the modulation depth to make sure that it does not reach 100% when $a(t)$ is negative, or the receiver output becomes distorted. We will see this in our measurements.

We can use a diode circuit to detect the audio signal (Figure 2.29). The circuit is a half-wave rectifier with a capacitor. Whenever the diode is on, the capacitor charges up to the input voltage. The bleeder resistor adjusts how fast the current leaks out of the capacitor. This current needs to be small enough that the capacitor voltage does not drop much during an RF cycle, but it should allow the voltage to follow the audio signal.

The function-generator settings should be for a 1-MHz, 5-Vpp sine wave with a modulating frequency of 1 kHz and a modulation depth of 70%. You need to connect a cable from the sync output of the function generator to the oscilloscope trigger input and to use external triggering. The modulated waveforms that you see will be difficult to trigger on if you do not do this. Adjust the scope controls for a good display of the modulated waveform on channel 1. Now connect the detector circuit, using the breadboard, with the output on channel 2. The detector output should be a 1-kHz sine wave like the modulating waveform.

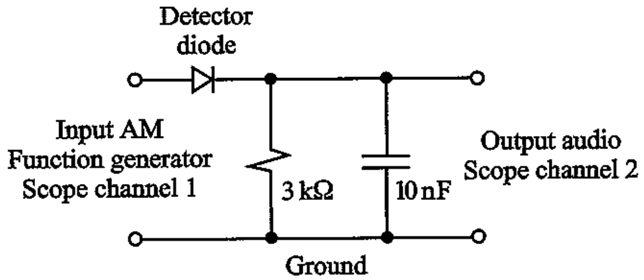


Figure 2.29. An AM detector circuit.

A. The RC time constant τ needs to be considerably less than the period of the modulating waveform, or the output will not be able to follow the modulation. Calculate τ and compare it with the period of the modulating waveform.

B. Compare the maximum voltage of the input AM signal with the maximum voltage of the output audio. It is convenient to use the same voltage scale and zero-voltage reference line on the scope for both the input and output. What would you expect the difference of these voltages to be?

C. The time constant τ should be considerably longer than the period of the carrier, or the voltage will droop between each cycle. To see this effect, reduce the carrier frequency to 100 kHz. Measure the voltage droop. Now calculate what you would expect for the droop.

D. Return the carrier frequency to 1 MHz. Adjust the modulation depth to 100% and sketch the distorted output waveform. Why does this distortion occur?

PROBLEM 5 – INDUCTORS

Inductors come in small packages that resemble resistors. Inside is a small magnetic rod with fine wire wrapped around it. Inductors use the same color code as resistors, except that the units are μH rather than Ω. We will measure the time it takes current to build up and decay in an inductor. Make the connections shown in Figure 2.30, using function-generator settings for a 1-kHz, 5-Vpp square wave. At the input, use a tee to connect channel 1 of the scope. At the output, use a tee at the scope to connect a 50-Ω load.

A. At the output, you should see a square wave with rounded corners. The zero-voltage line is the half-way point for current building up or for current decaying. Measure the time it takes to reach the zero-voltage line, t_2. Deduce the peak-to-peak inductor current from the voltage across the 50-Ω load.

B. Now calculate the peak-to-peak inductor current and the delay t_2 that we should expect.

C. Sketch the input voltage and interpret it.

In electronic circuits it is common to use transistors as switches. Transistors have three terminals, and by applying a current at one of the terminals, called the base, we can make the other two terminals, the collector and the emitter, act like a switch. There may be times when we want to switch the current in an inductor without producing a

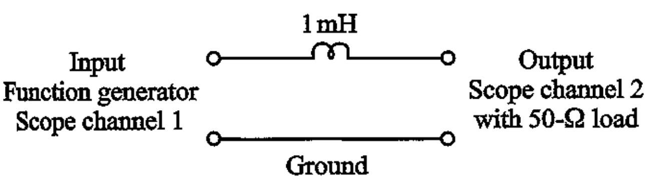


Figure 2.30. Circuit for observing current buildup and decay in inductors.

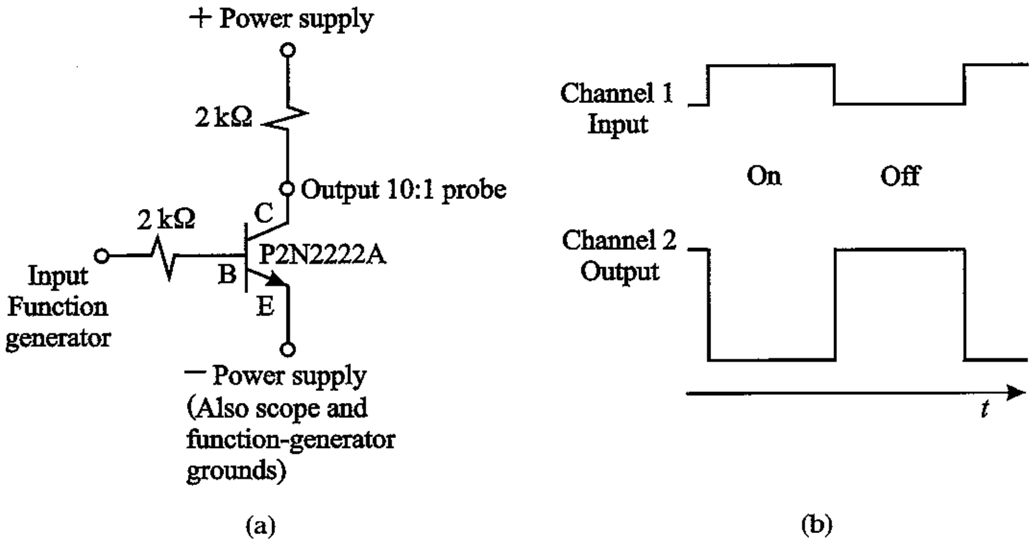


Figure 2.31. Making a transistor switch (a), and input and output scope traces (b).

large voltage. One example is in driving a relay. Relays are mechanical switches that are switched on and off by magnetic forces. The magnetic forces are generated by the current in a drive coil. Relays can control large voltages and currents with little loss, but the drive coils are quite inductive, and they generate large voltages if the drive current changes suddenly. You will make a transistor switch, first with a resistor load, and then with an inductor load, and finally, you will include a diode to prevent large inductive voltages.

To start, you will need to connect the circuit in Figure 2.31a on a breadboard. You should leave out the inductor and diode at first. Use a tee to connect the function generator to both channel 1 of the scope and your circuit at the same time. At the output, use a 10:1 probe on channel 2. The voltages in these measurements get rather large and will go off the oscilloscope screen unless you use a 10:1 probe. You will need to plug in a 12-V power supply.

When you put together circuits with a transistor and a power supply, there are many opportunities to destroy components. If your circuit does not work, and you have checked that the connections are correct, it is possible that a component has been destroyed. You can check resistors by measuring the resistance on a multimeter. You can also check an inductor by checking its resistance. The 1-mH inductors should have a resistance of around 10 Ω. The other inductors you will use have much smaller inductances, and their resistances are smaller also, an ohm or less. Inductors can either fail as open circuits if a wire melts or as short circuits if the insulation melts.

For testing diodes, some multimeters have a diode check setting. Other multimeters can provide a fixed current during a resistance measurement of 1 mA, and take the ratio of voltage to current. For example, for a diode with a forward voltage of 0.6 V, the reading would be 600 Ω. You do have to orient the diode correctly, by connecting the anode to the high-voltage terminal, and the cathode to the low-voltage terminal. The same approach

can be used to check transistors. The base–emitter and base–collector connections are diodes, so that a transistor can be checked as if it were two diodes.

The P2N2222A transistor comes in a plastic package with three terminals. ("P" is for plastic.) The base is the input, the collector is the output, and the emitter is the ground connection. Check the data sheet in Appendix D to identify the terminals. The base-to-emitter connection is a diode, so that only positive current flows into the base, and there is a forward voltage of about 0.6 V. When no current flows in the base, the resistance between the collector and emitter is high, and the switch is effectively open. When this happens, there is no current in the 2-kΩ collector resistor, and the output voltage is the same as the supply voltage. However, when enough current flows in the base, the resistance between the collector and emitter drops, and a large current can flow, effectively closing the switch. This causes a large voltage across the resistor, and it means that the output voltage will be small, nearly zero. The purpose of the 2-kΩ base resistor is to keep too much current from flowing in the base.

The function generator should be set for a 100-kHz, 100-mVpp square wave. Increase the input voltage until the switch begins to turn on. The output voltage should be low when the transistor is on. When the input-voltage setting is 1 Vpp, the scope traces should look like Figure 2.31b.

D. Now reduce the input voltage until the transistor does not turn on at all, and add a 1-mH inductor as shown in Figure 2.32a. Increase the function-generator setting gradually to 1 Vpp again. Sketch the output. When the transistor switches off, the current drops rapidly, and this causes a large voltage in the inductor. This voltage can destroy a transistor. Measure the maximum voltage across the transistor. You should notice that when the transistor turns off, the output voltage oscillates. We call this *ringing*. The ringing comes from a resonance between the inductor and circuit capacitance.

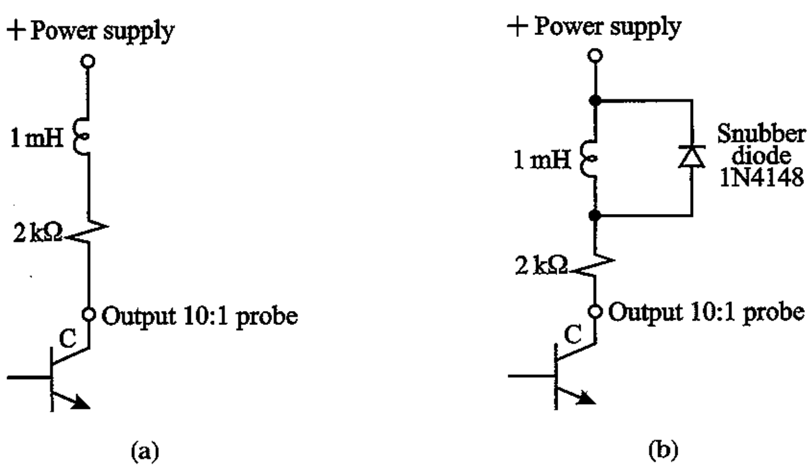


Figure 2.32. Adding an inductor to the switch (a), and a snubber diode to suppress ringing (b). These figures show only the collector circuit. The base and emitter connections do not change.

E. We can reduce the voltage by adding a diode across the inductor as in Figure 2.32b. This diode is called a *snubber*. The snubber diode limits the voltage across the inductor to the forward voltage of the diode. Sketch the transistor voltage with the snubber in the circuit.

PROBLEM 6 - DIODE SNUBBERS

A. Put together the circuit for the previous problem again, with the inductor in the circuit, but no snubber diode. Examine the ringing that you see when the transistor switches off. Measure the ringing frequency. You can do this by measuring the time it takes for several cycles of the oscillation to be completed. You should expand the time scale so that a few cycles of the oscillating waveform take up most of the scope display.

B. The ringing results from a resonance between the inductor and circuit capacitance. The circuit capacitance comes from many places, including the inductor itself, the 10 : 1 probe, the transistor, and the breadboard. Calculate the circuit capacitance from the following formula that relates the inductance and capacitance to the resonant frequency f_0:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}. \tag{2.64}$$

Sketch the inductor voltage with the snubber diode in the circuit, using a scale of 2 V per large division and 2 μs per large division. Identify the time when the diode is on and the time when the diode is off. This measurement takes some thought. When you measure the inductor voltage, it is important not to move a scope ground connection. The two oscilloscope grounds are connected by the scope case, and they are connected to the minus lead of the power supply through the wall plugs. If you move a scope ground clip to the inductor, you will put the full power-supply voltage across it. This will destroy the inductor. One way to approach the measurement is to use the plus lead of the power supply as a reference. Attach the scope probe to it and adjust the vertical position control to set the trace on the center line of your screen. This becomes your reference, and you can then move the probe to the other end of the inductor to measure the inductor voltage.

C. You should find that the snubber diode is on all of the time that the transistor is off. Measure the forward voltage of the diode during the middle of the off time. Calculate how long the diode would stay on if the current were allowed to run down completely. Start by setting the diode voltage equal to the inductor voltage so that you can calculate the derivative of the current. You can figure out the initial current by measuring the collector-resistor voltage.

D. Now decrease the square-wave frequency to 30 kHz so that you can see the diode turn off. How long does the diode actually stay on?

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3

Phasors

Complex numbers find one of their best applications in analyzing electronic circuits, because cosine signals can be efficiently represented by complex numbers called phasors. With phasors we can analyze circuits with inductors and capacitors almost as easily as resistor circuits, without worrying about calculating derivatives and integrals. In addition, we can use phasors to calculate average power and stored energy.

3.1 Complex Numbers

Complex numbers are often introduced in a mathematics class by writing $\sqrt{-1}$ as i. Electrical engineers use j instead of i, so that i can be reserved for current. It is a good idea to use j in electrical-engineering problems and i in mathematics and physics problems, because the fields follow different sign conventions. Typically

$$j = -i. \tag{3.1}$$

Using j will let people know that you are following the electrical-engineer's sign convention, and i will tell them that you are following the mathematician's or physicist's convention.

In electrical engineering, it may be best to start by thinking of a complex number as a pair of numbers that we call the *real* and *imaginary* parts. In this sense, a complex number is like a two-dimensional vector, and we can draw it like a vector in a plane (Figure 3.1a). We call this the *complex plane*. The horizontal axis is used for the real part, and the vertical axis for the imaginary part. We will use several different notations for writing a complex number, depending on what we want to emphasize. If we let z be a complex number, we can write

$$z = x + jy, \tag{3.2}$$

where x is the real part and y is the imaginary part. The *complex conjugate* z^* is given by

$$z^* = x - jy. \tag{3.3}$$

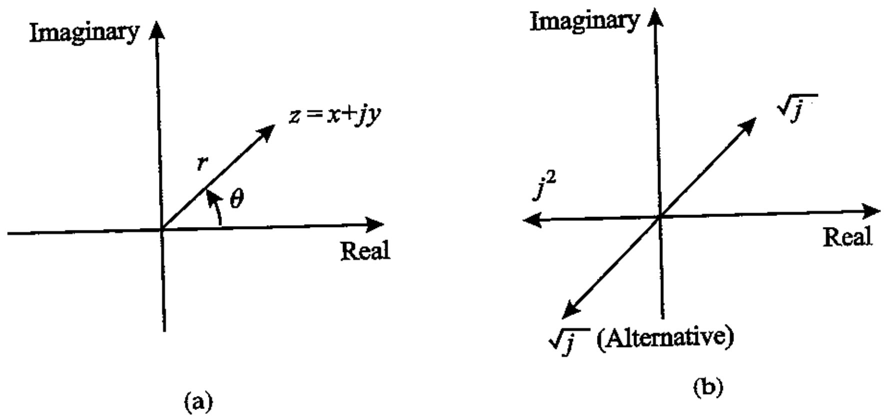


Figure 3.1. Representing a complex number z in a plane by drawing a line from the origin to the point $z = x + jy$ (a). The square and square roots of j (b).

We will also indicate that x and y are the real and imaginary parts of z by writing

$$\mathrm{Re}(z) = x, \tag{3.4}$$

$$\mathrm{Im}(z) = y. \tag{3.5}$$

We add and subtract complex numbers by adding and subtracting the real and imaginary parts separately. This is like vector addition.

We can also represent a complex number in terms of its magnitude and phase. The *magnitude* is the distance from the origin to z in the complex plane. We let the magnitude be given by r and calculate it from the Pythagorean theorem as

$$r = \sqrt{x^2 + y^2}. \tag{3.6}$$

The *phase* is the angle from the real axis, and we write it as θ (the Greek letter *theta*),

$$\theta = \tan^{-1}(y/x). \tag{3.7}$$

We may use either degrees or radians to represent the angle. As a shorthand notation we can write

$$z = r\angle\theta. \tag{3.8}$$

The number j itself can be written as

$$j = 1\angle 90^\circ \tag{3.9}$$

and -1 is given by

$$-1 = 1\angle 180^\circ. \tag{3.10}$$

With trigonometry, we can express the real and imaginary parts in terms of the magnitude and phase as

$$x = r\cos\theta, \tag{3.11}$$

$$y = r\sin\theta. \tag{3.12}$$

We can also indicate that r and θ are the magnitude and phase of z by writing

$$|z| = r, \tag{3.13}$$

$$\angle z = \theta. \tag{3.14}$$

So far complex numbers only seem to be a funny form of vector notation, and if this were all there was to it, we would not need complex numbers. A key difference is in how we multiply and divide. The magnitude and phase are convenient for this calculation. If we have two complex numbers s and t, we can write the magnitude and phase of the product as

$$|st| = |s|\,|t|, \tag{3.15}$$

$$\angle(st) = \angle s + \angle t. \tag{3.16}$$

This means that the magnitude of the product of two complex numbers is the product of the magnitudes, and the phase is the sum of the phases. For example, the product $-1 \cdot z$ is given by

$$|-z| = 1 \cdot |z| = |z|, \tag{3.17}$$

$$\angle(-z) = \angle z + 180^\circ. \tag{3.18}$$

Similarly the quotient s/t is given by

$$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}, \tag{3.19}$$

$$\angle(s/t) = \angle s - \angle t. \tag{3.20}$$

In words, the magnitude of the quotient is given by the quotient of the magnitudes, and the phase is the difference of the phases. As a special case, the quotient $1/s$ is given by

$$|1/s| = 1/|s|, \tag{3.21}$$

$$\angle(1/s) = -\angle s. \tag{3.22}$$

We can deduce the formulas for squares and square roots from the product formulas. We write z^2 as

$$|z^2| = |z|^2, \tag{3.23}$$

$$\angle(z^2) = 2\angle z. \tag{3.24}$$

For example, we can write j^2 as

$$|j^2| = 1, \tag{3.25}$$

$$\angle(j^2) = 180^\circ, \tag{3.26}$$

so that $j^2 = -1$ (Figure 3.1b). If we think of taking the square root as the inverse of squaring, we can write

$$|\sqrt{z}| = \sqrt{|z|}, \tag{3.27}$$

$$\angle(\sqrt{z}) = \frac{\angle z}{2}. \tag{3.28}$$

As in the ordinary square root of a positive number, we have a choice of two roots that differ only in sign. This is shown in Figure 3.1b. The other root can be written as

$$|\sqrt{z}| = \sqrt{|z|}, \tag{3.29}$$

$$\angle(\sqrt{z}) = \frac{\angle z}{2} + 180°. \tag{3.30}$$

For example, consider the square root of j:

$$|\sqrt{j}\,| = 1, \tag{3.31}$$

$$\angle(\sqrt{j}\,) = 45° \quad \text{or} \quad 225°. \tag{3.32}$$

In rectangular coordinates, we would write

$$\sqrt{j} = 1/\sqrt{2} + j/\sqrt{2} \quad \text{or} \quad -1/\sqrt{2} - j/\sqrt{2}. \tag{3.33}$$

3.2 Exponential Function

The exponential function $\exp(x)$ has a deep connection to the cosine and sine functions through complex numbers. We will start with a fundamental definition of the exponential function. There are two parts. First, the exponential function is its own derivative:

$$\frac{d\,\exp(x)}{dx} = \exp(x). \tag{3.34}$$

To completely determine the function, we must specify its value at some point, because any multiple of the exponential function also satisfies this equation. We set

$$\exp(0) = 1. \tag{3.35}$$

It is interesting to consider the exponential of an imaginary number $j\theta$. We can write the derivative with the chain rule as

$$\frac{d\,\exp(j\theta)}{d\theta} = j\,\exp(j\theta). \tag{3.36}$$

This expression indicates that the derivative has the same magnitude as the exponential, but the angle differs by 90°. If we start at $\theta = 0$, where the exponential is just 1, then the function will move up as θ increases. The interesting thing is that the function always moves at right angles to the arrow that represents it.

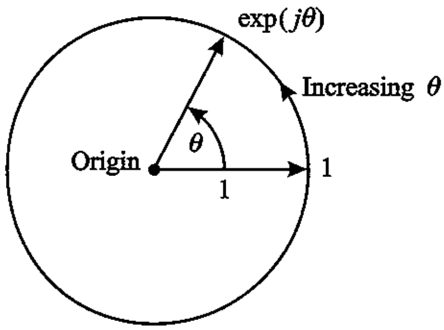


Figure 3.2. Locus of $\exp(j\theta)$ as θ increases from 0.

This causes the function to follow a circular path (Figure 3.2). It is as if the arrow were a rod pinned at one end, so that movement is always at right angles to the rod. More than this, the derivative always has magnitude 1, so that the distance traveled around the circle is equal to θ. Thus θ is the angle in radians as we move around the circle. The path followed by a function as its argument changes is called a *locus*. We would say that the locus of $\exp(j\theta)$ is a unit circle centered at the origin.

From Figure 3.2, we can see by trigonometry that the real and imaginary parts of $\exp(j\theta)$ are $\cos\theta$ and $\sin\theta$. We can write

$$\exp(j\,\theta) = \cos\theta + j\sin\theta. \tag{3.37}$$

This is Euler's formula, and it is one of the most elegant (and surprising) formulas in all of mathematics. We can use Euler's formula to represent the cosine and sine functions in terms of exponentials:

$$\cos(\theta) = \frac{\exp(j\theta) + \exp(-j\theta)}{2}, \tag{3.38}$$

$$\sin(\theta) = \frac{\exp(j\theta) - \exp(-j\theta)}{2j}. \tag{3.39}$$

If you are not familiar with these formulas, it is a good idea to work out the details by substituting Euler's formula into these expressions.

3.3 Phasors

Let us summarize the circuit relations that we have learned for resistors, capacitors, and inductors:

$$V(t) = RI(t), \tag{3.40}$$

$$V(t) = LI'(t), \tag{3.41}$$

$$I(t) = CV'(t). \tag{3.42}$$

The primes denote derivatives. Often it is not very convenient to work with derivatives. However, our radio signals can often be described by cosine functions,

and these have simple derivatives, particularly when we consider the relation between the cosine function and an exponential.

We may write a cosine voltage $V(t)$ as

$$V(t) = A\cos(\omega t + \theta), \tag{3.43}$$

where A is the *peak amplitude* in volts, ω (the Greek lower-case *omega*) is the *frequency* in radians per second, and θ is the *phase* in radians. The frequency in radians per second differs from the frequency in cycles per second, or hertz, by a factor of 2π, and so we can write

$$\omega = 2\pi f. \tag{3.44}$$

We will be careful to write the frequency in radians per second as ω and the frequency in hertz as f, so that we can distinguish them. We can write a current $I(t)$ at the same frequency in a similar form:

$$I(t) = B\cos(\omega t + \phi), \tag{3.45}$$

where ϕ (the Greek letter *phi*) is the phase of the current. If the current phase ϕ is different from the voltage phase, then the current can either be ahead of the voltage or behind it (Figure 3.3). If $\phi > \theta$, then we say the current *leads* the voltage, and if $\phi < \theta$, we say the current *lags* the voltage.

If the voltage $V(t) = A\cos(\omega t + \theta)$ is applied to a capacitor C, we can write the current $I(t)$ as

$$I(t) = CV'(t) = -CA\omega\sin(\omega t + \theta) = CA\omega\cos(\omega t + \theta + \pi/2). \tag{3.46}$$

We would say that the current in a capacitor leads the voltage by $\pi/2$, or 90°. In an inductor, the situation is reversed, and the current lags the voltage. An interesting thing happens if we use Euler's formula to express the cosine as the real part of an exponential and repeat this calculation. We write

$$V(t) = \text{Re}[A\exp(j\omega t + j\theta)]. \tag{3.47}$$

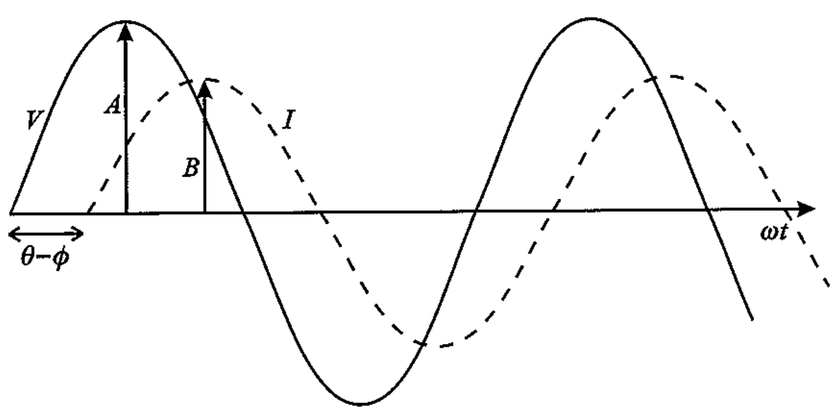


Figure 3.3. Cosine voltage $V(t)$ with a current $I(t)$ lagging it.

We can write the current as

$$I(t) = CV'(t) = \mathrm{Re}\left(C\frac{d}{dt}\left[A\ \exp(j\omega t + j\theta)\right]\right). \tag{3.48}$$

We take the derivative of the exponential by multiplying by $j\omega$. This gives us

$$I(t) = \mathrm{Re}[j\omega CA\ \exp(j\omega t + j\theta)]. \tag{3.49}$$

This formula is equivalent to Equation 3.46, and you should work out the details to show this. The exponential allowed us to replace a derivative with a multiplication by $j\omega$. We can put this approach on firmer ground if we define complex numbers V and I, given as

$$V = A\ \exp(j\theta), \tag{3.50}$$

$$I = B\ \exp(j\phi). \tag{3.51}$$

In terms of the magnitude and phase, we can write

$$|V| = A, \tag{3.52}$$

$$\angle V = \theta, \tag{3.53}$$

$$|I| = B, \tag{3.54}$$

$$\angle I = \phi. \tag{3.55}$$

V and I are called *phasors*. Because they are fixed complex numbers rather than functions of time, t does not appear.

The magnitude of the phasor is equal to the peak amplitude of the original cosine voltage or current, and the phase is the same. To recover the cosine function, we multiply by $\exp(j\omega t)$ and take the real part:

$$V(t) = \mathrm{Re}[V\ \exp(j\omega t)] = |V|\cos(\omega t + \angle V), \tag{3.56}$$

$$I(t) = \mathrm{Re}[I\ \exp(j\omega t)] = |I|\cos(\omega t + \angle I). \tag{3.57}$$

Taking the derivative with respect to time is equivalent to multiplying by $j\omega$ for a phasor. For example, for a capacitor we write

$$I = j\omega CV \tag{3.58}$$

as the phasor equivalent of $I(t) = CV'(t)$. We can write a similar relation between current and voltage phasors for an inductor, if we repeat these steps. This gives us

$$V = j\omega LI \tag{3.59}$$

as the equivalent of $V(t) = LI'(t)$. For a resistor we have

$$V = RI \tag{3.60}$$

for phasors, which looks the same as before.

3.4 Impedance

We will be writing voltage and current as phasors most of the time. The ratio of V and I is called the *impedance* and it is written as Z:

$$V = ZI. \tag{3.61}$$

The units of impedance are ohms, like resistance. However, because V and I are complex numbers, the impedance is a complex number with real and imaginary parts. It is traditional to write the real and imaginary parts as

$$Z = R + jX, \tag{3.62}$$

where R is the resistance and X is the *reactance*. We can compare this formula to Equation 3.59, and say that the reactance of an inductor is given by

$$X = \omega L. \tag{3.63}$$

The reactance of an inductor is positive. It is trickier to get the reactance of a capacitor. If we invert Equation 3.58, we get

$$V = \frac{I}{j\omega C}, \tag{3.64}$$

and so we would say that the reactance of a capacitor is given by

$$X = -1/\omega C. \tag{3.65}$$

The minus sign takes the j in the denominator into account. The reactance of a capacitor is negative. Be forewarned: We will often work with the absolute value of the reactance, given by

$$|X| = 1/\omega C. \tag{3.66}$$

People often call this quantity "the reactance," even though it is positive. This is ambiguous but convenient. You have to get the sign from the context.

Impedance is a powerful idea, because it lets us include inductors and capacitors in our analysis without having to take derivatives and integrals. The arithmetic is like that for resistors, except that we use complex numbers, although we have to be careful to remember that impedance is only used for cosine voltages and currents. For example, the impedance Z_s of a series connection of components is the sum of the impedances,

$$Z_s = \sum_i Z_i, \tag{3.67}$$

and the impedance Z_p of a parallel connection is given by the formula

$$\frac{1}{Z_p} = \sum_i \frac{1}{Z_i}. \tag{3.68}$$

You can also find Thevenin and Norton equivalent circuits and voltage and current dividers for impedances in just the same manner that you did for resistances. Even

the name impedance suggests the same idea as resistance – a large impedance will *impede* current.

We will often use the inverse of impedance. This is the *admittance*, and the units are siemens (S). We write admittance with a Y:

$$I = YV. \tag{3.69}$$

The real and imaginary parts of the admittance are traditionally written as

$$Y = G + jB, \tag{3.70}$$

where G is the conductance and B is the *susceptance*. We say that the susceptance of a capacitor is ωC and the susceptance of an inductor is $-1/\omega L$. Admittances behave like conductances, so that we write

$$Y_p = \sum_i Y_i \tag{3.71}$$

for components in parallel. Using admittances in parallel circuits is convenient because we can just add the admittances. For components in series we get

$$\frac{1}{Y_s} = \sum_i \frac{1}{Y_i}. \tag{3.72}$$

3.5 RC Filters

We can use phasors to analyze the RC circuits that we build in the lab. These act as low-pass or high-pass filters, selecting either the high or low frequencies in a signal. For example, the circuit in Figure 3.4a allows signals at low frequencies through but blocks higher frequencies. This is a low-pass filter. We find the response of the circuit with phasors and impedances.

We can write the current in terms of the input voltage V_i as

$$I = \frac{V_i}{Z} = \frac{V_i}{R + 1/(j\omega C)}. \tag{3.73}$$

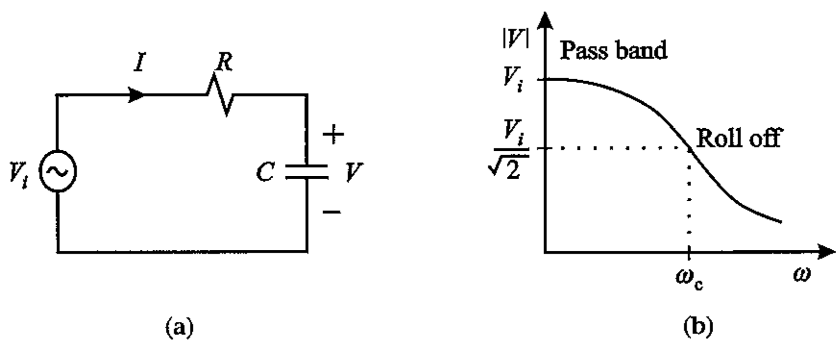


Figure 3.4. RC low-pass filter (a), and the response (b).

The output voltage V is given by

$$V = \frac{I}{j\omega C} = \frac{V_i}{1 + j\omega RC} = \frac{V_i}{1 + j\omega\tau}, \tag{3.74}$$

where $\tau = RC$ is the time constant. Figure 3.4b is a plot of $|V|$. In the pass band, where $\omega\tau \ll 1$, the output voltage is close to the input voltage. When $\omega\tau = 1$, the output voltage is given by

$$|V| = \frac{V_i}{|1 + j|} = \frac{V_i}{\sqrt{2}}. \tag{3.75}$$

This means that the output voltage has dropped by a factor of $\sqrt{2}$. Because power is proportional to the square of the voltage, we can think of this as the half-power frequency, or the 3-dB frequency. This means that we can write the cut-off frequency as

$$\omega_c = 1/\tau. \tag{3.76}$$

In the roll-off region above the cut-off frequency, the response drops as the frequency increases. For $\omega \gg \omega_c$, we can write

$$V \approx \frac{V_i}{j\omega\tau}. \tag{3.77}$$

For phasors, multiplying by $j\omega$ is equivalent to differentiating, and dividing by $j\omega$ is equivalent to integrating. This means that in the roll-off region, this circuit acts as an integrator. One application of this filter would be in an audio system to remove the hiss that you often hear. The hiss comes primarily from frequencies that are higher than the frequency range we use for speaking. A filter with a cut-off frequency of about 3 kHz can remove hiss without hurting speech quality.

The circuit in Figure 3.5a acts as a high-pass filter, letting high frequencies through and blocking low frequencies. We can write the response with the potential-divider formula:

$$V = \frac{V_i R}{R + 1/(j\omega C)} = \frac{V_i}{1 + 1/(j\omega RC)} = \frac{V_i}{1 + 1/(j\omega\tau)}. \tag{3.78}$$

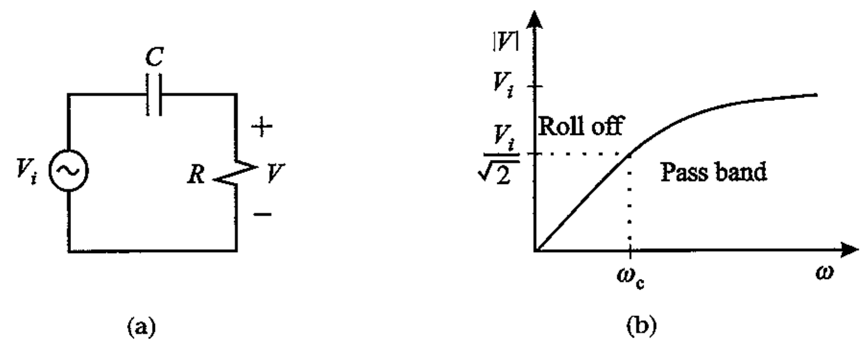


Figure 3.5. RC high-pass filter (a), and the response (b).

This is a high-pass response (Figure 3.5b). The cut-off frequency is the same as for the low-pass filter. This time, however, the pass band is above the cut-off frequency, and the stop band is below it. In the stop band, where $\omega \ll \omega_c$, we can write

$$V \approx j\omega\tau V_i. \tag{3.79}$$

The roll off is proportional to frequency. A circuit like this could be used in an audio system to remove hum. Hum is the low-frequency buzzing that is associated with the AC wall supply.

3.6 Series Resonance

Consider a voltage source with an inductor, capacitor, and load resistor (Figure 3.6a). This is a common circuit for band-pass filters that select signals near a particular frequency. We can use the potential-divider formula to write the output voltage as

$$V = \frac{V_i R}{Z}, \tag{3.80}$$

where Z is the circuit impedance, given by

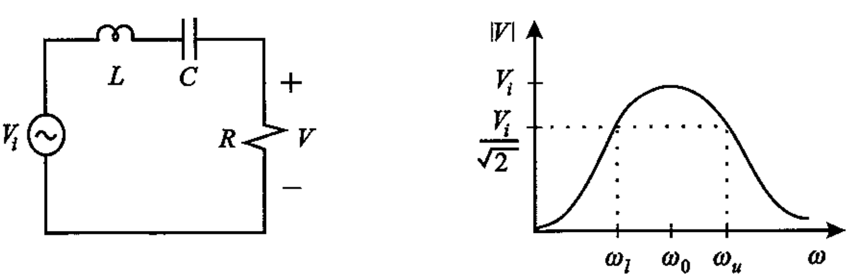
$$Z = R + jX = R + j\omega L + 1/(j\omega C). \tag{3.81}$$

Let us consider the reactance X first, which is the imaginary part:

$$X = \omega L - 1/(\omega C). \tag{3.82}$$

At low frequencies, the capacitive reactance dominates, and the reactance is large and negative. At high frequencies, the inductive reactance dominates, and the reactance is large and positive. The frequency where the reactance is zero is called the *resonant* frequency, and we write it as

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{3.83}$$



(a) (b)

Figure 3.6. Series resonant circuit with a source (a), and the response (b).

At the resonant frequency, the inductive and capacitive reactance cancel, and Equation 3.80 becomes

$$V = V_i \tag{3.84}$$

so that the output voltage is equal to the input voltage.

Away from the resonant frequency, the reactance increases, and the output voltage drops (Figure 3.6b). When the reactance and resistance are equal, the output voltage is given by

$$|V| = \frac{V_i}{|1 \pm j|} = \frac{V_i}{\sqrt{2}}. \tag{3.85}$$

This means that the upper and lower half-power frequencies ω_u and ω_l are where the reactance and resistance are equal. We can find formulas for these frequencies by setting $X = \pm R$:

$$\omega_u L - 1/(\omega_u C) = +R, \tag{3.86}$$

$$\omega_l L - 1/(\omega_l C) = -R. \tag{3.87}$$

Working with these formulas is messy, but we need to go through the details because the results are important. Let us divide through by the resonant inductive reactance $\omega_0 L$ and substitute $1/(\omega_0^2 L)$ for C. We can write

$$\omega_u/\omega_0 - \omega_0/\omega_u = +R/(\omega_0 L), \tag{3.88}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -R/(\omega_0 L). \tag{3.89}$$

The ratio of reactance to resistance in a series circuit is called the *quality factor*, or Q for short:

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}. \tag{3.90}$$

We will see later that this corresponds physically to the ratio of the energy stored in reactive elements to the energy lost in the resistor. We can write the quality factor in terms of either the inductive reactance or capacitive reactance, but it is important to realize that in a resonant circuit, it is not the total reactance that we are talking about, but one or the other. The idea of Q is useful because we can relate it to the bandwidth in a simple way. In terms of Q our formula becomes

$$\omega_u/\omega_0 - \omega_0/\omega_u = +1/Q, \tag{3.91}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -1/Q. \tag{3.92}$$

If you study these formulas, you can see that ω_u and ω_l must be related by

$$\omega_u/\omega_0 = \omega_0/\omega_l. \tag{3.93}$$

We can rewrite this relation as

$$\sqrt{\omega_l \omega_u} = \omega_0. \tag{3.94}$$

In words, the resonant frequency is the geometric mean of the upper and lower half-power frequencies. Now we substitute Equation 3.93 back in Equation 3.91 to get

$$\omega_u/\omega_0 - \omega_l/\omega_0 = 1/Q. \tag{3.95}$$

I have skipped arithmetic here, but you should fill in the details. It is easier to relate this formula to measurements if we rewrite it in terms of the frequency f by dividing ωs by 2π. We get

$$Q = \frac{\omega_0}{\omega_u - \omega_l} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}, \tag{3.96}$$

where $\Delta\omega$ is the half-power bandwidth in radians per second and Δf is the half-power bandwidth in hertz. In words, Q is the ratio of the resonant frequency to the bandwidth. If we want a selective filter with a small bandwidth, then we need a large Q. The Q of the resonant circuits that you build with inductors and capacitors is rather low, less than 100. Later on, we will study quartz crystal resonators that have Qs in the range of 50,000 to 100,000; these make extremely selective filters.

Now consider the behavior of the circuit in the stop band, far away from the resonant frequency. At high frequencies, where $\omega L \gg 1/(\omega C)$ and $\omega L \gg R$, the inductive reactance dominates the circuit, and we can write the circuit impedance approximately as

$$Z \approx j\omega L. \tag{3.97}$$

The output voltage becomes

$$V = \left(\frac{R}{j\omega L}\right) V_i = \frac{V_i}{j\omega\tau_l}, \tag{3.98}$$

where $\tau_l = L/R$ is the inductive time constant. This resembles the equation for the roll off in a low-pass filter (Equation 3.77).

At low frequencies, where $1/\omega C \gg \omega L$ and $1/\omega C \gg R$, the capacitive reactance dominates, and we can write the circuit impedance approximately as

$$Z \approx 1/j\omega C. \tag{3.99}$$

The output voltage becomes

$$V = j\omega RC V_i = j\omega\tau_c V_i, \tag{3.100}$$

where $\tau_c = RC$ is the capacitive time constant. This resembles the equation for the roll off in a high-pass filter (Equation 3.79). We can use Equations 3.98 and 3.100 to predict the rejection ratio of filters at different frequencies in the stop band.

3.7 Parallel Resonance

We have learned that if we want a small bandwidth, we need a large Q. For a large Q in a series resonant circuit, the reactance must be large compared to the resistance. This makes it convenient to use series circuits when the resistances are low, like the 50-Ω input of an antenna and a receiver. However, if the resistance is large, it becomes more difficult to make a high-Q series resonant circuit. For example, the input resistance of the mixers in the NorCal 40A is 1,500 Ω, and a high-Q filter would require extremely large reactances. A parallel resonant circuit may be a good choice in this case. This is because a high-Q parallel circuit requires that the reactance be small compared to the resistance. Let us consider a current source with an inductor, capacitor, and load resistor in parallel (Figure 3.7a). Our analysis will be like that of the series resonance, except that we use admittance instead of impedance. We write the output voltage as

$$V = I/Y \tag{3.101}$$

and the load admittance Y as

$$Y = G + jB = G + j\omega C + 1/(j\omega L). \tag{3.102}$$

We start with the susceptance B, which is the imaginary part of the admittance:

$$B = \omega C - 1/(\omega L). \tag{3.103}$$

At low frequencies, the inductive susceptance dominates, and the susceptance is large and negative. At high frequencies, the capacitive susceptance dominates, and the susceptance is large and positive. The susceptance is zero at the resonant frequency ω_0 given by

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{3.104}$$

This is the same formula we found for series resonant circuits. At the resonant frequency, the inductive and capacitive susceptance cancel, and we have the

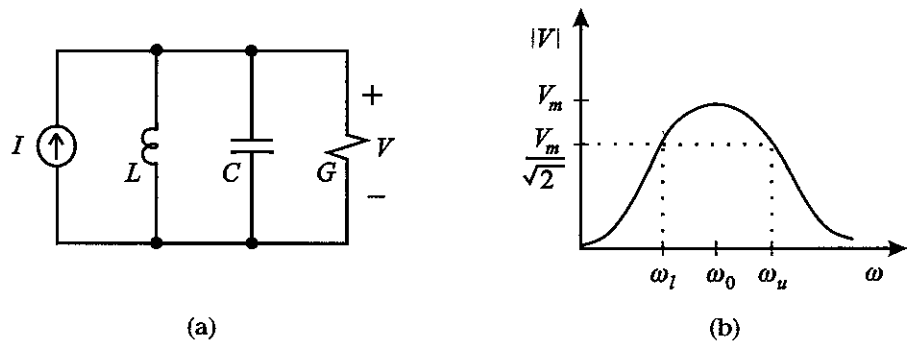


Figure 3.7. Parallel resonant circuit (a), and the response (b).

maximum output voltage. We can write it as

$$V_m = I/G. \tag{3.105}$$

Away from the resonant frequency, the susceptance increases, and the voltage falls (Figure 3.7b). The arithmetic for finding the upper and lower half-power frequencies is similar to that for the series circuit. The half-power frequencies occur when the susceptance equals the conductance. We can write the upper and lower half-power frequencies as

$$\omega_u/\omega_0 - \omega_0/\omega_u = +G/(\omega_0 C), \tag{3.106}$$

$$\omega_l/\omega_0 - \omega_0/\omega_l = -G/(\omega_0 C). \tag{3.107}$$

We define the Q for a parallel circuit as the ratio of susceptance to conductance:

$$Q_p = \frac{\omega_0 C}{G} = \frac{1}{\omega_0 L G}. \tag{3.108}$$

We write this as Q_p, with p standing for parallel. We can rewrite these as

$$Q_p = \frac{R}{\omega_0 L} = \omega_0 R C, \tag{3.109}$$

where $R = 1/G$. This is the inverse of the series formula (Equation 3.90). This means that for high Q in a parallel resonant circuit, we need small reactances. This is different from series resonance. The bandwidth formula, however, is the same as before:

$$Q_p = \frac{f_0}{\Delta f}. \tag{3.110}$$

3.8 Phasor Power

We can write the instantaneous power as

$$P(t) = V(t)I(t). \tag{3.111}$$

Here power is a function of time. This expression includes power that is dissipated as heat in resistors or radiated from antennas and power that goes into inductors and capacitors. In Equation 2.9, we wrote the average power for a resistor with a cosine current as

$$P_a = I_p^2 R/2, \tag{3.112}$$

where I_p is the peak amplitude. We can also write the power in terms of phasors. We define the *complex power* P as

$$P = VI^*/2, \tag{3.113}$$

where * denotes the complex conjugate. We substitute in terms of the circuit impedance $V = ZI$ and get

$$P = ZII^*/2. \tag{3.114}$$

We can write this in terms of the magnitude $|I|$ as

$$P = Z|I|^2/2. \tag{3.115}$$

We can interpret this expression if we write Z in terms of the resistance R and the reactance X:

$$P = R|I|^2/2 + jX|I|^2/2. \tag{3.116}$$

The first term on the right side is real. It is equal to the average power (Equation 3.112), so that we can write

$$P_a = \mathrm{Re}(P) = \mathrm{Re}(VI^*/2). \tag{3.117}$$

The second term on the right side of Equation 3.116 is imaginary. This is the *reactive power*, and it is related to the energy stored in the inductors and capacitors. We can illustrate this for a series combination of an inductor and capacitor. We write the reactive power P_r as

$$P_r = \mathrm{Im}(P) = \frac{\omega L|I|^2}{2} - \frac{|I|^2}{2\omega C} = \omega\left(\frac{L|I|^2}{2} - \frac{C|V_c|^2}{2}\right), \tag{3.118}$$

where V_c is the capacitor voltage. We can rewrite this in terms of the stored energy as

$$P_r = \omega(E_l - E_c), \tag{3.119}$$

where E_l is the peak energy stored in the inductor and E_c is the peak energy stored in the capacitor. This calculation is for a series RLC circuit, but the result also holds for more complicated circuits. The reactive power is proportional to the difference between the peak magnetic energy and the peak electric energy. At resonance, the reactive power is zero, and the peak electric energy equals the peak magnetic energy.

Equation 3.119 allows us to develop a more general formula for Q that includes the series and parallel circuits as special cases. We can rewrite the series Q as

$$Q = \omega\frac{L}{R} = \omega\frac{L|I|^2/2}{R|I|^2/2}, \tag{3.120}$$

or

$$Q = \omega\frac{E_l}{P_a}. \tag{3.121}$$

At resonance the peak inductor energy E_l is equal to the peak capacitor energy E_c, and this energy oscillates back and forth between the inductor and capacitor. When the stored energy in the inductor is at its peak, the stored energy in the capacitor is zero, and this means that E_l is actually the total energy stored in the circuit. We drop the subscript and get

$$Q = \omega\frac{E}{P_a}, \tag{3.122}$$

where E is the total stored energy. This says that Q is proportional to the ratio of the stored energy to the average power. To raise Q, we should increase the stored energy or decrease the loss. You should verify that this general formula is equivalent to the Q_p we defined for parallel resonant circuits. We will also apply the formula to resonant transmission lines in the next chapter.

FURTHER READING

Complex numbers are a fascinating part of mathematics, and students who would like to learn more should read Paul Nahin's *An Imaginary Tale: The Story of* $\sqrt{-1}$, published by Princeton University Press, on the history and application of complex numbers. Nahin has developed geometric interpretations that provide powerful insights into the solution of many physics and engineering problems. The classic textbook on complex numbers is *Theory of Functions of a Complex Variable,* by A. I. Markushevich, published by Chelsea Publishing Company.

PROBLEM 7 – PARALLEL-TO-SERIES CONVERSION

A. It is often useful in discussing circuits to be able to convert a parallel combination of reactance and resistance to an equivalent series combination. Starting with the parallel circuit in Figure 3.8a, find expressions for the components in a series circuit (Figure 3.8b) that give the same impedance. One way to approach this problem is to define a Q for each circuit that is the ratio of the reactance to the resistance. We let

$$Q_s = X_s/R_s \tag{3.123}$$

and

$$Q_p = R_p/X_p. \tag{3.124}$$

First show that if the two circuits are to have the same impedance, the two Qs must be the same. This means that in the rest of the problem, you can drop the subscripts, and just write Q.

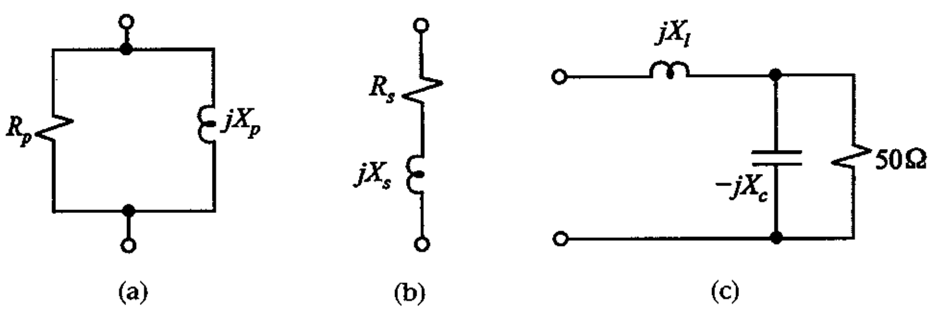


Figure 3.8. Parallel circuit with a resistance R_p and a reactance X_p (a), equivalent series circuit with R_s and X_s (b), and matching circuit with a parallel capacitor and a series inductor (c).

B. Find an approximate formula for X_s when Q is large. Find an approximate formula for R_s when Q is small.

There is one thing you should think about. The Q we define here is not quite the same as the one we use for resonant circuits. This Q involves the total reactance, whereas the resonant circuit Q uses only one of the two reactances. In practice, people use the same letter Q for both situations, and you have to figure out which is intended by the context.

Many transmitters have a low output impedance so that the output power varies inversely with the load resistance. For example, if an amplifier has an output of 1 W with a 50-Ω load, we would hope for 10 W with a 5-Ω load.

C. We will use the network in Figure 3.8c to transform a 50-Ω antenna to 5 Ω. We need our parallel–series conversion formulas. The first step is to find the capacitor reactance X_C. When the capacitor and resistor are converted to a series circuit, the resistance should be 5 Ω. Next choose the inductor reactance X_l to cancel the capacitive reactance. What capacitance (in nF) and inductance (in nH) are required at a frequency of 7 MHz?

PROBLEM 8 - SERIES RESONANCE

In this problem, we solder an inductor and capacitor on the NorCal 40A circuit board and make measurements. It is convenient to mount the board in an electronics vise for soldering. The components mount on the side with the white lettering, and the solder is applied to the other side. Insert the parts that you plan to solder. They should be close to the board, but you may want to leave a millimeter of space so that you can hook up scope probes conveniently. You may need to bend the wires a bit so that the parts do not fall out. Before you solder, check that the parts are in the right holes. They can be unsoldered if you make a mistake, but this is difficult if the part has more than two leads.

Before you start, put some water on a sponge. Turn on a soldering iron, and when it is warm, apply solder to the tip of the iron to tin it. Wipe the tip on the sponge to remove the excess solder. This wiping leaves a shiny surface on the tip that heats up parts much better than a tip without solder. Apply the tip and solder at the same time to the hole and the wire. Be alert when soldering parts with plastic packages, or the plastic will melt. Do not use more solder than you need to flow through the hole and coat the wire, or you run the risk that there will be short circuits to other holes. Clip off the wire ends close to the board after you finish so that they will not touch other wires. Inspect the hole and the wire. The solder should flow completely through the hole and coat the wire. If the wire is not hot enough, the solder will not coat the wire well. This is called a cold solder joint. Cold solder joints often cause open circuits.

If you do make a mistake and put the parts in the wrong holes, be careful when you take them out so that you do not damage the parts or the board. I like to remove solder with wick before I remove the part. Solder wick is a copper braid that absorbs molten solder. Melt the solder with the iron and coax the solder into the wick. Cut off pieces of the wick that get solder on them and throw the pieces away. When you have taken off as much solder as you can, apply the soldering iron at the joints to melt the remaining solder and loosen the part with pliers. You may have to do this repeatedly with each lead

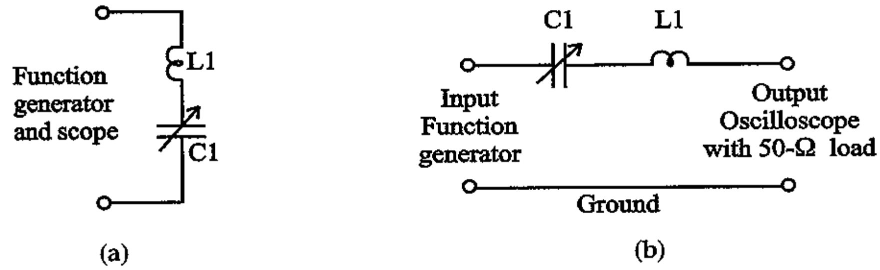


Figure 3.9. Connections for observing a series resonance (a), and band-pass filter connection (b).

until the part comes out. Finally, use the solder wick to remove the solder from the hole before you insert another part.

Solder contains lead, which is dangerous if you eat it; therefore you should wash your hands with soap and water after soldering. It is a good idea to remember to turn off the soldering iron when you have finished working. Leaving the irons on is hard on the tips.

Install C1 and L1 on the NorCal 40A board. C1 is a variable capacitor with a nominal range from 8 to 50 pF, and L1 is an inductor with an inductance of 15 μH. The variable capacitor has meshing metal vanes separated by ceramic insulators. It is best adjusted with a plastic screwdriver. Metal screwdrivers have extra capacitance that shifts the resonant frequency. We will tune the capacitor for resonance at 7 MHz. The inductor and capacitor form part of the RF Filter. Look inside the front cover of this book to see where this circuit goes in the receiver. Appendix D also has data on the components that you use.

A. Make the connections shown in Figure 3.9a, with a tee connection to the scope. Set the function generator for a 7-MHz sine wave with an amplitude of 1 Vpp. When a circuit is resonant, the capacitive reactance cancels the inductive reactance, leaving us with the source resistance of the function generator (50 Ω) and the resistance of the capacitor and the inductor. If we tune the capacitor through resonance, we will see a dip in the input voltage when the circuit is resonant. Adjust the capacitor for minimum input voltage. This sets the resonant frequency to 7 MHz. What is this voltage? Use the voltage to calculate the total resistance of the capacitor and inductor.

B. Now make the connections for the band-pass filter shown in Figure 3.9b, with a 50-Ω scope load. Adjust the capacitor for maximum output voltage. What is this voltage? Calculate the voltage that you expect.

C. Find the half-power bandwidth by measuring the frequencies f_u and f_l where the output voltage has dropped by a factor of $\sqrt{2}$. One way to do this is to use a larger amplitude setting of 1.41 Vpp, and look for the two frequencies that give the same output voltage as 1 Vpp at 7 MHz.

D. Now we will calculate the half-power bandwidth Δf that we expect. Start by finding the resonant inductive reactance. Calculate the Q from the inductive reactance and the total circuit resistance, and then calculate the half-power bandwidth.

E. Return the amplitude setting to 1 Vpp. Measure the output voltage at 1-MHz intervals from 1 MHz to 15 MHz and make a plot. The response changes dramatically

between 6 MHz and 8 MHz, and you will need some additional data points to keep them from getting too far apart.

AM radio transmitters in the frequency range from 0.5 to 1.5 MHz are a big problem for receivers because they are usually close and powerful. For example, some broadcasters use 50 kW, and they may only be a few miles away. The NorCal 40A is for 2-W stations that might be a thousand miles away. We will find the AM *voltage rejection factor* R_{am}, given by

$$R_{am} = V_{rf}/V_{am}, \tag{3.125}$$

where V_{rf} is the output at 7 MHz and V_{am} is the output at 1 MHz.

F. Measuring V_{am} is tricky, because the output is small at 1 MHz. One way to approach the problem is to use an amplitude setting of 10 Vpp at 1 MHz. This increases the output voltage by a factor of 10, making it easier to measure. You will need to divide your output voltage by a factor of 10 to take this into account before you compare it with the 7-MHz voltage. It is not a good idea to use an amplitude of 10 Vpp at 7 MHz, because the voltages on the inductor and capacitor get large enough to change their response. What is R_{am}?

G. Use the low-frequency approximation (Equation 3.100) to calculate the value of R_{am} that we would expect.

PROBLEM 9 – PARALLEL RESONANCE

In the NorCal 40A, the transmitter signal is produced by mixing the VFO at 2.1 MHz with the Transmit Oscillator at 4.9 MHz. The transmitter frequency is the sum of these two frequencies, 7.0 MHz. The Transmit Mixer also produces other frequencies that are removed by the Transmit Filter. This filter uses a parallel resonance. A parallel resonance is a good choice for a band-pass filter if the source and load resistances are large, because we can easily make capacitors and inductors with much smaller reactances to give high Q. This filter is made up of C37, C38, C39, and L6. You should study the endpaper to see how this circuit works in the transmitter.

Start by soldering C37 (5-pF disk) and C38 (100-pF disk) on the board. Do not include the variable capacitor C39 yet – we will make some measurements first. L6 is the first inductor that you make yourself by winding wire on a toroidal core. *Toroidal* means donut-shaped. This shape is good for radio inductors because it keeps the magnetic field inside the magnetic material. Compared with the smaller rod inductors we have worked with so far, the toroidal inductors have a better Q and can operate at higher power. L6 uses a T37–2 core. "T" indicates toroidal core, 37 is the outside diameter in hundredths of an inch, and 2 refers to the particular mix of material. Material #2 is an iron powder mix that is useful from 1 to 30 MHz. #2 cores are traditionally painted red to distinguish them from other mixes.

The L6 coil has 28 turns of #28 wire. Cut a 40-cm length and wrap it around the core, being careful with the count (Figure 3.10a). It is easy to be low by one turn. For example, the figure shows a core with 6 turns, not 5. After you finish winding, spread the turns evenly around the core, leaving a gap between the first and last turn so that the

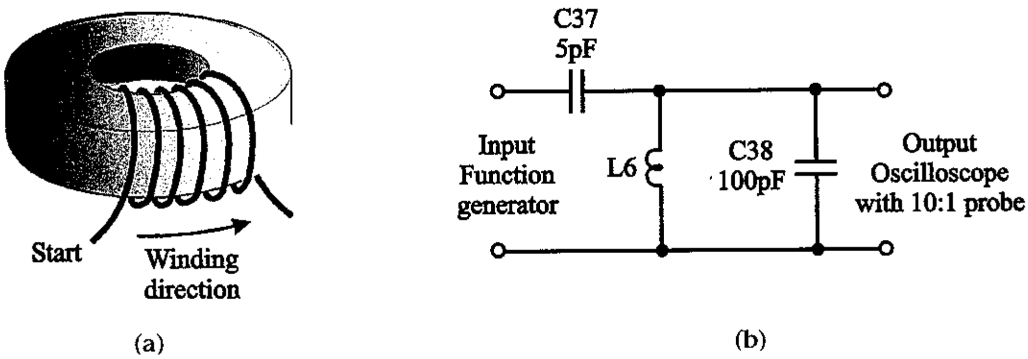


Figure 3.10. (a) Winding a toroid. Six turns are shown. (b) Initial band-pass filter connections.

wire ends line up with the holes in the board. Strip the wire ends using a cigarette lighter to burn the enamel. Sand the ends until the enamel is completely gone. If the enamel is not completely removed, the solder may not stick to the wire. If you are lucky, you get an open circuit. More likely is an intermittent contact that depends on temperature, pressure, and the phase of the moon.

Solder L6 onto the board. Connect the function generator and 10:1 probe as shown in Figure 3.10b. The coil wire is thin, and it is a bad idea to attach probes to it. It will take a little practice to follow the traces on the circuit board so that you can tell where to connect the probes. Most of the traces are on the solder side of the board, but there are also a few on the component side. Moreover, most of the component side is a single connected ground plane. If you see solder pads that do not appear to lead anywhere, they are likely to be ground connections.

A. The function generator should be set for a sine wave, with an amplitude setting of 1 Vpp. Find the resonant frequency f_0 that gives the largest output voltage. From f_0 and the total capacitance (C37, C38, and the probe capacitance), calculate the inductance of the coil that you wound.

B. We will discuss inductance calculations in Chapter 6, but for now you need to know that the inductance is proportional to the square of the number of turns. We write

$$L = A_l N^2, \tag{3.126}$$

where A_l is an inductance constant and N is the number of turns. Core manufacturers provide the inductance constant in their data sheets. For the T37–2 core, A_l is 4.0 nH/turn^2. Calculate the inductance that you expect for L6.

C. Now solder the variable capacitor C39 into the circuit. Set the frequency to 7 MHz and adjust the capacitor carefully for maximum output. Record the output voltage. Measure the half-power bandwidth Δf, and calculate the Q.

D. Calculate the inductor reactance X at 7 MHz. Use this reactance and the Q you measured to find the effective parallel resistance R. This resistance is not a separate component but is associated with the inductor, the capacitors, the function generator, and the scope probe.

E. We can also calculate the output voltage that we expect. One way to start is to find a Norton equivalent circuit for the series combination of the function generator and the 5-pF capacitor. The output voltage can be calculated from the Norton current and the effective parallel resistance R.

F. In addition to the sum frequency at 7 MHz, the mixer produces a strong difference-frequency signal at 2.8 MHz. We do not want to transmit the difference frequency, because it might interfere with other services. Measure the response of the filter at the difference frequency. Express the difference-frequency voltage rejection factor R_- as

$$R_- = V_{rf}/V_-, \tag{3.127}$$

where V_{rf} is the 7-MHz voltage and V_- is the difference voltage. At 2.8 MHz, you should turn up the function generator to 10 Vpp to make the output signal as large as you can, and you should take this into account in calculating the voltage ratio. The output signal will be quite small, and the trace will become fuzzy because of scope noise. You need to be careful to measure at the same place in the noise at the top and bottom of the sine wave.

G. Although dB are units for comparing power levels, we can also write dB expressions in terms of voltage or current if we take into account the fact that the power is proportional to the square of the voltage or current. We write

$$10\log(P_1/P_2) = 20\log(V_1/V_2) = 20\log(I_1/I_2)\ \text{dB}. \tag{3.128}$$

For example, if V_1 is twice V_2, then we would say that the first signal is 6 dB bigger than the second. For these voltage and current formulas to make sense, the resistance associated with each power must be the same, because the power depends on the resistance. This is appropriate for the rejection factor of a filter. Now express the rejection factor as a dB difference, using the formula

$$R_- = 20\log(V_{rf}/V_-)\ \text{dB}. \tag{3.129}$$

H. Calculate what the difference-frequency rejection should be. You will need to consider how the circuit quantities vary with frequency.

I. What would the Q of the filter be if the 5-pF input capacitor (C37) were bypassed and the 50-Ω function generator were connected directly to C38?