ECE 5654 Lecture 4
Digital Modulation Schemes

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Feb 3, 2014

Learning Objectives

At the end of this lecture, the student should be able to:

- Describe the main memoryless methods for digital modulation (such as ASK, PSK, QAM, FSK).
- Create the signal space representation (constellation diagram) for any memoryless modulation scheme and determine the minimum distance between signal points for memoryless modulation in terms of symbol or bit energy.

Classification of Digital Modulation Schemes

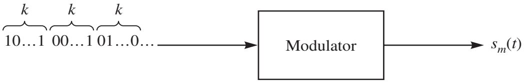


FIGURE 3.1–1
Block diagram of a memoryless digital modulation scheme.

- Digital data are usually in form of a binary stream (or bit stream).
- We need to map the bit stream to signals which match the characteristics of the communication channel.
- This mapping can be either memoryless or with memory.
  - Memoryless modulation
  - Modulation with memory.
- Besides memory, modulation schemes can also be classified as
  - Linear modulation schemes
  - Non-linear modulation schemes

Classification based on Memory

- Memoryless modulation
  - binary sequence is divided into subsequences of length k
  - each length k sequence is mapped to a signal $s_m(t)$, $1 \leq m \leq 2^k$
  - this mapping does not depend on the past (hence memoryless)
  - Modulation = mapping of 2^k messages to 2^k possible signals.
  - $M = 2^k$
- Modulation with memory
  - mapping is from current k-bits and past $(L-1)k$ bits to the set of $M = 2^k$ messages
  - signal sent for the current block of bits depends on the current k bits as well as the past $L-1$ blocks (each of k bits)
  - parameter L is called constraint length
  - $L = 1$ corresponds to memoryless modulation

Symbol rate, Bit rate

- Every block of k bits is called a symbol.
- Mapping of symbols (or k bits) to signal waveforms $s_m(t)$, $m = 1, 2, \ldots, M$, where $M = 2^k$
- Signaling interval: T_s
  (a signal is transmitted every T_s seconds)
- Signaling rate (or Symbol rate): $R_s = \frac{1}{T_s}$
  (# of symbols sent per second)
- Bit interval: $T_b = \frac{T_s}{k} = \frac{T_s}{\log_2 M}$
  (a bit is transmitted every T_b seconds)
- Bit rate: $R = kR_s = R_s \log_2(M)$
  (# of bits sent per second)

Energy & Power

- If energy of $s_m(t) = E_m$, then average signal energy is

$$E_{avg} = \sum_{m=1}^{M} p_m E_m$$

- p_m: probability of mth signal being sent.
- For $p_m = 1/M$ (equiprobable signals), $E_{avg} = \frac{1}{M}\sum_{m=1}^{M} E_m$
- Average energy per bit:

$$E_{bavg} = \frac{E_{avg}}{k} = \frac{E_{avg}}{\log_2 M}$$

- Average Power (sent by the transmitter):

$$P_{avg} = \frac{E_{bavg}}{T_b} = \underbrace{R}_{\text{bit rate}} \times \underbrace{E_{bavg}}_{\text{avg bit energy}}$$

Memoryless Modulation Schemes

- Linear modulation schemes:
  - Pulse Amplitude Modulation (PAM)
    (amplitude modulation)
  - Phase Shift Keying (PSK)
    (phase modulation)
  - Quadrature Amplitude Modulation (QAM)
    (amplitude & phase modulation)
- Non-linear modulation schemes:
  - Frequency Shift Keying (FSK)
    (frequency modulation)

Pulse Amplitude Modulation (PAM)

- PAM signal waveforms are represented as

$$s_m(t) = A_m p(t), \quad 1 \leq m \leq M$$

- $p(t)$: (real valued) pulse of duration T
- $\{A_m\}$: set of $M = 2^k$ amplitudes
- Usually, $A_m = 2m - 1 - M, \quad m = 1, 2, \ldots, M$,
  i.e., amplitudes are $\pm 1, \pm 3, \pm 5, \ldots, \pm (M-1)$
- Energy of mth signal: $E_m = A_m^2 E_p$
- Average signal energy: $E_{avg} = \frac{(M-1)^2 E_p}{3}$
- Average bit energy: $E_{bang} = \frac{(M-1)^2 E_p}{3 \log_2 M}$
- Carrier modulated bandpass PAM (ASK): $s_m(t) = A_m p(t) \cos(2\pi f_c t)$

Signal Space Diagram for PAM

- PAM signals are one-dimensional (all are multiples of the same signal)
- Basis for a PAM signal:

$$\phi(t) = \frac{p(t)}{\sqrt{E_p}}$$

- Signal representation in terms of basis:

$$s_m(t) = A_m \sqrt{E_p} \phi(t), m = 1, \ldots, M$$

- One-dimensional vector representation:

$$s_m = A_m \sqrt{E_p}, \quad A_m = \pm 1, \pm 3, \ldots, \pm (M-1)$$

PAM constellation

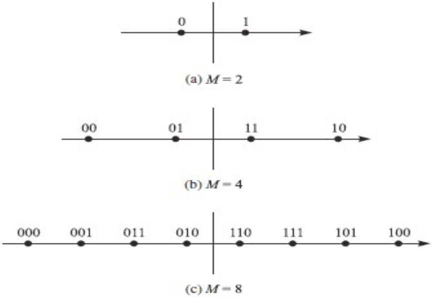


FIGURE 3.2–1
Constellation for PAM signaling.

- Mapping of $M = 2^k$ bits to M amplitudes.
- Gray mapping: adjacent bits differ by 1 binary digit.
- Distance between any pair of signal points:

$$d_{mn} = \sqrt{||s_m - s_n||^2} = |A_m - A_n|\sqrt{E_p}$$

- Minimum distance ($d_{\min}$): smallest d_{mn} over all possible (m, n) pairs

Phase Modulation

- In phase modulation, the M signal waveforms are represented as:

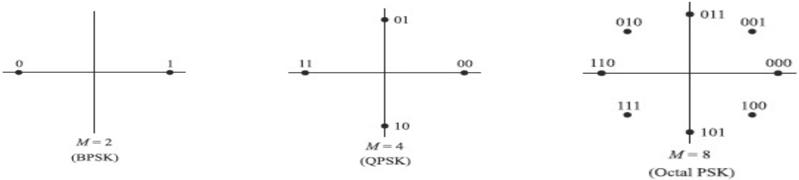
$$s_m(t) = g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right], \quad m = 1, 2, \ldots, M$$

- $g(t)$: signal pulse shape
- $\theta_m = \frac{2\pi(m-1)}{M}$ are the M possible phases of the carrier which convey the transmitted information. (PSK)
- All M waveforms have equal energy, hence $E_{avg} = E_m = \frac{E_g}{2}$.
- Basis Functions: (go to board)

$$\phi_1(t) = \sqrt{\frac{2}{E_g}}g(t)\cos(2\pi f_c t); \quad \phi_2(t) \quad = -\sqrt{\frac{2}{E_g}}g(t)\sin(2\pi f_c t)$$

- Signal space dimensionality $N = 2$ (does not depend on M)

PSK Vector Representation



- Using the basis functions, we can write the vector representations as

$$s_m = \left(\sqrt{\frac{E_g}{2}} \cos(\theta_m), \sqrt{\frac{E_g}{2}} \sin(\theta_m) \right), \quad m = 1, 2, \ldots, M$$

- $\theta_m = 2\pi(m-1)/M$
- Distance: $d_{mn} = \sqrt{||s_m - s_n||^2} = \sqrt{E_g \left[1 - cos(2\pi(m-n)/M)\right]}$
- Minimum distance $d_{\min} = \sqrt{E_g \left[1 - cos(2\pi/M)\right]}$ (when $|m-n| = 1$)

Quadrature Amplitude Modulation (QAM)

- Signal waveforms are expressed as

$$s_m(t) = r_m g(t)\cos(2\pi f_c t + \theta_m), \quad m = 1, \ldots, M$$

- modulation in both amplitude and phase.
- Can also be equivalently written as:

$$s_m(t) = A_{mi} g(t)\cos(2\pi f_c t) - A_{mq} g(t)\sin(2\pi f_c t), m = 1, \ldots, M$$

- Where, $r_m = \sqrt{A_{mi}^2 + A_{mq}^2}$, and $\theta_m = \tan^{-1}(A_{mq}/A_{mi})$.
- What are the basis functions for QAM signals ?
- What is the dimensionality of the QAM signal space ?

Orthonormal basis for QAM

- Similar to PSK, the orthonormal basis functions for QAM are:

$$\phi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t); \quad \phi_2(t) \quad = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t)$$

- Using these, we can write the signal $s_m(t)$ as

$$s_m(t) = A_{mi} \sqrt{\frac{E_g}{2}} \phi_1(t) + A_{mq} \sqrt{\frac{E_g}{2}} \phi_2(t)$$

- Therefore, the vector representation is

$$s_m = \left(A_{mi} \sqrt{\frac{E_g}{2}}, A_{mq} \sqrt{\frac{E_g}{2}} \right)$$

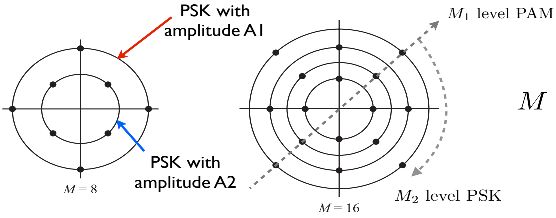
QAM continued

- Signal energy: $E_m = ||s_m||^2 = (E_g/2)(A_{mi}^2 + A_{mq}^2)$
- Distance between two QAM signals:

$$d_{mn} = \sqrt{||s_m - s_n||^2} = \sqrt{(E_g/2)[(A_{mi} - A_{ni})^2 + (A_{mq} - A_{nq})^2]} \quad (1)$$

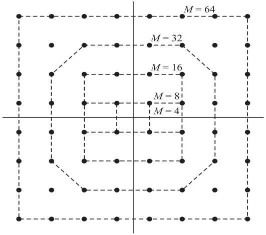
- Rectangular QAM: when signal amplitudes are $\{(2m - 1 - M), m = 1, 2, \ldots, M\}$, then the signal space diagram is rectangular.

Combined PAM-PSK



$$M = M_1 M_2$$

Rectangular QAM



Summary of PAM, PSK & QAM

- In general these signaling schemes can be written as:

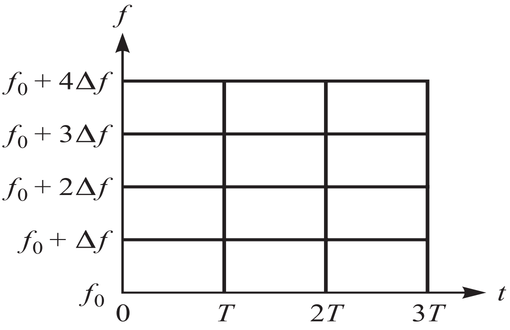
$$s_m(t) = \text{Real}\left[A_m g(t) e^{j2\pi f_c t}\right], \quad m = 1, 2, \ldots, M$$

  where A_m depends on the signaling scheme.
- Quiz: What is this representation ? and what is $A_m g(t)$?
- PAM: A_m is real, and usually $\pm 1, \pm 3, \ldots, \pm(M-1)$
- PSK: A_m is complex and equal to $e^{\frac{j2\pi(m-1)}{M}}$
- QAM: A_m is a general complex number $(A_{mi} + jA_{mq})$
- In all these schemes, the dimensionality of the signal space is low ($N = 1$ for PAM, and $N = 2$ for PSK, QAM) and independent of the constellation size M.

Multidimensional signaling

- Modulation of the carrier amplitude and phase allows us to construct signal waveforms that correspond to $N = 2$ dimensional vectors and signal space diagrams.
- We can also use the time domain, frequency domain or both time and frequency domains to increase the dimensionality.
- Suppose we have N-dimensional vectors.
- We subdivide a time interval of length T_1 into N intervals, $T = T_1/N$. In each sub-interval of length T, we can use PAM to transmit one element of the N-dimensional signal vector.
- Similarly, we can divide a frequency band of width $N\Delta f$ into N frequency slots, each of width Δf. We can then modulate the amplitude of N carriers, one in each of the N frequency slots.

Multidimensional signaling



- We can also use the time domain, frequency domain or both time and frequency domains to increase the dimensionality.

Orthogonal Signaling

- Definition: Orthogonal signals are a set of equal energy (E) signals $s_m(t)$, $m = 1, 2, \ldots, M$, such that $\langle s_m(t), s_n(t) \rangle = 0$.
- Orthogonal signals are linearly independent and hence $N = M$
- Orthonormal basis for representing $\{s_m(t)\}$:

$$\phi_m(t) = \frac{s_m(t)}{\sqrt{E}}, \quad m = 1, \ldots, M$$

- Vector representation:

$$\begin{aligned} s_1 &= \left(\sqrt{E}, 0, 0, \ldots, 0\right) \\ s_2 &= \left(0, \sqrt{E}, 0, \ldots, 0\right) \\ \vdots &= \vdots \\ s_M &= \left(0, 0, \ldots, 0, \sqrt{E}\right) \end{aligned}$$

- $d_{mn} = \sqrt{2E}$ (for $m \neq n$) and hence $d_{\min} = \sqrt{2E}$

Frequency-Shift Keying (FSK)

- FSK falls in the class of orthogonal signaling, in which the orthogonal signal waveforms differ in frequency.
- FSK waveforms are represented as

$$s_m(t) = \sqrt{\frac{2E}{T}} \cos(2\pi(f_c + m\Delta f)t), \quad m = 1, 2, \ldots, M$$

- Coefficient $\sqrt{\frac{2E}{T}}$ guarantees that signal energy is E.
- We can also write $s_m(t) = \text{Real}\left[s_{ml}(t)e^{j2\pi f_c t}\right]$, where

$$s_{ml}(t) = \sqrt{\frac{2E}{T}} e^{j2\pi m\Delta f t}$$

$s_{ml}(t)$ is the lowpass equivalent of $s_m(t)$.

Orthogonality of FSK signals

- For this set of signals to be orthogonal, we must have $\text{Real}[\langle s_{ml}(t), s_{nl}(t)\rangle] = 0$ for all $m \neq n$
- This inner product evaluates to:

$$\begin{aligned}\langle s_{ml}(t), s_{nl}(t)\rangle &= \frac{2E}{T}\int_0^T e^{j2\pi(m-n)\Delta f t}\,dt \\ &= \frac{2E}{j2\pi(m-n)\Delta f T}\left[e^{j2\pi(m-n)\Delta f T} - 1\right]\end{aligned}$$

and the real part is

$$\begin{aligned}\text{Real}[\langle s_{ml}(t), s_{nl}(t)\rangle] &= \frac{2E\sin(2\pi(m-n)\Delta f T)}{2\pi(m-n)\Delta f T} \\ &= 2E\cdot \operatorname{sinc}\left(2(m-n)\Delta f T\right)\end{aligned}$$

- Hence signals are orthogonal if $\Delta f = k/2T$, for some integer k.
- $\Delta f = 1/2T$ is the minimum frequency separation to guarantee orthogonality.

Linear vs. Non-linear modulation

- Recall that for PAM, PSK, QAM modulation, the lowpass equivalent signal is of the form $A_m g(t)$.
- Sum of two lowpass QAM signals is the lowpass equivalent of another QAM signal.
- In this sense, sum of two QAM signals is another QAM signal.
- Thus, these schemes fall in the category of linear modulation schemes.
- Does FSK satisfy this property ?
- No, hence FSK is a non-linear modulation scheme.

Lecture Summary

- In this lecture, we have studied various memoryless modulation schemes, such as PAM, PSK, QAM and FSK.
- Next lecture, we will focus on non-linear modulation schemes and also study the Power spectrum of digitally modulated signals.