ECE 5654 Lecture 8
Union Bound and Probability of Error

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Feb 19, 2014

Learning Objectives

At the end of this lecture, the student should be able to:

- Describe the functional blocks for implementing the optimal receiver in an AWGN channel
- Union bound for probability of error in AWGN.
- Probability of error for various modulation schemes.
- Power vs Bandwidth tradeoffs.

MAP Detector for AWGN

- MAP for AWGN:

$$g_{MAP}(r) = \underset{1\leq m\leq M}{argmax} \; \frac{N_0}{2}\log(p_m) - \frac{||r - s_m||^2}{2}$$
$$= \underset{1\leq m\leq M}{argmax} \; \frac{N_0}{2}\log(p_m) - \left(\frac{||r||^2}{2} + \frac{||s_m||^2}{2} - \frac{2r \cdot s_m}{2}\right)$$

- $||r||^2$ does not depend on m so it can be ignored

$$g_{MAP}(r) = \underset{1\leq m\leq M}{argmax} \; \frac{N_0}{2}\log(p_m) - \frac{||s_m||^2}{2} + r \cdot s_m$$
$$= \underset{1\leq m\leq M}{argmax} \; \underbrace{\frac{N_0}{2}\log(p_m) - \frac{E_m}{2}}_{\eta_m\text{: signal dependent bias}} + r \cdot s_m$$
$$= \underset{1\leq m\leq M}{argmax} \; \eta_m + r \cdot s_m$$

Decision regions for MAP

- MAP receiver:

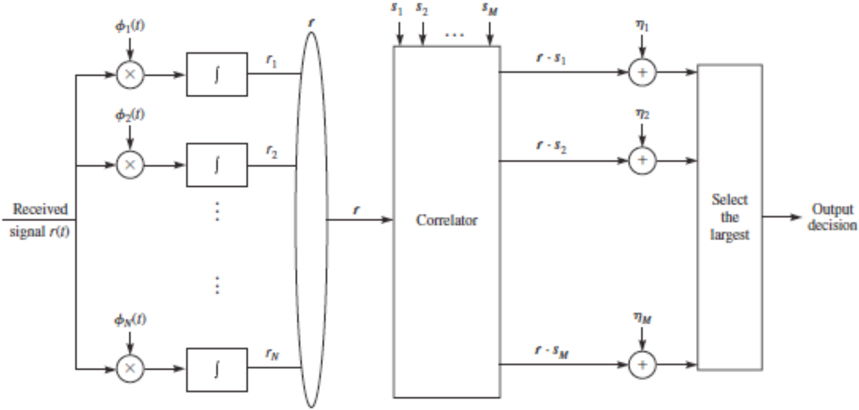
$$g_{MAP}(r) = \underset{1 \le m \le M}{argmax} \quad \eta_m + r \cdot s_m$$

- Signal dependent bias: $\eta_m = \frac{N_0}{2} \log(p_m) - \frac{E_m}{2}$
- Output dependent term: $r \cdot s_m = \int_{-\infty}^{\infty} r(t) s_m(t)\, dt$
- Decision region D_m:

$$D_m = \{r : \eta_m + r \cdot s_m > \eta_{m'} + r \cdot s_{m'}, \forall m^{'} \neq M\}$$

The Correlation Receiver

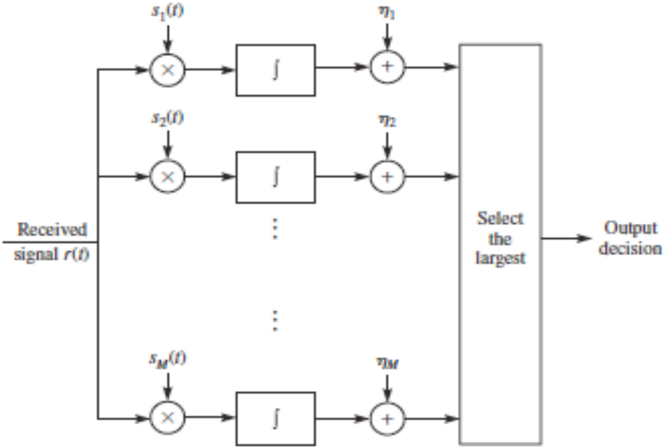
- For each m, the receiver needs to obtain $\eta_m + r \cdot s_m$
- Receiver observes $r(t)$
- First step is to obtain the vector r from the received signal $r(t)$
- $r_j = \int_{-\infty}^{\infty} r(t)\phi_j(t)\,dt$, $j = 1, 2, \ldots, N$
- $r = (r_1, r_2, \ldots r_N)$
- Using r and s_m, computes the inner product: $r \cdot s_m$
- Adds the bias term η_m to get $\eta_m + r \cdot s_m$
- Compare the results and choose m which maximizes this result.



N correlators

N: signal space dimensionality

Correlation Receiver Implementation 2



M correlators $\quad$ M: number of signals

Typically $N \ll M$ $\quad$ Implementation 1 is preferable over 2

The Matched Filter Receiver

- In both correlation receiver implementations, the receiver computes

$$r_x = \int_{-\infty}^{\infty} r(t)x(t)\, dt$$

  where $x(t)$ could either be
  - Implementation 1: $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$
  - Implementation 2: $s_1(t), s_2(t), \ldots, s_M(t)$
- Matched filter: if we define $h(t) = x(T - t)$ and consider a filter with impulse response $h(t)$, this filter is called a filter matched to $x(t)$, or a matched filter.

Matched Filter

- If input $r(t)$ is applied to the matched filter $h(t) = x(T - t)$, then its output is

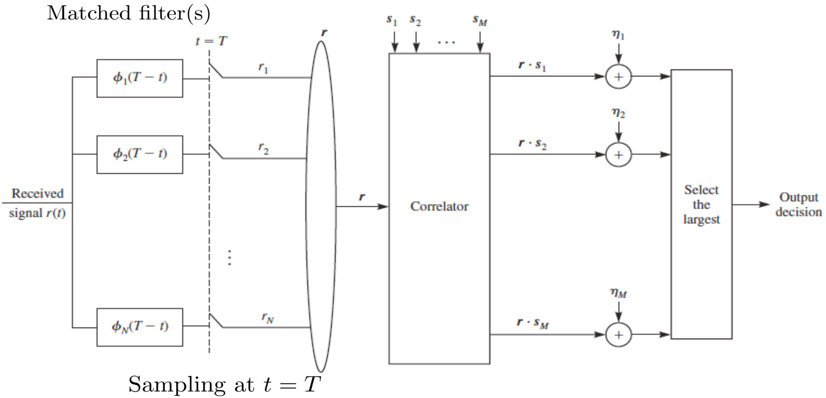
$$
\begin{aligned}
y(t) = r(t) * h(t) &= \int_{-\infty}^{\infty} r(\tau)h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} r(\tau)x(T - t + \tau)\, d\tau
\end{aligned}
$$

- If we sample the output of the matched filter at $t = T$, we obtain

$$
r_x = y(T) = \int_{-\infty}^{\infty} r(\tau)x(\tau)\, d\tau
$$

- Hence, the correlator r_x can be obtained by sampling the output of the matched filter at time $t = T$.
- T could be arbitrary but the sampling must be done exactly at $t = T$, where T is the value used in the design of the matched filter.

Matched Filter Receiver



From a practical perspective, T must be chosen such that

- the resulting filters are causal, i.e., $h(t) = 0$ for $t < 0$.
- this puts a practical limit on possible values of T

Frequency Domain Interpretation of Matched Filter

- MF of a signal $s(t)$ is $h(t) = s(T - t)$
- Fourier transform of $h(t) = H(f) = S^*(f)e^{-j2\pi fT}$
- Magnitude remains the same, $|S(f)| = |H(f)|$
- Phase of $H(f) = -$Phase of $S(f)$ $-2\pi fT$
- Represents the sampling delay of T

Matched Filter maximizes SNR

- We will now show that MF has an interesting property that it maximizes the signal-to-noise ratio (SNR) at the receiver.
- Suppose that $r(t) = s(t) + n(t)$ (the received signal) is passed through a filter with impulse response $h(t)$
- The output is

$$\begin{aligned} y(t) = r(t) * h(t) &= [s(t) + n(t)] * h(t) \\ &= s(t) * h(t) + n(t) * h(t) \\ &= y_s(t) + \nu(t) \end{aligned}$$

- Output has two parts:
  - Signal part: $y_s(t)$, with Fourier transform $H(f)S(f)$
  - Noise part: $\nu(t)$, with PSD $\frac{N_0|H(f)|^2}{2}$
- We sample the output at time $t = T$

MF maximizes SNR

- Sampling these components at $t = T$ results in: (book has a Typo)

$$y_s(T) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi fT}\, df$$

and a zero-mean Gaussian noise component $\nu(T)$, which has the variance:

$$Var[\nu(T)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df = \frac{N_0}{2} \cdot E_h$$

- E_h is the energy in $h(t)$.
- Let us define the SNR at the filter output as:

$$SNR = \frac{y_s^2(T)}{Var[\nu(T)]}$$

MF maximizes SNR

- Using Cauchy-Schwartz inequality:

$$
\begin{aligned}
y_s^2(T) &= \left(\int_{-\infty}^{\infty} H(f) S(f) e^{j2\pi f T}\, df \right)^2 \\
&\leq \int_{-\infty}^{\infty} |H(f)|^2\, df \cdot \int_{-\infty}^{\infty} |S(f) e^{j2\pi f T}|^2\, df \\
&= E_h \cdot E_s
\end{aligned}
$$

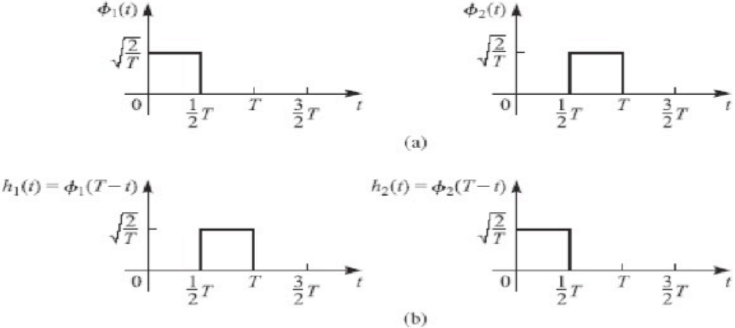
  which holds with equality if and only if $H(f) = \alpha S^*(f) e^{-j2\pi f T}$ for some complex constant α.

- So we can upper bound the SNR as

$$
SNR = \frac{y_s^2(T)}{Var[\nu(T)]} \leq \frac{E_h E_s}{\frac{N_0}{2} E_h} = \frac{2E_s}{N_0}
$$

- This shows that MF achieves this upper bound and maximizes SNR.

Example 1 Biorthogonal signaling

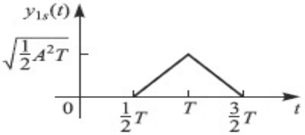


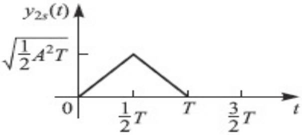
- Biorthogonal signaling
- $M = 4$ signals
- $s_1(t) = A\sqrt{T/2}\phi_1(t) = [A\sqrt{T/2} \quad 0]$; $s_3(t) = -s_1(t)$
- $s_2(t) = A\sqrt{T/2}\phi_2(t) = [0 \quad A\sqrt{T/2}]$; $s_4(t) = -s_2(t)$

Example 1

- We have two matched filters, one for each basis function.
- Suppose $s_1(t)$ is transmitted, then we can find the (noise-free) responses of the two matched filters.

$$y_{1s}(t) = \int_{-\infty}^{\infty} h_1(\tau)s_1(t-\tau)\,d\tau, \quad y_{2s}(t) = \int_{-\infty}^{\infty} h_2(\tau)s_1(t-\tau)\,d\tau$$





- We sample these outputs at $t = T$;
  $y_1 s(T) = \sqrt{A^2T/2}$ and $y_{2s}(T) = 0$
- Hence, output SNR $= (A^2T/2)/(N_0/2) = \frac{A^2T}{N_0} = 2E_1/N_0$

Probability of Error (for ML Receiver)

- Recall that for equiprobable symbols $p_m = 1/M$,
- MAP reduces to ML and hence ML receiver is the optimal receiver.
- Error Probability for ML detection becomes:

$$
\begin{aligned}
P_e &= \frac{1}{M}\sum_{m=1}^{1} P_{e|m} \\
&= \frac{1}{M}\sum_{m=1}^{1} \sum_{1\le m' \le M, m' \neq m} \int_{D_{m'}} p(r|s_m)\, dr
\end{aligned}
$$

- For very specific constellations, the decision regions D_m's are regular enough that the integrals can be computed in closed form.
- Hence, it is useful to have upper bounds for the error probability.

Union Bound

- The union bound is the simplest and widely used bound which gives a good approximation (particularly) at high SNR.
- We will start with the union bound for a general communication channel (described by a probabilistic transformation) and then return to the AWGN channel as a special case.
- Let us look at the term $P_{e|m}$ and specifically the integral

$$\int_{D_{m'}} p(r|s_m)\, dr$$

- Let's take a closer look at the decision region $D_{m'}$

Union Bound

- The decision region $D_{m'}$ is defined (for ML) as

$$D_{m'} = \{r : p(r|s_{m'}) > p(r|s_k), 1 \leq k \leq M, k \neq m'\}$$

- Let's define a new region $D_{mm'}$ as

$$D_{mm'} = \{r : p(r|s_{m'}) > p(r|s_m)\}$$

- $D_{mm'}$ is the decision region for the symbol m' in a binary equiprobable system with two signals, m and m'
- Comparing $D_{m'}$ and $D_{mm'}$, it is clear that $D_{m'}$ is contained in the region $D_{mm'}$, i.e., $D_{m'} \subseteq D_{mm'}$. This implies that

$$\int_{D_{m'}} p(r|s_m)\, dr \leq \int_{D_{mm'}} p(r|s_m)\, dr$$

Union Bound Continued..

- So we have

$$\int_{D_{m'}} p(r|s_m)\, dr \leq \int_{D_{mm'}} p(r|s_m)\, dr$$

- The r.h.s. above is the error probability of a binary equiprobable system with signals s_m and $s_{m'}$, when the signal s_m is sent.
- If we denote this error probability as

$$P_{m \to m'} = \int_{D_{mm'}} p(r|s_m)\, dr$$

we can then upper bound the probability $P_{e|m}$ as

$$P_{e|m} = \sum_{m' \neq m} \int_{D_{m'}} p(r|s_m) \leq \sum_{m' \neq m} P_{m \to m'}$$

- Putting everything together, we have the union bound for a general communication channel

$$P_e = \frac{1}{M} \sum_{m=1}^{M} P_{e|m}$$
$$\leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} P_{m \to m'}$$

- In summary, union bound upper bounds the probability by the pairwise error probabilities, which are often easier to obtain.
- In previous lecture, for the AWGN channel, we obtained the error probability for a binary equiprobable signaling scheme was shown to be

$$P_{m \to m'} = P_b = Q\left(\sqrt{\frac{d_{mm'}^2}{2N_0}}\right)$$

Union Bound for AWGN

- For the AWGN channel, the union bound gives

$$P_e \leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} Q\left(\sqrt{\frac{d_{mm'}^2}{2N_0}}\right)$$

- $d_{mm'}$ is the distance between constellation points s_m and $s_{m'}$
- If the distance structure of the constellation has symmetries, this bound can be further simplified.
- The $Q(.)$ function can be upper bounded as $Q(x) \leq \frac{1}{2}e^{-x^2/2}$.
- Using this, we can further bound

$$P_e \leq \frac{1}{2M} \sum_{m=1}^{M} \sum_{m' \neq m} e^{-d_{mm'}^2/4N_0}$$

Impact of Minimum Distance

- Let us define d_{min}, the minimum distance of a constellation as

$$d_{min} = \min_{m,m',m\neq m'} ||s_m - s_{m'}||$$

- Then, as the $Q(.)$ function is decreasing,

$$Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}}\right) \leq Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$$

- Using this, the upper bound can be further relaxed to

$$P_e \leq \frac{1}{M}\sum_{m=1}^{M}\sum_{m'\neq m} Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}}\right) \leq \frac{1}{M}\sum_{m=1}^{M}\sum_{m'\neq m} Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$$

$$= (M-1)Q\left(\sqrt{\frac{d^2_{min}}{2N_0}}\right)$$

Lower bound on Error Probability

- Recall that

$$\begin{aligned} P_e &= \frac{1}{M}\sum_{m=1}^{M} P_{e|m} \\ &= \frac{1}{M}\sum_{m=1}^{M}\int_{D_m^c} p(r|s_m)\,dr \end{aligned}$$

where D_m^c the complementary decision region of message m.

- $D_m^c = \cup_{m' \neq m} D_{m'}$
- Hence $D_{m'}$ is contained in D_m^c for all $m^{'} \neq m$
- This implies that for all $m^{'} \neq m$

$$\int_{D_m^c} p(r|s_m)\,dr \geq \int_{D_{m'}} p(r|s_m)\,dr$$

Lower bound on Error Probability

- Recall that $P_e = \frac{1}{M}\sum_{m=1}^{M}\int_{D_m^c} p(r|s_m)\,dr$
- For all $m^{'} \neq m$

$$\int_{D_m^c} p(r|s_m)\,dr \geq \int_{D_{m^{'}}} p(r|s_m)\,dr$$

- The r.h.s can be interpreted as the probability of error of a binary signaling scheme with messages s_m and $s_{m^{'}}$ when s_m is sent.
- hence, we can now obtain a lower bound on the error probability as

$$\begin{aligned} P_e &\geq \frac{1}{M}\sum_{m=1}^{M}\int_{D_{m^{'}}} p(r|s_m)\,dr \\ &= \frac{1}{M}\sum_{m=1}^{M} Q\left(\sqrt{\frac{d^2_{mm^{'}}}{2N_0}}\right) \end{aligned}$$

- Note that this bound holds for all $m^{'} \neq m$ and can be optimized.

Lower bound on Error Probability

- The optimized lower bound on the error probability is then given as

$$P_e \geq \frac{1}{M} \sum_{m=1}^{M} \max_{m' \neq m} Q\left(\sqrt{\frac{d^2_{mm'}}{2N_0}}\right)$$

- As we saw before $Q(.)$ is a decreasing function, choosing m' that gives the maximum value is equivalent to finding such m' with minimum distance from m
- If we denote d^m_{min} as the distance from m to its nearest neighbor, then

$$P_e \geq \frac{1}{M} \sum_{m=1}^{M} Q\left(\frac{d^m_{min}}{\sqrt{2N_0}}\right)$$

Lower bound on Error Probability

- Now, note that

$$Q\left(\frac{d_{min}^m}{\sqrt{2N_0}}\right) \geq \begin{cases} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) & \text{if there exists at least one} \\ & \text{signal at distance } d_{min} \text{ from } m \\ 0 & \text{otherwise} \end{cases}$$

- Hence, if we denote N_{min} as the number of message points which are at a distance d_{min} from at least one other point in the constellation, then we have

$$P_e \geq \frac{N_{min}}{M} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

- In summary, we have

$$\frac{N_{min}}{M} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \leq P_e \leq (M-1) Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$

Conclusions

- Next lecture, we will look at optimal detection and error probability for band-limited signaling schemes (such as ASK, PSK and QAM).
- We will also do the same for Power-limited signaling schemes (such as orthogonal signaling and FSK).
- Bandwidth vs Power tradeoffs.