ECE 5654 Lecture 7
Optimum Receivers for AWGN Channels

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Learning Objectives

At the end of this lecture, the student should be able to:

- Describe the AWGN channel in both waveform and vector forms
- Formulate the optimal receiver for an AWGN channel
- Describe the functional blocks for implementing the optimal receiver in an AWGN channel

Impairments

- In previous lectures, we have seen various modulation schemes.
- Communication channels suffer from a variety of impairments:
  - Noise
  - Attenuation & distortion
  - Fading
  - Interference
  - ...
- Noise is one of the major impairments.
- We will try to understand the effect of noise on reliability and performance.

Waveform Channel Model

- AWGN Channel model:

$$r(t) = s_m(t) + n(t)$$

- $s_m(t)$: transmitted signal $(1 \leq m \leq M)$
- $n(t)$ is a sample waveform of zero-mean, WGN process; PSD $= N_0/2$
- $r(t)$ is the received waveform.
- Using $r(t)$, receiver must make an optimal decision as to which message m was sent.
- Optimal decision: a decision (or a rule) which minimizes

$$\text{Probability of error } = P_e = P[\hat{m} \neq m]$$

Why do we need to study the AWGN Model

- AWGN model (i.e., communication in AWGN) may seem limiting
- It is worthwhile to study the performance in AWGN as a warmup.
- Noise is the major impediment in communication channels.
- AWGN can model:
  - Thermal noise (from vibrations of atoms in conductors)
  - From Central limit theorem, summation of several r.p.'s converges to a Gaussian r.p.
  - Good model for satellite and deep space comms.
- Mathematically tractable
- Other connections (also known as a the worst case/ maximum entropy noise).

Waveform Model to a Vector Model

- The waveform model is described by

$$r(t) = s_m(t) + n(t)$$

- Recall that we can use the Gram-Schmidt procedure to obtain an orthonormal basis $\{\phi_j(t), 1 \leq j \leq N\}$ for the set of M signals $\{s_1(t), s_2(t), \ldots, s_M(t)\}$.
- The vector representation of the signals is then $\{s_1, s_2, \ldots, s_M\}$, where

$$s_m(t) = \sum_{i=1}^{N} s_{mj}\phi_j(t), \quad m = 1, 2, \ldots, M$$

- Can we represent $n(t)$ in terms of **this** orthonormal basis ?

Vector representation of Noise

- We decompose the noise process $n(t)$ into two components:
  - $n_1(t)$: part which can be expressed in terms of $\{\phi_j(t)\}$.
  - $n_2(t)$: part which cannot be expressed in terms of $\{\phi_j(t)\}$.
  - $n(t) = n_1(t) + n_2(t)$
- $n_1(t) = \sum_{j=1}^{N} n_j \phi_j(t), \quad n_j = \langle n(t), \phi_j(t) \rangle$
- n_j is the projection of the noise onto $\phi_j(t)$.
- n_j's are random variables.
- We now focus on these random variables if $n(t)$ is a WGN process (with PSD $N_0/2$),

Vector representation of AWGN

- $n_j = \langle n(t), \phi_j(t) \rangle = \int_{-\infty}^{\infty} n(t)\phi_j(t)\, dt$
- Mean of n_j: $E[n_j] = \int_{-\infty}^{\infty} E[n(t)]\phi_j(t)\, dt = 0$
- Covariance of n_i and n_j:

$$
\begin{aligned}
Cov(n_i, n_j) &= E[(n_i - 0)(n_j - 0)] \\
&= E\left[\int_{-\infty}^{\infty} n(t)\phi_i(t)\, dt \int_{-\infty}^{\infty} n(s)\phi_j(s)\, ds\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[n(t)n(s)]\phi_i(t)\phi_j(s)\, dt\, ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(t-s)\phi_i(t)\phi_j(s)\, dt\, ds \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} \phi_i(s)\phi_j(s)\, ds \quad = \begin{cases} N_0/2, & i = j \\ 0, & i \neq j \end{cases}
\end{aligned}
$$

Vector Representation continued

- Using all this, we first write

$$\begin{aligned} r(t) &= s_m(t) + n(t) \\ &= s_m(t) + n_1(t) + n_2(t) \\ &= \sum_{j=1}^{N} (s_{mj} + n_j)\phi_j(t) + n_2(t) \end{aligned}$$

- If we define $r_j = s_{mj} + n_j$, then

$$r(t) = \sum_{j=1}^{N} r_j \phi_j(t) + n_2(t), \quad r_j = \langle r(t), \phi_j(t) \rangle$$

Equivalence of Waveform and Vector channel models

- At any given time t, we can show that

$$Cov(n_j, n_2(t)) = E[n_j n_2(t)] = 0$$

  which implies that $n_2(t)$ is uncorrelated with all n_j's (and since they are jointly Gaussian), it is independent of $n_1(t)$.
- $n_2(t)$ is independent of $s_m(t)$
- $n_2(t)$ is also independent of $n_1(t)$
- Recall that $r(t) = \sum_{j=1}^{N} r_j \phi_j(t) + n_2(t)$
- Hence $\sum_j r_j \phi_j(t)$ and $n_2(t)$ are also independent.
- Thus, the second component $n_2(t)$ cannot provide any information about the transmitted signal.
- It has no effect in the detection process and can be ignored without sacrificing the optimality of the detection process.
- Hence, the waveform model $r(t) = s_m(t) + n(t)$ is equivalent to the vector model $r = s_m + n$ (sufficient statistic)

Optimum Signal Detection for a Vector Channel

- After projection onto an orthonormal basis, we have the vector channel,

$$r = s_m + n$$

- All vectors are N-dimensional real vectors.
- Message m is chosen with probability p_m, $m = 1, 2, \ldots, M$.
- For AWGN: i.i.d. noise components $n_j \sim \mathcal{N}(0, N_0/2)$
- PDF of the AWGN noise vector n:

$$p(n) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{n_1^2+n_2^2+\ldots+n_N^2}{N_0}} = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{||n||^2}{N_0}}$$

- PDF of AWGN noise vector depends on its norm

Signal Detection for General Vector Channels

- We will consider detection over a more general model.
- Abstract *noise* through a probabilistic transformation on the input
- Input : vector s_m
- Channel: $p(r|s_m)$ (may not be additive nor Gaussian)
- Output: vector r
- Goal: given a received vector r, make a decision $g(r)$ and detect the message m
- Decision rule: $g(r) : \mathbb{R}^N \to \{1, 2, \ldots, M\}$
- Suppose $g(r) = \hat{m}$ (receiver's estimate about the message m)
- $\mathrm{P}[\text{Correct decision}|r] = \mathrm{P}[\hat{m} \text{ is sent}|r]$

MAP Receiver for General Vector Channels

- P[Correct decision] $= \int_r \mathrm{P}[\hat{m} \text{ is sent}|r]p(r)\,dr$
- P[Correct decision] is maximized if for each r, P[$\hat{m}$ is sent$|r$] is maximized
- Optimum Decision rule: upon receiving r, decide in favor of message m for which $P[m|r]$ is maximum.
- This receiver is known as the Maximum a posteriori probability (MAP) receiver, and the decision rule is known as the MAP decision rule.

$$\begin{aligned}\hat{m} = g_{MAP}(r) &= \underset{1 \leq m \leq M}{argmax}\ P[m|r] \\ &= \underset{1 \leq m \leq M}{argmax}\ P[s_m|r]\end{aligned}$$

Maximum Likelihood (ML) Receiver

- Let's simplify the MAP receiver (using Baye's rule):

$$P[s_m|r] = \frac{P[s_m, r]}{P[r]} = \frac{P[s_m]P[r|s_m]}{P[r]} = \frac{p_m P[r|s_m]}{P[r]}$$

- Hence, MAP is equivalent to

$$g_{MAP}(r) = \underset{1 \leq m \leq M}{argmax}\ p_m P[r|s_m]$$

- This description of MAP makes much more (intuitive) sense as it is expressed in terms of the prior probabilities p_m and the probabilistic description of the channel $P[r|s_m]$ (both are known)
- If all messages are equiprobable (a priori), i.e., $p_m = 1/M$ for all M, then MAP reduces to Maximum Likelihood rule (or ML rule)

$$g_{ML}(r) = \underset{1 \leq m \leq M}{argmax}\ P[r|s_m]$$

- If p_m's are not uniform, (or not known), then ML is a popular choice (it is not optimal in general unless $p_m = 1/M$ for all m).

Decision Regions

- Any detector, (including MAP or ML) partitions the output space $\mathbb{R}^N$ into M regions: $D_1, D_2, \ldots, D_M$.
- These partitions are done such that if $r \in D_m$, then $\hat{m} = g(r) = m$
- The region D_m is called as the decision region for the message m
- Set of all received vectors (outputs) in region D_m are mapped into message m.
- If s_m is sent, an error occurs when the received vector r is not in the decision region D_m.

Error Probability

- For a receiver with decision regions $\{D_1, D_2, \ldots, D_M\}$, the symbol error probability can be obtained as

$$P_e = \sum_{m=1}^{M} p_m P[r \notin D_m | s_m \text{ is sent}] = \sum_{m=1}^{M} p_m P_{e|m}$$

where $P_{e|m}$ is the error prob. when message m is sent:

$$P_{e|m} = \int_{D_m^c} p(r|s_m)\, dr = \sum_{1 \leq m' \leq M, m' \neq m} \int_{D_{m'}} p(r|s_m)\, dr$$

- P_e: symbol error (or the message error) probability.
- P_b: bit error probability $P_b \leq P_e \leq kP_b$, which implies
- $P_e / \log_2(M) \leq P_b \leq P_e$
- We will later see how to obtain bounds on these probabilities..

Example 1

- Consider two equiprobable signal vectors: $s_1 = (0,0)$ and $s_2 = (1,1)$.
- The received vector is $r = s_m + n$, where $n = (n_1, n_2)$, the additive noise elements have exponential PDF (and are independent of each other).
- $p(n) = \begin{cases} e^{-n}, & n \geq 0 \\ 0, & otherwise \end{cases}$
- For equiprobable signaling, MAP reduces to ML.
- Write down ML rule: (board)
- Draw the decision region(s) for the detector: (board)
- Find the error probability (board)

MAP Detector for Vector AWGN Channel

- For the vector AWGN channel, we have

$$r = s_m + n$$

- MAP detector: $\hat{m} = \underset{1 \leq m \leq M}{argmax}\, p_m P[r|s_m]$
- Density $P(r|s_m)$

$$P(r|s_m) = P(r - s_m|s_m) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{||n||^2}{N_0}}$$
$$= \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{||r - s_m||^2}{N_0}}$$

- log(.) is an increasing function, so we can maximize $\log(p_m P[r|s_m])$
- Taking the log(.) of $p_m P[r|s_m]$, we get

$$\log\left(p_m P[r|s_m]\right) = \log(p_m) - \frac{||r - s_m||^2}{N_0} - \frac{N}{2}\log(\pi N_0)$$

MAP/ML for Vector AWGN Channel

- The term $-\frac{N}{2}\log(\pi N_0)$ is a constant and can be ignored

$$g_{MAP}(r) = \underset{1 \leq m \leq M}{argmax} \ \log(p_m) - \frac{||r - s_m||^2}{N_0}$$

- Multiply by a constant $N_0/2$:

$$g_{MAP}(r) = \underset{1 \leq m \leq M}{argmax} \ \frac{N_0}{2}\log(p_m) - \frac{||r - s_m||^2}{2}$$

- If $p_m = 1/M$ (i.e., equiprobable a priori messages), then MAP reduces to ML which can be further simplified to:

$$g_{ML}(r) = \underset{1 \leq m \leq M}{argmin} \ ||r - s_m||$$

- This detector is also known as the nearest-neighbor or minimum-distance detector.
- Upon receiving r, the receiver looks among all s_m to find the one which is closest to r using standard Euclidean distance.

Continuing with MAP

- MAP for AWGN:

$$g_{MAP}(r) = \underset{1\leq m\leq M}{argmax} \ \frac{N_0}{2}\log(p_m) - \frac{||r - s_m||^2}{2}$$

$$= \underset{1\leq m\leq M}{argmax} \ \frac{N_0}{2}\log(p_m) - \left(\frac{||r||^2}{2} + \frac{||s_m||^2}{2} - \frac{2r\cdot s_m}{2}\right)$$

- $||r||^2$ does not depend on m so it can be ignored

$$g_{MAP}(r) = \underset{1\leq m\leq M}{argmax} \ \frac{N_0}{2}\log(p_m) - \frac{||s_m||^2}{2} + r\cdot s_m$$

$$= \underset{1\leq m\leq M}{argmax} \ \underbrace{\frac{N_0}{2}\log(p_m) - \frac{E_m}{2}}_{\eta_m\text{: signal dependent bias}} + r\cdot s_m$$

$$= \underset{1\leq m\leq M}{argmax} \ \eta_m + r\cdot s_m$$

Decision regions for MAP

- MAP receiver:

$$g_{MAP}(r) = \underset{1 \leq m \leq M}{argmax} \quad \eta_m + r \cdot s_m$$

- Decision region D_m:

$$D_m = \{r : \eta_m + r \cdot s_m > \eta_{m'} + r \cdot s_{m'}, \forall m' \neq M\}$$

- Each region is described by (at most) $M - 1$ inequalities
- Each decision boundary is of the form $r \cdot (s_m - s_{m'}) > \eta_{m'} - \eta_m$ (these boundaries are hyperplanes)
- $r \cdot s_m = \int_{-\infty}^{\infty} r(t) s_m(t)\, dt$ and $E_m = \int_{-\infty}^{\infty} s_m^2(t)\, dt$

Example 2: Binary Antipodal Signaling

- Binary antipodal signaling: $s_1(t) = s(t)$ and $s_2(t) = -s(t)$
- (a priori) probabilities: $p_1 = p$ and $p_2 = (1-p)$
- Dimensionality: $N = 1$ (orthonormal basis only has one element)
- Vector space representation: $s_1 = \sqrt{E}$, and $s_2 = -\sqrt{E}$
- $\eta_1 = \frac{N_0}{2}\log(p) - \frac{E}{2}$; $\eta_2 = \frac{N_0}{2}\log(1-p) - \frac{E}{2}$
- Decision regions:
- $D_1 = \{r : r\sqrt{E} + \frac{N_0}{2}\log(p) - \frac{E}{2} > -r\sqrt{E} + \frac{N_0}{2}\log(1-p) - \frac{E}{2}\}$
- Simplifies to

$$D_1 = \left\{ r : r > \frac{N_0}{4\sqrt{E}} \log\left(\frac{1-p}{p}\right) \triangleq r_{th} \right\}$$

- If $r > r_{th}$, receiver selects $\hat{m} = 1$ otherwise selects $\hat{m} = 2$

Example 2: Binary Antipodal Signaling

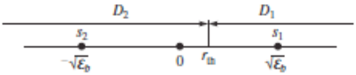


FIGURE 4.2–2
The decision regions for antipodal signaling.

- For $p = 1/2$, $r_{th} = 0$; MAP reduces to ML (nearest neighbor rule).
- Probability of symbol (or message error)

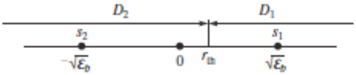
$$P_e = \sum_{m=1}^{2} P_m \sum_{m' \neq m} \int_{D'_m} p(r|s_m)\, dr$$

$$= p \cdot \int_{D_2} p(r|s = \sqrt{E})\, dr + (1-p) \cdot \int_{D_1} p(r|s = -\sqrt{E})\, dr$$

$$= p \cdot \int_{-\infty}^{r_{th}} p(r|s = \sqrt{E})\, dr + (1-p) \cdot \int_{r_{th}}^{\infty} p(r|s = -\sqrt{E})\, dr$$

$$= p \cdot P\left[\mathcal{N}(\sqrt{E}, N_0/2) < r_{th}\right] + (1-p) \cdot P\left[\mathcal{N}(-\sqrt{E}, N_0/2) > r_{th}\right]$$

Example 2: Binary Antipodal Signaling



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- For $p = 1/2$, $r_{th} = 0$; MAP reduces to ML (nearest neighbor rule).
- Probability of symbol (or message error)

$$P_e = p \cdot P\left[\mathcal{N}(\sqrt{E}, N_0/2) < r_{th}\right] + (1-p) \cdot P\left[\mathcal{N}(-\sqrt{E}, N_0/2) > r_{th}\right]$$

$$= p \cdot Q\left(\frac{\sqrt{E} - r_{th}}{\sqrt{N_0/2}}\right) + (1-p) \cdot Q\left(\frac{\sqrt{E} + r_{th}}{\sqrt{N_0/2}}\right)$$

- For $p = 1/2$, this simplifies to $Q\left(\sqrt{\frac{2E}{N_0}}\right)$
- What is the Probability of bit error ?

Equiprobable Binary Signaling

- Suppose the Transmitter sends either $s_1(t)$ or $s_2(t)$ over AWGN
- Equally likely signals, MAP reduces to ML (minimum distance rule)
- Decision region: separated by perpendicular bisector of line joining s_1 and s_2.
- Probability of error: (by symmetry)

$$
\begin{aligned}
P_e &= P(||r - s_1|| > ||r - s_2|| \quad |s_1 \text{ is sent}) \\
&= P(||n + s_1 - s_1|| > ||n + s_1 - s_2|| \quad |s_1 \text{ is sent}) \\
&= P(||n|| > ||n - (s_2 - s_1)|| \quad |s_1 \text{ is sent}) \\
&= P(||n|| > ||n - (s_2 - s_1)||) \\
&= P(||n||^2 > ||n - (s_2 - s_1)||^2) \\
&= P\left(\langle n, (s_2 - s_1)\rangle > \frac{d_{12}^2}{2}\right)
\end{aligned}
$$

Equiprobable Binary Signaling

- Probability of error:

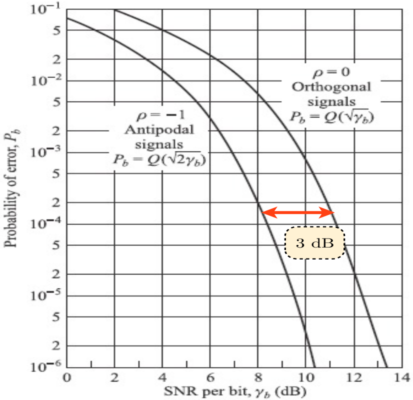
$$P_e = P\left(\langle n, (s_2 - s_1)\rangle > \frac{d_{12}^2}{2}\right) = P\left(\mathcal{N}\left(0, \frac{d_{12}^2 N_0}{2}\right) > \frac{d_{12}^2}{2}\right)$$
$$= Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

- $d_{12}^2 = \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 \, dt = E_1 + E_2 - 2\langle s_1(t), s_2(t)\rangle$
- For $E_1 = E_2 = E$, we have $d_{12}^2 = 2E(1-\rho)$
  ($\rho =$ cross correlation between signals)
- $P_e = Q\left(\sqrt{\frac{E(1-\rho)}{N_0}}\right)$ which is minimum for $\rho = -1$
- Antipodal signaling minimizes probability of error.
- Is this the unique signaling scheme which does so ?

Equiprobable Binary Signaling

- Probability of error: $P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$
- $d_{12}^2 = E_1 + E_2 - 2\langle s_1(t), s_2(t)\rangle$
- For $E_1 = E_2 = E$, we have $d_{12}^2 = 2E(1-\rho)$
  ($\rho=$ cross correlation between signals)
- This derivation is very general and applies to all binary equiprobable signaling schemes (regardless of the signal shapes).
- Orthogonal signaling: $s_1 = (\sqrt{E}, 0)$ $s_2 = (0, \sqrt{E})$; $d_{12}^2 = 2E$
- Orthogonal signaling: $P_e = Q\left(\sqrt{\frac{E}{N_0}}\right)$
- Antipodal signaling: $P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$
- Which scheme is more efficient ?

Error Probability Comparison



- Signal-to-noise ratio (SNR per bit): $\gamma = \frac{E}{N_0}$
- Orthogonal $P_e = Q(\sqrt{SNR})$ Antipodal: $P_e = Q(\sqrt{2SNR})$
- Antipodal signaling is power efficient.
  (lower error with same signal energy)

Conclusions

- We have described the AWGN channel in waveform and vector forms.
- Formulated the optimum receiver for an AWGN channel.
- Next lecture, we will study how to implement this receiver.
- We will also look at the performance of various linear modulation schemes in AWGN and the corresponding tradeoffs.
- Bandwidth vs Power tradeoffs.