ECE 5654 Lecture 9
Probability of Error for Modulation Schemes

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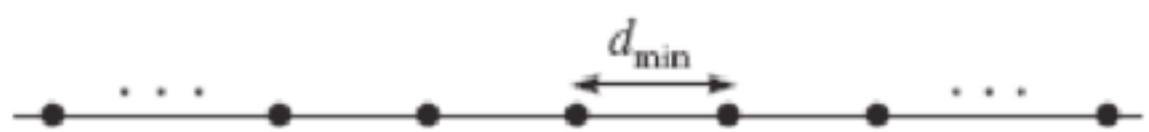
Optimum Detection and Error Probability for Bandlimited Signaling

Amplitude-Shift Keying (ASK) or PAM

- Let d_{min} be the minimum distance between adjacent PAM constellations

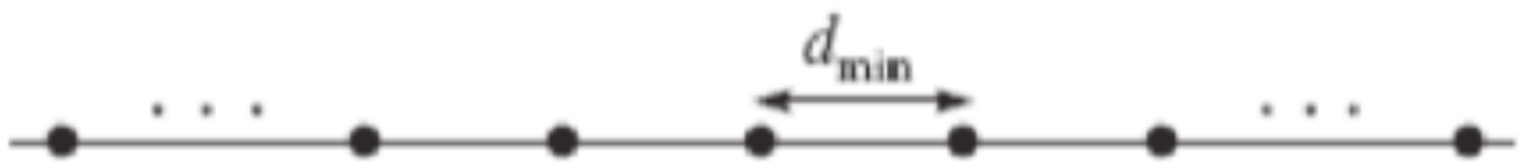
- Consider the signal constellations.

$$\mathcal{S} = \left\{ \pm\frac{1}{2}d_{\text{min}}, \pm\frac{3}{2}d_{\text{min}}, \cdots, \pm\frac{M-1}{2}d_{\text{min}} \right\}$$



- The average bit signal energy is

$$\mathcal{E}_{\text{bavg}} = \frac{1}{\log_2(M)}\mathbb{E}[|\mathbf{s}|^2] = \frac{M^2-1}{12\log_2(M)}d_{\text{min}}^2$$



There are two types of error events (under AWGN):

- Inner points with error probability P_{ei} (M-2) such points

$$P_{ei} = \Pr\left\{|\mathbf{n}| > \frac{d_{min}}{2}\right\} = 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

- Outer points with error probability P_{eo}: only one end causes errors: 2 such points

$$P_{eo} = \Pr\left\{\mathbf{n} > \frac{d_{min}}{2}\right\} = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

The symbol error probability is given by

$$
\begin{aligned}
P_e &= \frac{1}{M}\sum_{m=1}^{M} \Pr\{\text{error}|m \text{ sent}\} \\
&= \frac{1}{M}\left[(M-2)\cdot 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) + 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right]
\end{aligned}
$$

(M-2) inner points　　　2 outer points

$$
\begin{aligned}
&= \frac{2(M-1)}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\
&= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2(M)}{(M^2-1)}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)
\end{aligned}
$$

(Substituting dmin in terms of Ebavg)

$$
\approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right) \qquad \text{for large } M
$$

Efficiency of PAM

$$P_e \approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right) \qquad \text{for large } M$$

- To increase rate by 1 bit (i.e., $M \to 2M$), we need to double M.
- To keep (almost) the same P_e, we need $\mathcal{E}_{\text{bavg}}$ to quadruple.

M	2	4	8	16
$\frac{6\log_2(M)}{(M^2-1)}$	2	$\frac{4}{5}$	$\frac{2}{7}$	$\frac{8}{85}$

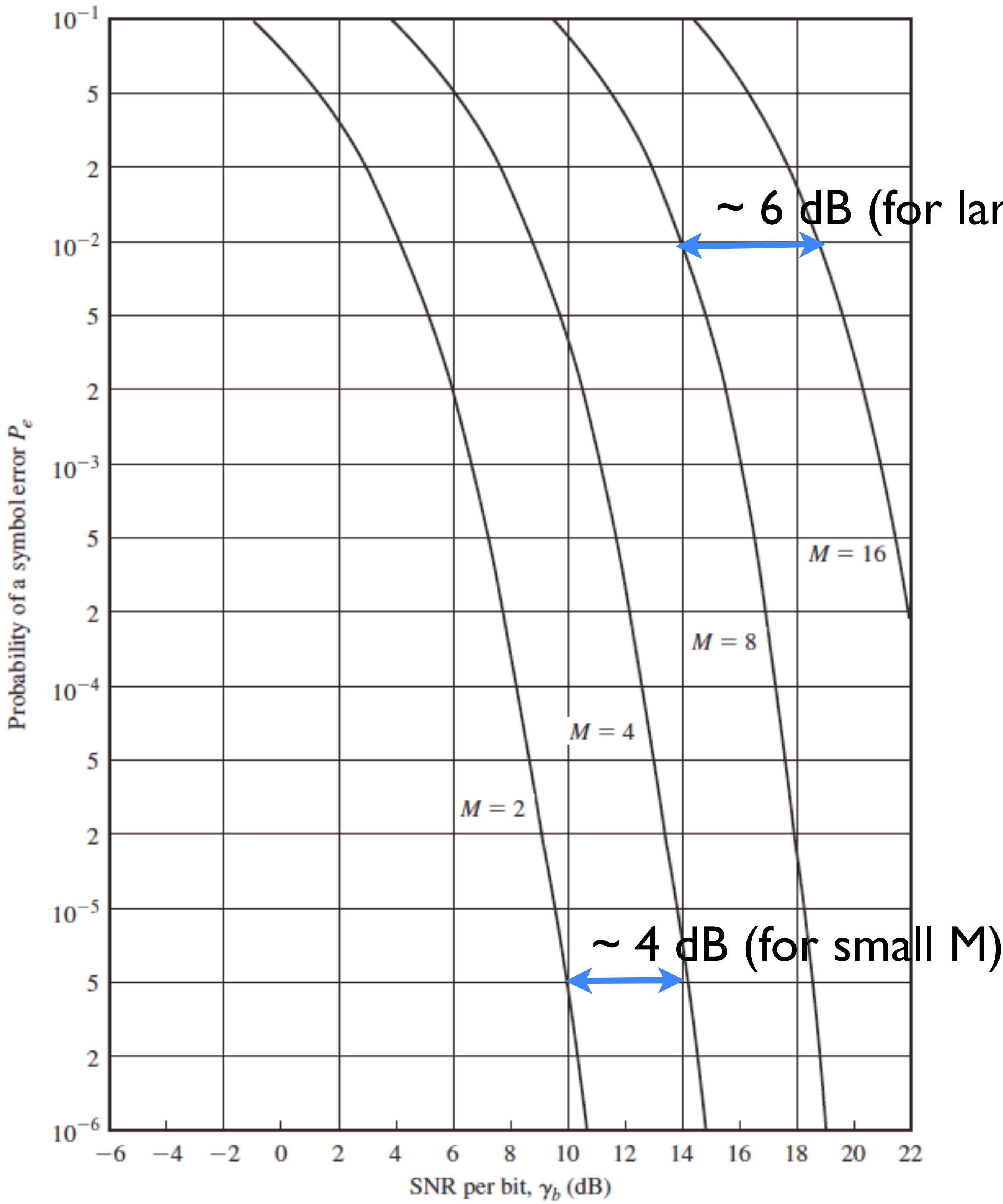
2→4	4→8	8→16	$\cdots$	$M \to 2M$ as M large
2.5	2.8	3.0	$\cdots$	4

$$\frac{6\log_2(M)}{(M^2-1)}\frac{\mathcal{E}_{\text{bavg}}}{N_0} \approx \frac{6\log_2(2M)}{((2M)^2-1)}\frac{\mathcal{E}_{\text{bavg}}^{(\text{new})}}{N_0}$$

$$\Rightarrow \mathcal{E}_{\text{bavg}}^{(\text{new})} \approx 4\mathcal{E}_{\text{bavg}}$$

- Increase rate by 1 bit $\Longrightarrow$ increase $\mathcal{E}_{\text{bavg}}$ by 6 dB

PAM Performance



The larger the M is,
the worse the symbol performance

Phase-Shift Keying (PSK) Signaling

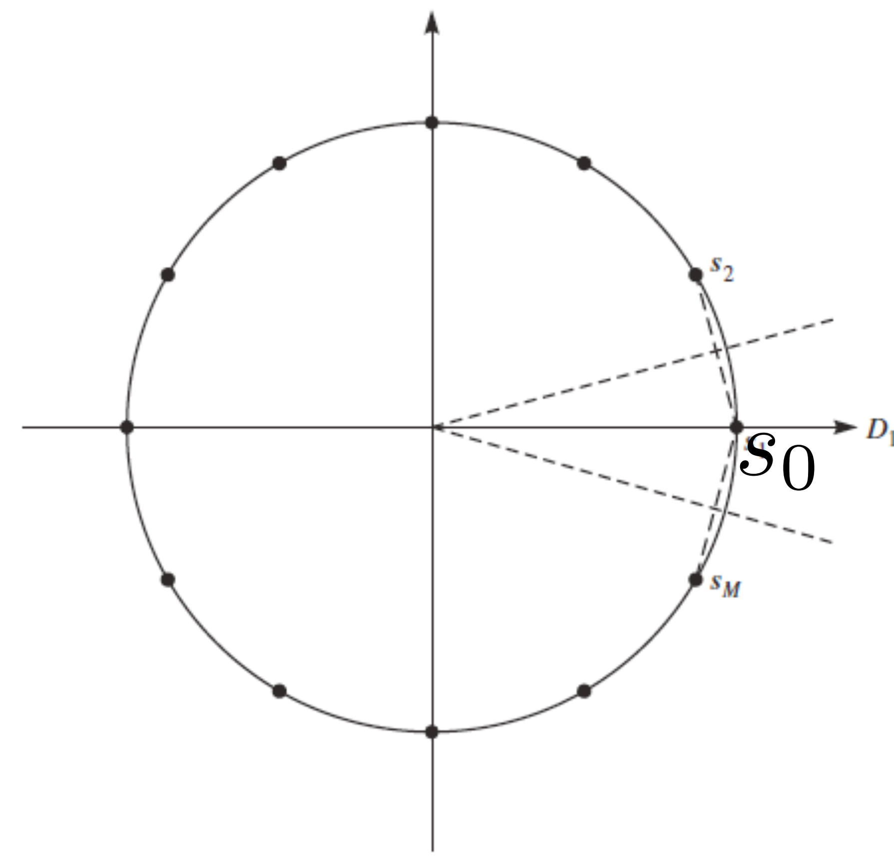
Signal constellation for M-ary PSK is

$$\mathcal{S} \;=\; \left\{ \mathbf{s}_k = \sqrt{\mathcal{E}}\left(\cos\left(\frac{2\pi k}{M}\right), \sin\left(\frac{2\pi k}{M}\right)\right) \;:\; k = 0, 1, \cdots, M-1 \right\}$$

- By symmetry we can assume $\mathbf{s}_0 = \left(\sqrt{\mathcal{E}}, 0\right)$ was transmitted.

- The received signal vector $\mathbf{r}$ is

$$\mathbf{r} \;=\; \left(\sqrt{\mathcal{E}} + \mathbf{n}_1, \mathbf{n}_2\right)^{\mathsf{T}}$$



- Assume Gaussian random process with $R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau)$

$$f(n_1, n_2) = \frac{1}{\pi N_0} \exp\left(-\frac{n_1^2 + n_2^2}{N_0}\right)$$

- Thus we have

$$f(\mathbf{r} = (r_1, r_2)|\mathbf{s}_0) = \frac{1}{\pi N_0} \exp\left(-\frac{(r_1 - \sqrt{\mathcal{E}})^2 + r_2^2}{N_0}\right)$$

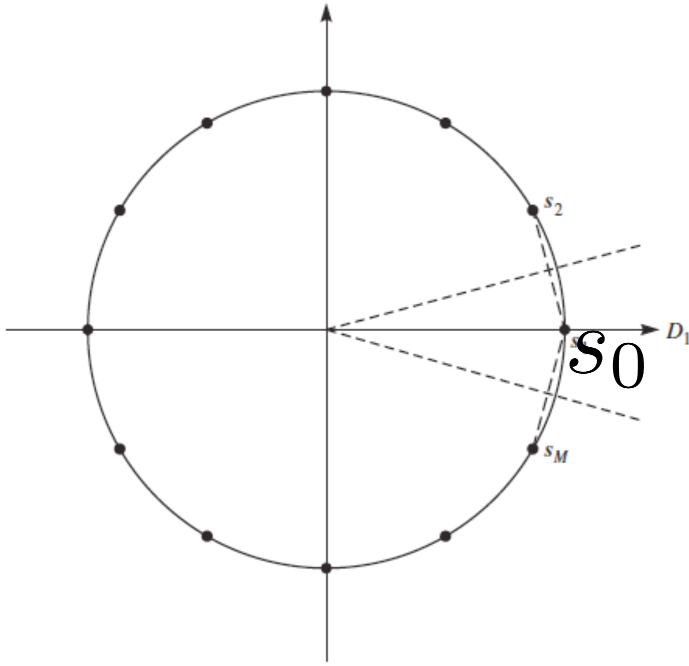
For PSK, the Decision region can be more conveniently described using polar coordinates

- Define $V = \sqrt{r_1^2 + r_2^2}$ and $\Theta = \arctan \frac{r_2}{r_1}$

$$f(v, \theta|\mathbf{s}_0) = \frac{v}{\pi N_0} \exp\left(-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}} v \cos\theta}{N_0}\right)$$

The ML decision region for $\mathbf{s}_0$ is

$$\mathcal{D}_0 = \left\{ \mathbf{r} : -\frac{\pi}{M} < \theta < \frac{\pi}{M} \right\}$$



The probability of erroneous decision given $\mathbf{s}_0$ is

$$\begin{aligned}
&\Pr\{\text{error}|\mathbf{s}_0\} \\
&= 1 - \iint_{\mathcal{D}_0} f(v, \theta|\mathbf{s}_0)\, dv\, d\theta \\
&= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \underbrace{\int_0^{\infty} \frac{v}{\pi N_0} \exp\left(-\frac{v^2 + \mathcal{E} - 2\sqrt{\mathcal{E}} v \cos\theta}{N_0} \right) dv}_{f(\theta|\mathbf{s}_0)}\, d\theta
\end{aligned}$$

Marginal PDF of *phase* of r

$$= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} f(\theta|\mathbf{s}_0) d\theta$$

PDF of received vector r (amplitude, phase)

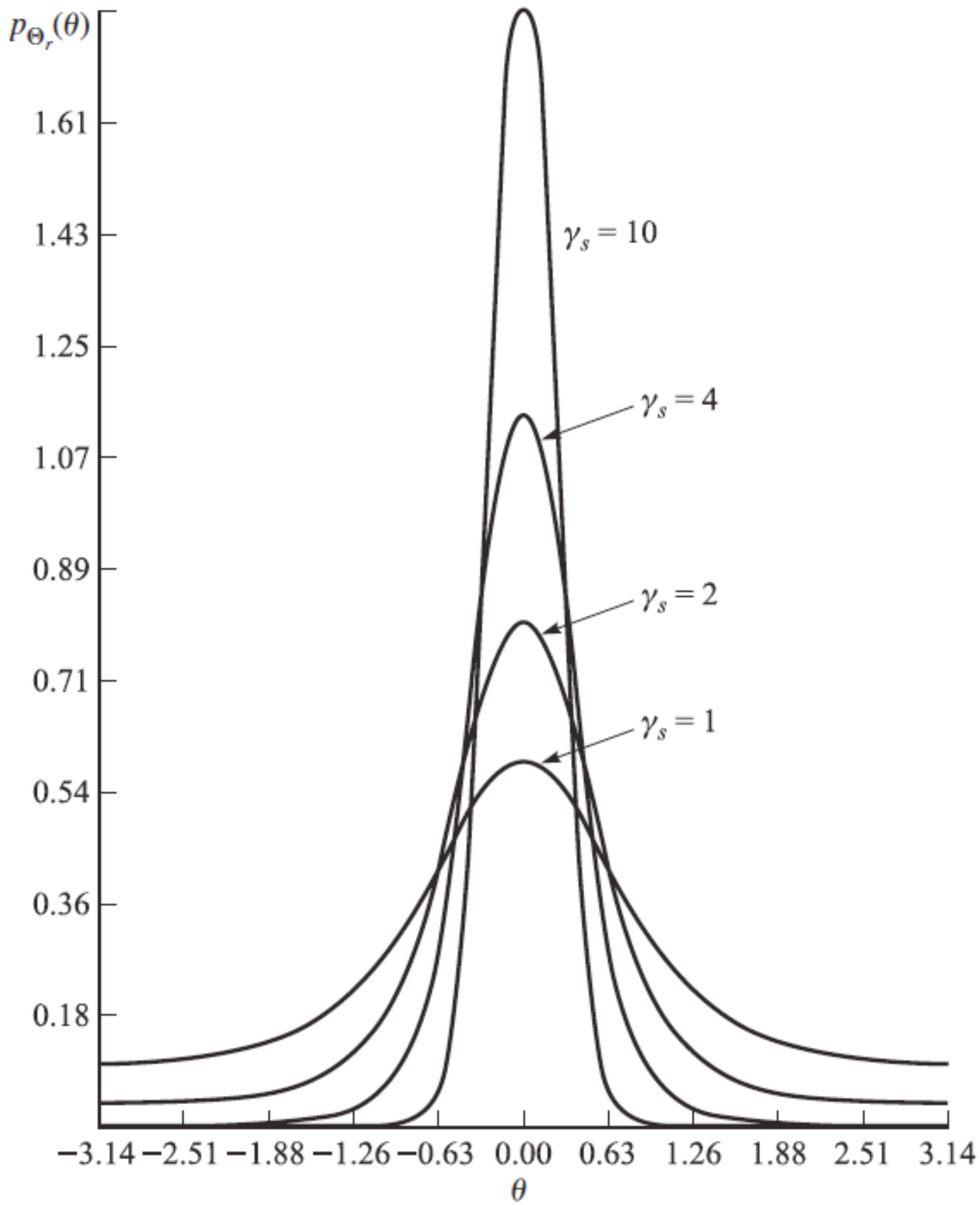
$$f(v,\theta|\mathbf{s}_0) = \frac{v}{\pi N_0}\exp\left(-\frac{v^2+\mathcal{E}-2\sqrt{\mathcal{E}}v\cos\theta}{N_0}\right)$$

Marginal PDF of *phase* of r

$$\begin{aligned}
f(\theta|\mathbf{s}_0) &= \int_0^\infty \frac{v}{\pi N_0}\exp\left(-\frac{v^2+\mathcal{E}-2\sqrt{\mathcal{E}}v\cos\theta}{N_0}\right)dv \\
&= \int_0^\infty \frac{v}{\pi N_0}\exp\left(-\frac{(v-\sqrt{\mathcal{E}}\cos\theta)^2+\mathcal{E}\sin^2\theta}{N_0}\right)dv \\
&= \frac{1}{2\pi}\exp\left(-\gamma_s\sin^2\theta\right)\int_0^\infty t\exp\left(-\frac{(t-\sqrt{2\gamma_s}\cos\theta)^2}{2}\right)dt
\end{aligned}$$

where $\gamma_s = \mathcal{E}/N_0$ and $t = v/\sqrt{N_0/2}$

SNR per symbol or Symbol SNR



$$P_e = 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} f(\theta|\mathbf{s}_0)\,d\theta$$

The larger the γ_s,
the narrower the $f(\theta|\mathbf{s}_0)$,
and the smaller the P_e.

- When $M = 2$, binary PSK is antipodal ($\mathcal{E} = \mathcal{E}_b$):

$$P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- When $M = 4$, it is QPSK ($\mathcal{E} = 2\mathcal{E}_b$).

$$P_e = 1 - \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]^2$$

- When $M > 4$, no simple Q-function expression for P_e?

  However, we can obtain a good approximation.

$$f(\theta|\mathbf{s}_0) \geq \sqrt{\frac{\gamma_s}{\pi}} e^{-\gamma_s \sin^2\theta} \cos\theta$$

This is a good approximation for large values of M, and large SNR

Thus

$$\begin{aligned} P_e &= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} f(\theta|\mathbf{s}_0)\, d\theta \\ &\leq 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \sqrt{\frac{\gamma_s}{\pi}} e^{-\gamma_s \sin^2\theta} \cos\theta\, d\theta \\ &\leq 1 - \int_{-\sqrt{2\gamma_s}\sin(\pi/M)}^{\sqrt{2\gamma_s}\sin(\pi/M)} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \quad (u = \sqrt{2\gamma_s}\sin\theta) \\ &= 2Q\left(\sqrt{2\gamma_s}\sin(\pi/M)\right) \end{aligned}$$

Efficiency of PSK

For large M, we can approximate $\sin(\pi/M) \leq \pi/M$ and $\gamma_s = \frac{\mathcal{E}}{N_0} = \log_2(M)\frac{\mathcal{E}_b}{N_0}$,

$$P_e \approx 2Q\left(\sqrt{2\pi^2 \frac{\log_2 M}{M^2}\frac{\mathcal{E}_b}{N_0}}\right)$$

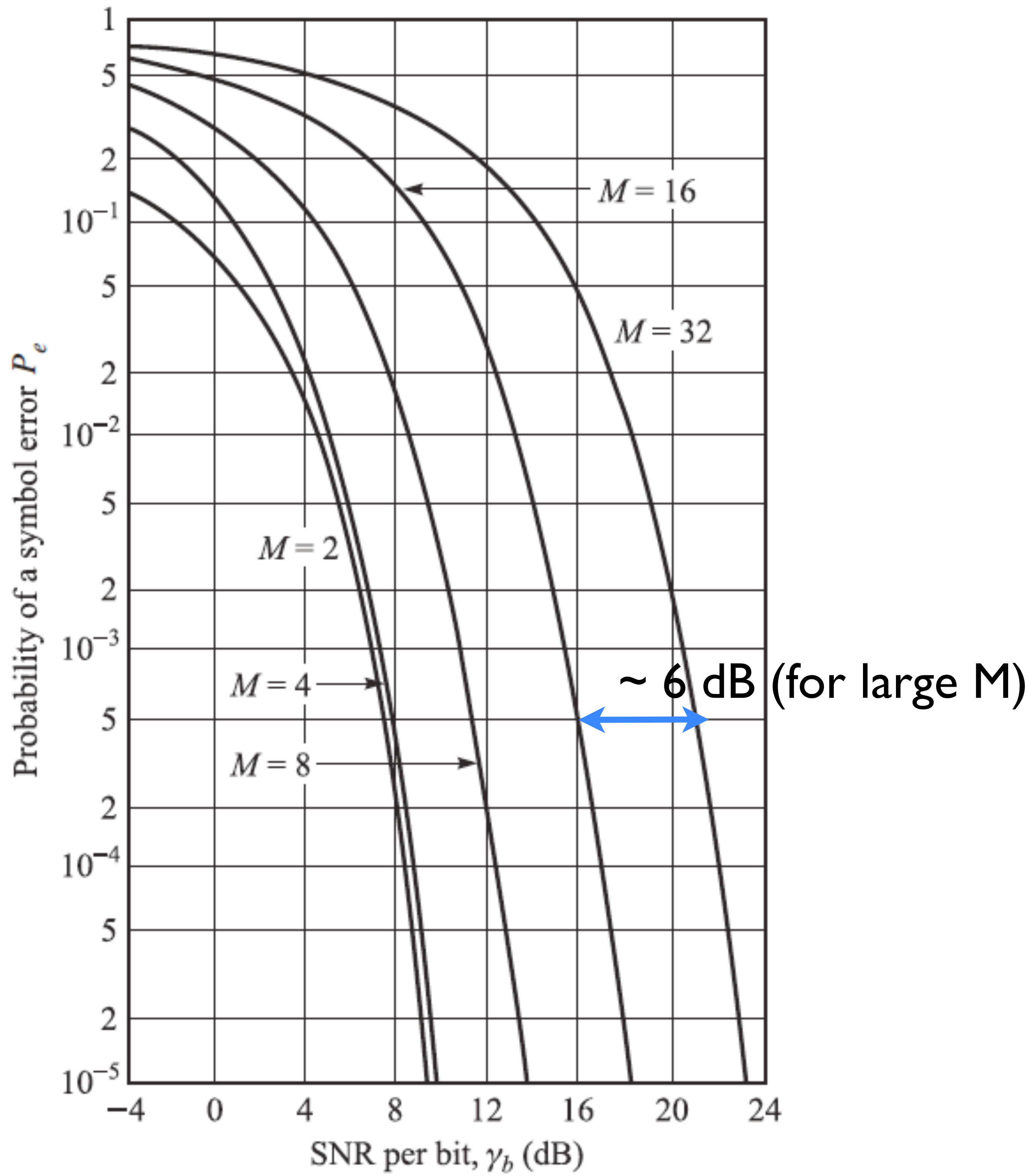
- To increase rate by 1 bit, we need to double M.
- To keep (almost) the same P_e, we need $\mathcal{E}_{\text{bavg}}$ to quadruple.

M	2	4	8	16
$\frac{\log_2(M)}{M^2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{64}$	$\frac{1}{64}$

$2 \to 4$	$4 \to 8$	$8 \to 16$	$\cdots$	$M \to 2M$ as M large
2	2.67	3	$\cdots$	4

- Increase rate by 1 bit $\Longrightarrow$ increase $\mathcal{E}_{\text{bavg}}$ by 6 dB as M large

PSK Performance



Same as PAM

The larger the M is,
the worse the symbol performance

M-ary (rectangular) QAM Signaling

- M is usually a product number, $M = M_1 M_2$
- M-ary QAM is composed of two independent M_i-ary PAM (because the noise is white)

$$\begin{aligned} \mathcal{S}_{PAM_i} &= \left\{ \pm\frac{1}{2} d_{\min}, \pm\frac{3}{2} d_{\min}, \cdots, \pm\frac{M_i - 1}{2} d_{\min} \right\} \\ \mathcal{S}_{QAM} &= \{ (x, y) \, : \, x \in \mathcal{S}_{PAM_1} \text{ and } y \in \mathcal{S}_{PAM_2} \} \end{aligned}$$

$$\mathbb{E}[|x|^2] = \frac{M_1^2 - 1}{12} d_{\min}^2 \quad \text{and} \quad \mathbb{E}[|y|^2] = \frac{M_2^2 - 1}{12} d_{\min}^2$$

Thus for M-ary QAM we have

$$\mathcal{E}_{\text{bavg}} = \frac{\mathbb{E}[|x|^2] + \mathbb{E}[|y|^2]}{\log_2(M)} = \frac{(M_1^2 - 1) + (M_2^2 - 1)}{12 \log_2 M} d_{\min}^2$$

Hence

$$\begin{aligned} P_{e,M\text{-QAM}} &= 1-(1-P_{e,M_1\text{-PAM}})(1-P_{e,M_2\text{-PAM}}) \\ &= P_{e,M_1\text{-PAM}} + P_{e,M_2\text{-PAM}} - P_{e,M_1\text{-PAM}} P_{e,M_2\text{-PAM}} \end{aligned}$$

Recall from Slide 5 on PAM

$$P_{e,M_i\text{-PAM}} = 2\left(1-\frac{1}{M_i}\right) Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \le 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

we have

$$\begin{aligned} P_{e,M\text{-QAM}} &\le P_{e,M_1\text{-PAM}} + P_{e,M_2\text{-PAM}} \le 4Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \\ &= 4Q\left(\sqrt{\frac{6\log_2 M}{M_1^2+M_2^2-2}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right) \end{aligned}$$

Efficiency of QAM

When $M_1 = M_2$,

$$P_{e,M\text{-QAM}} \;\le\; 4Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{\mathcal{E}_{\text{bavg}}}{N_0}}\right)$$

- To increase rate by 2 bit, we need to quadruple M.

- To keep (almost) the same P_e, we need $\mathcal{E}_{\text{bavg}}$ to double.

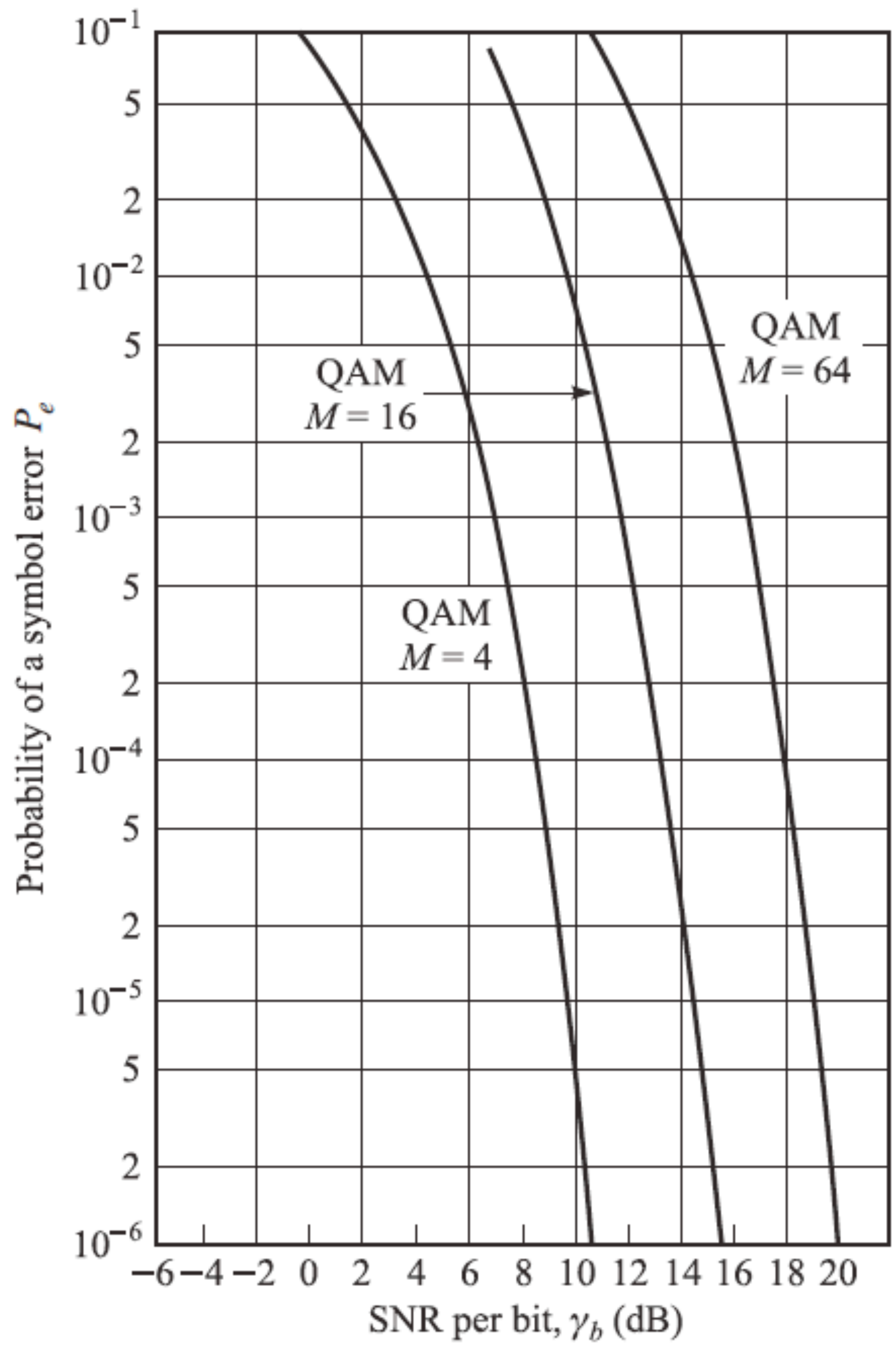
M	4	16	64
$\frac{\log_2(M)}{M-1}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{2}{21}$

$4\to16$	$16\to64$	$\cdots$	$M \to 4M$ as M large
2.5	2.8	$\cdots$	4

- Equivalently, increase rate by 1 bit $\Longrightarrow$ increase $\mathcal{E}_{\text{bavg}}$ by 3 dB as M large.

- QAM is more power efficient than PAM and PSK.

Performance of QAM



Optimal Detection and Error Probability for Power Limited Signaling

Orthogonal (FSK) Signaling

(Bandpass) Signal constellation of M-ary orthogonal signaling (OS)

$$\mathcal{S}_{OS} = \left\{ \mathbf{s}_1 = [\sqrt{\mathcal{E}}, 0, \cdots, 0]^\top, \cdots, \mathbf{s}_M = \left[0, \cdots, 0, \sqrt{\mathcal{E}}\right]^\top \right\}$$

where the dimension N is equal to M.

Given $\mathbf{s}_1$ transmitted, the received signal vector is

$$\mathbf{r} = \mathbf{s}_1 + \mathbf{n}$$

with ($\mathbf{n}$ being the bandpass projection noise and)

$$\begin{aligned} r_1 &= \sqrt{\mathcal{E}} + n_1 \\ r_2 &= n_2 \\ \vdots &\quad \vdots \quad \vdots \\ r_M &= n_M \end{aligned}$$

By assuming the signals $\mathbf{s}_m$ are equiprobable, the (bandpass) MAP/ML decision is

$$\hat{m} = \arg \max_{1 \le m \le M} \mathbf{r}^\top \mathbf{s}_m$$

Hence given $\mathbf{s}_1$ transmitted, we need for correct decision

$$\langle \mathbf{r}, \mathbf{s}_1 \rangle = \mathcal{E} + \sqrt{\mathcal{E}} n_1 \; > \langle \mathbf{r}, \mathbf{s}_m \rangle = \sqrt{\mathcal{E}} n_m, \quad 2 \le m \le M$$

It means

$$\Pr\{\text{Correct}|\mathbf{s}_1\} = \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\}$$

By symmetry, we have

$$\Pr\{\text{Correct}\} = \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\}$$

$$
\begin{aligned}
P_c &= \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M\right\} \\
&= \int_{-\infty}^{\infty} \Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2, \cdots, \sqrt{\mathcal{E}} + n_1 > n_M \middle| n_1\right\} f(n_1)\, dn_1
\end{aligned}
$$

Condition on n1, and average

$$
= \int_{-\infty}^{\infty} \left(\Pr\left\{\sqrt{\mathcal{E}} + n_1 > n_2 \middle| n_1\right\}\right)^{M-1} f(n_1)\, dn_1
$$

n2, n3, ... are all i.i.d.

$$
= \int_{-\infty}^{\infty} \left[Q\left(\frac{0 - (n_1 + \sqrt{\mathcal{E}})}{\sqrt{\frac{N_0}{2}}}\right)\right]^{M-1} f(n_1)\, dn_1
$$

$$
\Pr\left\{\mathcal{N}\left(m, \sigma^2\right) < r\right\} = Q\left(\frac{m-r}{\sigma}\right)
$$

$$
= \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}}\right)\right]^{M-1} f(n_1)\, dn_1
$$

Hence

$$
\begin{aligned}
P_e &= 1 - P_c \qquad \text{Prob of error = 1 - Prob. of correct decision} \\
&= 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} f(n_1)\, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q\left(\frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^{M-1} \right) \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_1^2}{N_0}}\, dn_1 \\
&= \int_{-\infty}^{\infty} \left(1 - \left[1 - Q(x) \right]^{M-1} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \sqrt{2k\gamma_b})^2}{2}}\, dx
\end{aligned}
$$

where $x = \frac{n_1 + \sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}}$, and $\gamma_b = \mathcal{E}_b / N_0$

Change of variable

Due to the complete symmetry of (binary) orthogonal signaling, the bit error rate P_b (see the red-color equation below) has a close-form formula.

$$\begin{cases} \Pr\{\hat{m} = i\} = P_c & \text{if } i = m \quad (e = \boxed{\text{0 bit error}}) \\ \Pr\{\hat{m} = i\} = \frac{P_e}{M-1} & \text{if } i \neq m \quad (e = \boxed{1 \sim k \text{ bits in error}}) \end{cases}$$

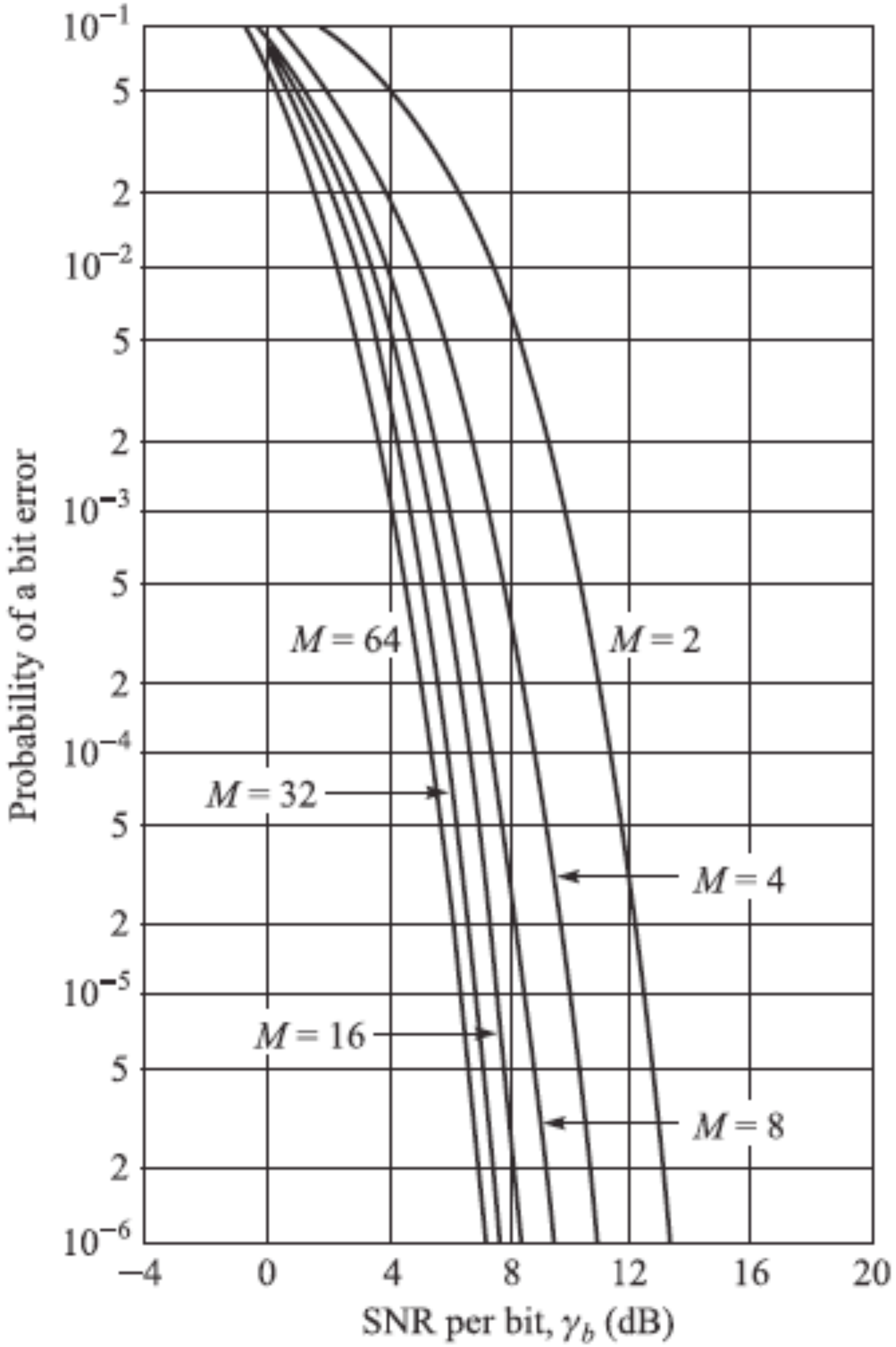
where $k = \log_2(M)$.

We then have

$$\begin{aligned} P_b &= \frac{E[e]}{k} = \frac{1}{k}\sum_{e=1}^{k} e \cdot \binom{k}{e}\frac{P_e}{M-1} \\ &= \frac{1}{2} \cdot \frac{2^k}{2^k - 1}P_e \approx \frac{1}{2}P_e \end{aligned}$$

(good approximation for large k)

Performance of Orthogonal Signaling



Different from PAM/PSK

The larger the M is,
the better the performance!!!

For example, to achieve $P_b = 10^{-5}$
one needs γ_b = 12 dB for M = 2;

but it only requires γ_b = 6 dB
for M = 64;

a 6 dB save in transmission power

Error Probability Upper Bound

Since P_b decreases with respect to M, is it possible that

$$\lim_{M\to\infty} P_e = \lim_{M\to\infty} P_b = 0?$$

Shannon limit of the AWGN channel:

1. If $\gamma_b > \log(2) \approx -1.6$ dB, then $\lim_{M\to\infty} P_e = 0$.
2. If $\gamma_b < \log(2) \approx -1.6$ dB, then $\inf_{M\geq 1} P_e > 0$.

Shannon's channel coding theorem

In 1948, Shannon proved that

- if $R < C$, then P_e can be made arbitrarily small (by extending the code size);
- if $R > C$, then P_e is bounded away from zero,

where $C = \max_{P_X} I(X;Y)$ is the channel capacity, and R is the code rate.

- For AWGN channels,

$$C = W \log_2\left(1 + \frac{P}{N_0 W}\right) \quad \textit{bit/second}$$

Note that W is in Hz=1/second, N_0 is in Joule (so $N_0 W$ is in Joule/second=Watt), and P is in Watt.

Since P (Watt) = R (bit/second) $\times$ $\mathcal{E}_b$ (Joule/bit), we have

$$\boxed{R >} C = W \log_2\left(1 + \frac{R\mathcal{E}_b}{N_0 W}\right) = \boxed{W \log_2\left(1 + \frac{R}{W}\gamma_b\right)}$$

$$\Leftrightarrow \boxed{\gamma_b < \frac{2^{R/W}-1}{R/W}}$$

For M-ary (orthogonal) FSK, $W = \frac{M}{2T}$ and $R = \frac{\log_2(M)}{T}$

Hence, $R/W = \frac{2\log_2(M)}{M} = \frac{2k}{2^k}$

This gives that

$$\boxed{\text{If } \gamma_b < \lim_{k\to\infty} \frac{2^{2k/2^k}-1}{2k/2^k} = \log(2), \text{ then } P_e \text{ is bounded away from zero.}}$$

Conclusions

•We have seen the probability of error calculations and performance of bandlimited and power limited schemes

• Bandlimited schemes (such as ASK, PSK, QAM) need more power (or energy per bit) to guarantee the same error probability as M (rate) increases

• For same energy per bit, the performance of B-L schemes degrades as M increases

• On the contrary, for power limited (orthogonal) signaling schemes, the performance improves as M increases, but this comes at the cost of higher bandwidth