Lecture 5:

Continuing with orthogonal Signaling (from Lecture 4)

<u>Hadamard Signals</u> — orthogonal signals from Hadamard matrices

— Hadamard matrices : $H_n : 2^n \times 2^n$ (size) $n = 1, 2, \ldots$

define H_n by the recursion $H_0 = [\,1\,]$

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

$$H_0 = [\,1\,] \; ; \quad H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \; ; \quad H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

— rows of H_n are orthogonal (inner product $= 0$)

Using Hadamard matrices, we can generate orthogonal signal

eg use $H_2 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \Rightarrow$

$$s_1 = (\sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}})$$
$$s_2 = (\sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}})$$
$$s_3 = (\sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}})$$
$$s_4 = (\sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}} \;\; -\sqrt{\mathcal{E}} \;\; \sqrt{\mathcal{E}})$$

& then these can be used to modulate any 4-dim orthonormal basis $\{\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)\}$

$$S_m(t) = \sum_{j=1}^{4} s_{mj}\, \phi_j(t) \; ; \quad m = 1, \ldots, 4.$$

Signaling Schemes with Memory

Differential - PSK (DPSK)

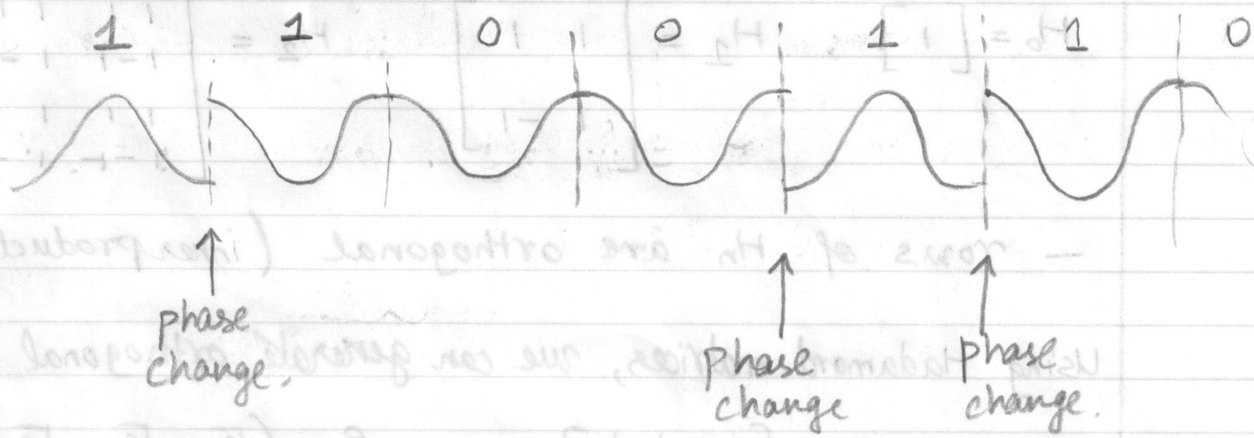
→ it conveys information by changing the phase of the carrier

Basic rule

0 ⟶ no change in phase

1 ⟶ Change phase by π

eg



→ this can be thought of as a modulation scheme with memory.

Continous - Phase FSK (CPFSK)

- in conventional FSK, we shift the carrier by $m\Delta f$
  $m = 1, 2, \ldots, M$.

Cons:
- switching from one frequency to other requires $M = 2^k$ oscillators tuned to desired freq.
- abrupt switching can lead to relatively large spectral lobes
  $\Rightarrow$ requires large Bandwidth.

— CPFSK - information bearing signal modulates a single carrier whose frequency is changed continuosly.
- phase-continuous & hence CPFSK.

We begin with a PAM signal:

$$d(t) = \sum_n I_n \underbrace{g(t - nT)}$$

rectangular pulse of amplitude $1/2T$; duration T

$I_n \in \{\pm 1 / \pm 3 \ldots / \pm(M-1)\}$

Using this,
Define a time-varying phase term:

$$\phi(t; I) = 4\pi T f_d \int_{-\infty}^{t} d(\tau)\, d\tau$$

— $d(t)$ is discontinuous
— $\int_{-\infty}^{t} d(\tau)\, d\tau$ is continuous.

then; the lowpass equivalent waveform is

$$v(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cdot e^{j[\phi_0 + \phi(t;I)]}$$

& the band pass signal is →

$$s(t) = \text{Real}\left[v(t)\, e^{j 2\pi f_c t} \right]$$

$$= \text{Real}\left[\sqrt{\frac{2\mathcal{E}}{T}}\, e^{j(\phi_0 + \phi(t;I))} \cdot e^{j 2\pi f_c t} \right]$$

$$s(t) = \sqrt{\frac{2\mathcal{E}}{T}} \cdot \cos\left[2\pi f_c t + \phi(t;I) + \phi_0 \right]$$

Let us find the phase of the carrier in the interval $nT \le t \le (n+1)T$.

$$\phi(t;I) = 4\pi T f_d \int_{-\infty}^{t} \left\{ \sum_n I_n \cdot g(\tau - nT) \right\} \cdot d\tau.$$

$$= 4\pi T f_d \cdot \left\{ \sum_{k \le n-1} I_k \int_{-\infty}^{t} g(\tau - kT)\, d\tau + I_n \cdot \int_{nT}^{t} g(\tau - nT)\, d\tau + 0 \right\}$$

$$= 4\pi T f_d \left\{ \frac{1}{2T} \times T \times \sum_{k=-\infty}^{n-1} I_k + I_n \cdot \frac{(t - nT)}{2T} \right\}$$

carrier
Phase in the interval

$nT \leq t \leq (n+1)T$

$$\Rightarrow \quad \phi(t; I) = \underbrace{2\pi f_d T \cdot \sum_{k=-\infty}^{n-1} I_k}_{\Downarrow} + 2\pi f_d (t - nT) \cdot I_n .$$

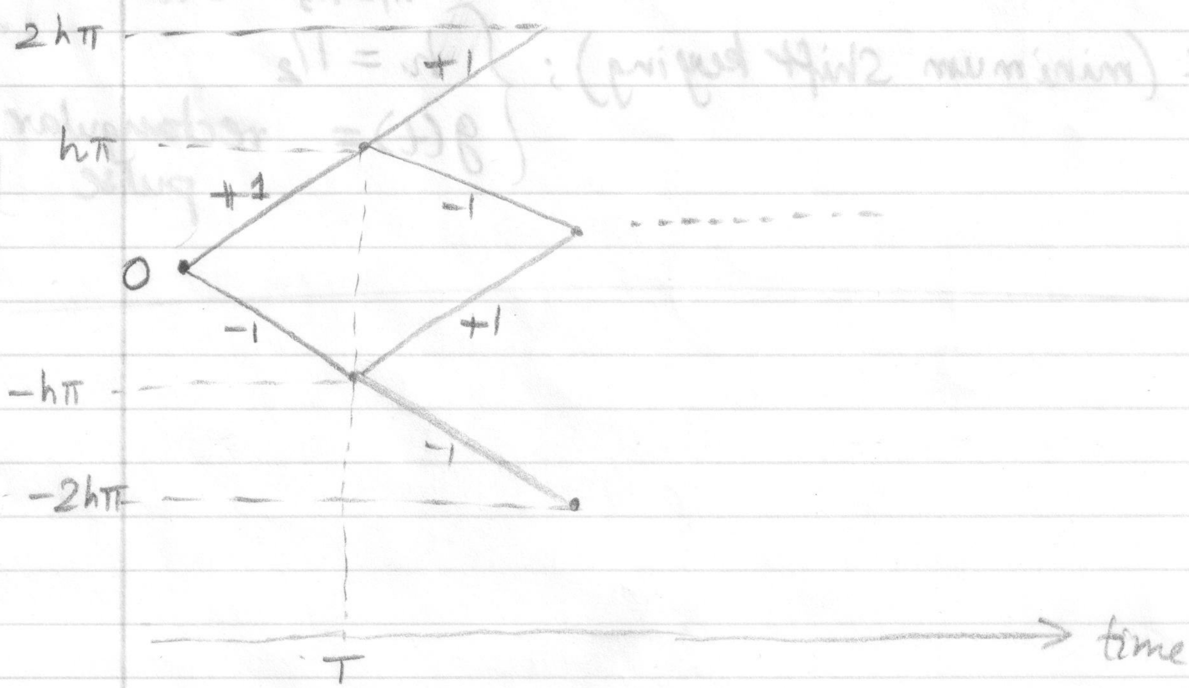
$h = 2 f_d T$ (modulation index)

$$= \theta_n + 2\pi f_d (t - nT) \cdot I_n$$

accumulation (or memory) of all symbols upto time $(n-1)T$.

Suppose $I_n = \pm 1$ { CP(B)FSK } or Binary CPFSK
(B ← binary.)

$\Rightarrow$ we can draw the phase trajectory (over time) $\rightarrow$ this is also called as a trellis diagram or Phase trellis.



Continuous Phase Modulation

→ a more general class than (CPFSK)

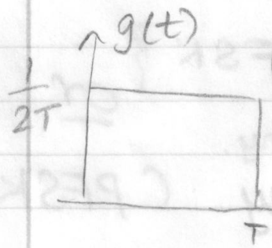
Carrier phase: $$\phi(t;I) = 2\pi \sum_{k=-\infty}^{n} I_k \cdot h_k \cdot q(t-kT)$$

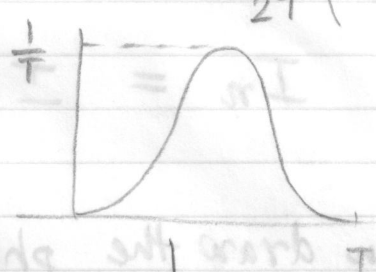
for $nT \le t \le (n+1)T$

$$q(t) = \int_0^t g(\tau)\,d\tau$$

(raised cosine pulse) $\frac{1}{2T}\left(1-\cos\left(\frac{2\pi t}{T}\right)\right)$

(rect. pulse)





smoother phase trellis

MSK (minimum shift keying): $\left\{ \begin{array}{l} h_1 = h_2 \cdots = h \\ h = 1/2 \\ g(t) = \text{rectangular pulse} \end{array} \right\}$

PSD of Linearly Modulated Signals

$$v_\ell(t) = \sum_{n=-\infty}^{\infty} I_n \, g(t - nT)$$

Lowpass equivalent

$\{I_n\}$ — stationary information sequence

$g(t)$ — modulation pulse.

Modulation Scheme	$\{I_n\}$
PAM	Real
PSK, QAM, combined PAM-PSK	Complex

$$R_{v_\ell}(t+\tau; t) = E[\, v_\ell(t+\tau)\, v_\ell^*(t)]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[I_n^* I_m] \cdot g^*(t-nT)\, g(t+\tau-mT)$$

$\{I_n\} \rightarrow$ stationary process with mean $= \mu$

$$R_I(m) = E[I_n^* I_{n+m}]$$

$$\Rightarrow \boxed{R_I(m-n) = E[I_n^* I_m]}$$

$$\Rightarrow R_{v_e}(t+\tau;t) = \sum_n \sum_m R_I(m-n) \cdot g^*(t-nT)\, g(t+\tau-mT)$$

Now change of variable: $m' = (m-n)$

$$= \sum_{m'} R_I(m') \cdot \sum_n g^*(t-nT)\, g(t+\tau-(m'+n)T)$$

Let $m' = m$ (relabling)

$$= \sum_m R_I(m) \sum_n g^*(t-nT)\, g(t+\tau-nT-mT)$$

Next, we observe that

$\boxed{g^*(t-nT)\, g(t+\tau-nT-mT)}$ is periodic with period $\underline{T}$.

$\Rightarrow$ R_{v_e} is also periodic with period T, i.e.

$$R_{v_e}(t+\tau;t) = R_{v_e}(t+T+\tau;t+T)$$

AF is periodic

$$E[v_e(t)] = E\{I_n\} \cdot \sum_{n=-\infty}^{\infty} g(t-nT)$$

$$= \mu \cdot \sum_{n=-\infty}^{\infty} g(t-nT)$$

mean is periodic

What kind of process is $v_e(t)$??

Detour

Cyclostationary Random Process (RP)

R.P. $X(t)$ is cyclostationary w. respect to time interval T if it is statistically indistinguishable from $X(t-kT)$ for any integer k.

Wide Sense Cyclostationary R.P.

2) mean & AF satisfy

$$m_X(t) = m_X(t-T) \quad \forall \, t$$

$$R_X(t_1, t_2) = R_X(t_1 - T, t_2 - T) \quad \forall \, (t_1, t_2).$$

PSD of a Cyclostationary Process (with period T)

Steps

- Calculate the autocorrelation func $R_X(t, t+\tau)$
- Average AF between 0 and T;

$$R_X(\tau) = \frac{1}{T} \int_0^T R_X(t, t+\tau) \, dt$$

- Calculate FT of averaged AF $R_X(\tau)$

Returning back;

$$R_{v_e}(t+\tau, t) = \sum_{m} R_I(m) \cdot \sum_{n=-\infty}^{\infty} g^*(t-nT)\, g(t+\tau-nT-mT)$$

⇒ Time Avg.

$$\overline{R}_{v_e}(\tau) = \frac{1}{T}\int_0^T R_{v_e}(t+\tau, t)\, dt$$

$$= \underset{\text{(Some manipulations)}}{\text{-----}} = \frac{1}{T}\sum_{m=-\infty}^{\infty} R_I(m) \cdot R_g(\tau - mT)$$

where, $R_g(\tau) = \int_{-\infty}^{\infty} g^*(t)\, g(t+\tau)\, dt$ (time AF of $g(t)$)

So, we have

$$\overline{R}_{v_e}(\tau) = \frac{1}{T}\sum_{m=-\infty}^{\infty} R_I(m)\, R_g(\tau - mT)$$

Taking F.T. $S_{v_e}(f) = \frac{1}{T}\, |G(f)|^2 \cdot S_I(f)$

$G(f)$: Fourier transform of $g(t)$

$S_I(f)$: PSD of information sequence.

$$S_I(f) = \sum_{m=-\infty}^{\infty} R_I(m) \cdot e^{-j2\pi f mT}$$

$$S_{v_e}(f) = \frac{1}{T} \underbrace{|G(f)|^2}_{\text{Shape of pulse.}} \underbrace{S_I(f)}_{\text{PSD of information sequence}}$$

2-main factors determine the Shape of PSD

<u>Example</u>: In a binary comm. system

$I_n = \pm 1$ with equal probability

& these are independent.

~~we~~ This stream modulates the pulse $g(t) = \Pi(t/T)$

& $v(t) = \sum_{k=-\infty}^{\infty} I_k \, g(t - kT)$.

$$\Rightarrow \underbrace{\text{PSD of } v(t)}_{\Downarrow} \equiv \frac{1}{T} |G(f)|^2 \cdot S_I(f).$$

$$S_v(f) = \frac{1}{T} \left| T \,\text{Sinc}(Tf) \right|^2 \cdot S_I(f)$$

$$S_I(f) = \sum_{k=-\infty}^{\infty} R_I(k) e^{-j2\pi k f T}$$

$$R_I(k) = E[I_{n+k} I_n^*] = \begin{cases} \overset{E[|I|^2]=1}{1} & k = 0 \\ E[I_{n+k}] \cdot \underset{0}{\underset{\shortparallel}{E[I_n^*]}} \cdot & k \neq 0 \end{cases}$$

$$\Rightarrow S_v(f) = T \,\text{Sinc}^2(Tf)$$