ECE 5654 Lecture 3
Representation of Digital Modulation

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Learning Objectives

At the end of this lecture, the student should be able to:

- Describe signal space representation of digital modulation
- Explain the major characteristics of a digital modulation scheme and what information the signal space representation tells us about these characteristics

Modulation Principles

- Almost all communication systems transmit digital data using a sinusoidal carrier waveform.
  - Electromagnetic signals propagate well
  - Choice of carrier frequency allows placement of signal in arbitrary part of spectrum
- Physical system implements modulation by
  - Processing digital information at baseband
  - Pulse shaping and filtering a digital waveform
  - Mixing the baseband signal with a carrier signal from an oscillator
  - Filtering and amplifying the RF signal and finally coupling it with an antenna

Conveying Digital Information

- We can modify amplitude, phase or frequency
- Amplitude Shift Keying (ASK) or On/Off Keying

$$
\begin{aligned}
1 &\Rightarrow A\cos(2\pi f_c t) \\
0 &\Rightarrow 0
\end{aligned}
$$

- Frequency Shift Keying (FSK):

$$
\begin{aligned}
1 &\Rightarrow A\cos(2\pi f_1 t) \\
0 &\Rightarrow A\sin(2\pi f_0 t)
\end{aligned}
$$

- Phase Shift Keying (PSK):

$$
\begin{aligned}
1 &\Rightarrow A\cos(2\pi f_c t + \theta_1) \\
0 &\Rightarrow A\cos(2\pi f_c t + \theta_0)
\end{aligned}
$$

Bandpass and Lowpass Signal Representation

- In most cases, information signal is a low frequency (baseband) signal.
- Available spectrum of communication channel is at higher frequencies.
- Up-conversion: convert baseband to bandpass.
- Any real, narrowband, high frequency signal (bandpass) - can be represented in terms of a complex, low frequency signal (lowpass equivalent).
  - Simplifies handling of bandpass signals.
  - Lower required sampling rates.

Properties of Fourier Transform (FT)

- FT provides information about frequency content (spectrum) of the signal.
- For a real signal $x(t)$, its Fourier Transform $X(f)$ satsifies
  - $X(-f) = X^*(f)$ [Hermitian symmetry]
  - Magnitude of $X(f)$ is even.
  - Phase of $X(f)$ is odd.
  - All information content is in positive (or negative) frequencies.
  - Positive spectrum: $X_+(f) = \begin{cases} X(f) & f \geq 0 \\ 0 & else. \end{cases}$
- Bandwidth: smallest W s.t. $X_+(f) = 0$ for $|f| \geq W$
- Bandwidth for a complex signal $=$ half of its frequency support.
- Thus, in general, Bandwidth $=$ half of frequency support.

FT continued..

- Lowpass (or baseband) signal has its spectrum located around zero frequency.
- Spectrum of bandpass signal located around some $\pm f_0$.
- f_0 = central frequency or carrier frequency.
- For positive frequencies, $X_+(f)$ is non-zero in $[f_0 - W/2, f_0 + W/2]$
- $X_+(f)$ has all information content about $X(f)$ (and thus $x(t)$)
- In practice, usually $W \ll f_0$ (bandwidth $\ll$ carrier frequency)

Lowpass Equivalent of Bandpass Signals

Bandpass signals (signals with small bandwidth compared to carrier frequency) can be represented in any of three standard formats:

- Complex Envelope Notation
- Quadrature Notation
- Magnitude and Phase Notation

Pre-envelope of $x(t)$

We start with $X_+(f)$ and obtain its inverse Fourier Transform: $x_+(t)$

- $x_+(t)$: Analytic signal (or pre-envelope) of $x(t)$
- Fourier transform of $x_+(t) = X_+(f)$

$$\begin{aligned} x_+(t) = \mathcal{F}^{-1}[X_+(f)] &= \mathcal{F}^{-1}[X(f)u_{-1}(f)] \\ &= x(t) * \left(\frac{1}{2}\delta(t) + j\frac{1}{2\pi t}\right) \\ &= \frac{1}{2}x(t) + \frac{j}{2}\hat{x}(t) \end{aligned}$$

- $x_+(t)$ is in general complex.
- $\hat{x}(t) = \frac{1}{\pi t} * x(t)$: Hilbert Transform of $x(t)$.
- $\mathcal{F}[\hat{x}(t)] = -j\text{sgn}(f)X(f)$

Complex Envelope Notation

- $x_l(t)$: Lowpass equivalent (or complex envelope) is the signal whose FT is $2X_+(f + f_0)$
- We can obtain $x_l(t)$ in terms of $x(t)$ as

$$\begin{aligned} x_l(t) &= \mathcal{F}^{-1}[2X_+(f + f_0)] \\ &= 2x_+(t)e^{-j2\pi f_0 t} \\ &= [x(t) + j\hat{x}(t)]e^{-j2\pi f_0 t} \end{aligned}$$

- $x_l(t)$ is called as the lowpass equivalent or complex envelope of $x(t)$.
- $x(t) = \mathrm{Re}[x_l(t)e^{j2\pi f_0 t}]$

Quadrature Notation

- $x_l(t)$: Lowpass equivalent is the signal whose FT is $2X_+(f+f_0)$

$$\begin{aligned} x_l(t) &= [x(t) + j\hat{x}(t)]e^{-j2\pi f_0 t} \\ &= \underbrace{(x(t)\cos(2\pi f_0 t) + \hat{x}(t)\sin(2\pi f_0 t))}_{\text{in-phase component } = x_i(t)} \\ &\quad + j\underbrace{(\hat{x}(t)\cos(2\pi f_0 t) - x(t)\sin(2\pi f_0 t))}_{\text{quadrature component } = x_q(t)} \end{aligned}$$

Quadrature Notation

- In-phase component: real part of $x_l(t)$
- Quadrature component: imaginary part of $x_l(t)$

Quadrature Notation (continued)..

- Thus, we can write

$$
\begin{aligned}
x_i(t) &= x(t)\cos(2\pi f_0 t) + \hat{x}(t)\sin(2\pi f_0 t))\\
x_q(t) &= \hat{x}(t)\cos(2\pi f_0 t) - x(t)\sin(2\pi f_0 t))
\end{aligned}
$$

- Solving for $x(t)$ in terms of $x_i(t)$ and $x_q(t)$:

$$
\begin{aligned}
x(t) &= x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)\\
\hat{x}(t) &= x_q(t)\cos(2\pi f_0 t) + x_i(t)\sin(2\pi f_0 t)
\end{aligned}
$$

- Any bandpass signal $x(t)$ can be represented in terms of two lowpass signals, namely, its in-phase and quadrature components.

Magnitude/Phase (or Polar) Notation

- $x_l(t)$: Lowpass equivalent is the signal whose FT is $2X_+(f+f_0)$

$$
\begin{aligned}
x_l(t) &= [x(t) + j\hat{x}(t)]e^{-j2\pi f_0 t} \\
&= \underbrace{(x(t)\cos(2\pi f_0 t) + \hat{x}(t)\sin(2\pi f_0 t))}_{\text{in-phase component } = x_i(t)} \\
&\quad + j\underbrace{(\hat{x}(t)\cos(2\pi f_0 t) - x(t)\sin(2\pi f_0 t))}_{\text{quadrature component } = x_q(t)}
\end{aligned}
$$

Magnitude and Phase Notation

- Envelope: $r_x(t) = \sqrt{x_i^2(t) + x_q^2(t)}$ (magnitude of $x_l(t)$)
- Phase: $\theta_x(t) = \tan^{-1}(x_q(t)/x_i(t))$ (phase of $x_l(t)$)
- $x_l(t) = r_x(t)e^{j\theta_x(t)}$

Energy Considerations

- Energy of a signal is defined as

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- By Rayleigh's relationship,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- Note that $X_+(f)X_-(f) = 0$, hence

$$E_x = \int_{-\infty}^{\infty} |X_+(f) + X_-(f)|^2 df = 2\int_{-\infty}^{\infty} |X_+(f)|^2 df = 2E_{x+}$$

- Also, in comparison to the lowpass equivalent signal,

$$E_x = 2\int_{-\infty}^{\infty} |X_+(f)|^2 df = 2\int_{-\infty}^{\infty} \left|\frac{X_l(f)}{2}\right|^2 df = \frac{1}{2}E_{x_l}$$

Inner Product & Signal Correlation

- Inner product of two signals $x(t)$ and $y(t)$ is defined as:

$$< x(t), y(t) > = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df$$

- From this definition, it should be clear that $E_x =< x(t), x(t) >$.
- Cross-correlation coefficient of $x(t)$ and $y(t)$: $\rho_{x,y}$

$$\rho_{x,y} = \frac{< x(t), y(t) >}{\sqrt{E_x E_y}}$$

- If $x(t), y(t)$ are bandpass signals, then $\rho_{x,y} = \text{Re}(\rho_{x_l,y_l})$
- Two signals are orthogonal if $\rho = 0$.
- If $\rho_{x_l,y_l} = 0$, then $\rho_{x,y} = 0$.
- Orthogonality in baseband $\Rightarrow$ Orthogonality in pass band.
- Reverse is not true in general.

Eg. 1

Consider a real baseband signal $m(t)$ (with bandwidth W), and define

$$x(t) = m(t)\cos(2\pi f_0 t), \quad y(t) = m(t)\sin(2\pi f_0 t) \quad (f_0 > W)$$

- What are in-phase & quadrature components of $x(t), y(t)$?
- What are the lowpass equivalent signals of $x(t), y(t)$?
- Are $x(t), y(t)$ orthogonal ?
- Are (lowpass equivalent signals), i.e., $x_l(t), y_l(t)$ orthogonal ?

Lowpass Equivalent of a Bandpass System

- A bandpass system is a system for which the transfer function $H(f)$ is located around f_0 (and its mirror image $-f_0$).
- Equivalently, a bandpass system has an impulse response $h(t)$, which is a bandpass signal.
- Since $h(t)$ is bandpass, its lowpass equivalent is $h_l(t)$

$$h(t) = \text{Re}[h_l(t)e^{2\pi f_0 t}]$$

- If a bandpass signal $x(t)$ passes through a bandpass system (with impulse response $h(t)$), then the output $y(t) = x(t) * h(t)$ is also a bandpass signal, and $Y(f) = X(f)H(f)$.

- Lowpass equivalent of the output: (go to board)

$$
\begin{aligned}
Y_l(f) &= 2Y(f+f_0)u_{-1}(f+f_0) \\
&= \frac{1}{2}X_l(f)H_l(f)
\end{aligned}
$$

- In time domain, we have

$$
y_l(t) = \frac{1}{2}x_l(t) * h_l(t)
$$

- Note that the relationship between (input-outputs) is similar for both bandpass and baseband, except that a factor of $1/2$ is introduced for baseband.

Signal Space Representation of Waveforms

- An n-dimensional vector $\mathbf{v} = [v_1 \;\; v_2 \;\; \ldots \;\; v_n]^t$
- Inner product: $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \sum_{i=1}^{n} v_{1i} v_{2i}^* = \mathbf{v}_2^H \mathbf{v}_1$
- For a matrix A, A^H denotes Hermitian transpose, (first transpose then conjugating its elements)
- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle^*$
- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{v}_2, \mathbf{v}_1 \rangle^* = 2\mathrm{Re}[\langle \mathbf{v}_1, \mathbf{v}_2 \rangle]$
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal if $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$
- A set of m vectors is orthogonal if every pair of vectors are orthogonal
- Norm of a vector $\mathbf{v}$: $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + \ldots + v_n^2}$

Orthonormal basis

- A set of m vectors is orthonormal if they are orthogonal and each vector has a unit norm
- A vector can be represented as a linear combination of orthogonal unit vectors (or orthonormal basis)

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

  where $\mathbf{e}_i$'s are orthogonal and each $\mathbf{e}_i$ has norm $= 1$
- $v_i = \langle \mathbf{v}, \mathbf{e}_i \rangle$ is the projection of $\mathbf{v}$ onto the unit vector $\mathbf{e}_i$
- Triangle inequality: $||\mathbf{v}_1 + \mathbf{v}_2|| \leq ||\mathbf{v}_1|| + ||\mathbf{v}_2||$
- Cauchy-Schwarz Inequality $|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle| \leq ||\mathbf{v}_1|| \cdot ||\mathbf{v}_2||$

Gram Schmidt Procedure

- We are given a set of n-dimensional vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$
- Gram-Schmidt procedure gives an orthonormal basis.
- Start by the first vector v_1:

$$u_1 = \frac{v_1}{||v_1||}$$

- Subtract the projection of v_2 onto u_1:

$$u_2^{'} = v_2 - (\langle v_2, u_1 \rangle)u_1; \quad u_2 = \frac{u_2^{'}}{||u_2^{'}||}$$

- Subtract the projections of v_3 onto u_1 and u_2:

$$u_3^{'} = v_3 - (\langle v_3, u_1 \rangle)u_1 - (\langle v_3, u_2 \rangle)u_2; \quad u_3 = \frac{u_3^{'}}{||u_3^{'}||}$$

- This leads to N orthonormal vectors, $N \leq \min(m, n)$.

Signal Space Concepts

- A parallel treatment can also be done for a set of signals.
- Inner product of two signals:

$$\langle x_1(t)x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

- Two signals are orthogonal if inner product is zero.
- Norm of a signal:

$$||x(t)|| = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2(t)dt} = \sqrt{E_x}$$

- A set of m signals are orthonormal if they are orthogonal & with unit norms.

Orthogonal Expansion of Signals

- We will show an equivalence between a signal waveform and its vector representation.
- Suppose that $s(t)$ is a deterministic signal with finite energy $E_s = \int_{-\infty}^{\infty} |s(t)|^2(t)dt$
- Also, suppose that there exists a set of orthonormal functions $\phi_k(t); k = 1, \ldots, K$, i.e.,

$$\langle \phi_n(t), \phi_m(t) \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

- We can approximate $s(t)$ by a weighted linear combination of these functions:

$$\hat{s}(t) = s_1\phi_1(t) + \ldots + s_K\phi_K(t)$$

- Approximation error: $e(t) = s(t) - \hat{s}(t)$

Orthogonal Expansion of Signals...

- Desirable: good approximation must have small error.
- We can optimize $\{s_1, \ldots, s_K\}$ to minimize the Energy of error signal.

$$E_e = \int_{-\infty}^{\infty} |s(t) - \hat{s}(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right|^2 \, dt$$

- Note that the Energy of error signal is the Mean Squared Error (MSE) & the goal is to find Minimum-MSE (or MMSE).
- There are two ways to do this optimization:
  - Method 1: Brute force, differentiate w.r.t. s_k and find optimum values.
  - Method 2: For MMSE estimator, error signal must be orthogonal to each function $\phi_n(t)$.

MMSE approach for Approximation Error Minimization

- Orthogonality of error signal to $\phi_n(t)$ implies that:

$$\int_{-\infty}^{\infty} \left| s(t) - \sum_{k=1}^{K} s_k \phi_k(t) \right| \phi_n^*(t)\, dt = 0, \quad n = 1, 2, \ldots, K$$

- Recall that $\{\phi_n(t)\}_{n=1}^{K}$ are orthonormal signals.
- Thus, we arrive at the solution for $\{s_1, s_2, \ldots, s_K\}$

$$s_n = \langle s(t), \phi_n(t) \rangle = \int_{-\infty}^{\infty} s(t)\phi_n^*(t)\, dt, \quad n = 1, 2, \ldots, K$$

- $\Rightarrow$ coefficients are projections of the signal $s(t)$ onto each of $\{\phi_n(t)\}$
- Resulting MMSE: (go to board)

$$\text{MMSE} = E_{\min} = \int_{-\infty}^{\infty} e(t)s^*(t)\, dt = E_s - \sum_{k=1}^{K} \left| s_k^2 \right|$$

Complete Set of Orthonormal Functions

$$\text{MMSE} = E_{\min} = E_s - \sum_{k=1}^{K} \left| s_k^2 \right|$$

- MMSE is zero (or $E_{\min} = 0$) when $E_s = \sum_{k=1}^{K} \left| s_k^2 \right|$
- In this case, we can express $s(t)$ as

$$s(t) = \sum_{k=1}^{K} s_k \phi_k(t)$$

  where the " $=$ " (equality) is in the sense that approximation error has zero energy.
- When every finite energy signal can be represented as a series expansion, then the set of orthonormal functions $\{\phi_n(t)\}$ is said to be complete.

Gram-Schmidt Procedure (for Signals)

- Given a set of finite energy signal waveforms $\{s_m, m = 1, \ldots, M\}$
- G-S procedure constructs a set of orthonormal waveforms.
- First orthonormal waveform: $\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$
- Second waveform: (compute projection of $s_2(t)$ onto $\phi_1(t)$)

$$c_{21} = \langle s_2(t), \phi_1(t) \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_1^*(t)\, dt$$

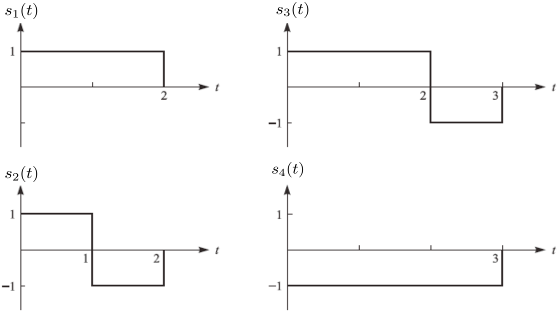
$$\gamma_2(t) = s_2(t) - c_{21}\phi_1(t); \quad \phi_2(t) = \frac{\gamma_2(t)}{\sqrt{E_{\gamma_2}}}$$

- nth waveform:

$$c_{ki} = \langle s_k(t), \phi_i(t) \rangle; \;\; \gamma_k(t) = s_k(t) - \sum_{i=0}^{k-1} c_{ki}\phi_i(t); \;\; \phi_k(t) = \frac{\gamma_k(t)}{\sqrt{E_{\gamma_k}}}$$

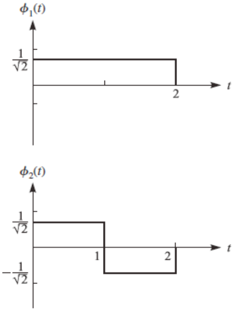
- Repeating this procedure gives $N \leq M$ orthonormal waveforms.

E.g. 2: Gram-Schmidt Procedure



- Let us apply the GS-procedure to above four waveforms.
- Go to board...

E.g. 2: Solution



$$\phi_1(t) = \frac{s_1(t)}{\sqrt{2}}, \quad \phi_3(t) = \begin{cases} -1 & 2 \leq t \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{2}}, \quad \phi_4(t) = 0$$

Signal Space Representation

- Once we have the set of orthonormal waveforms, we can then write

$$s_m(t) = \sum_{n=1}^{M} s_{mn}\phi_n(t), \quad m = 1, \ldots, M$$

- Each signal $s_m(t)$ can then be represented as a N-dimensional vector:

$$\mathbf{s}_m = [s_{m1}\ s_{m2} \cdots s_{mN}]^t$$

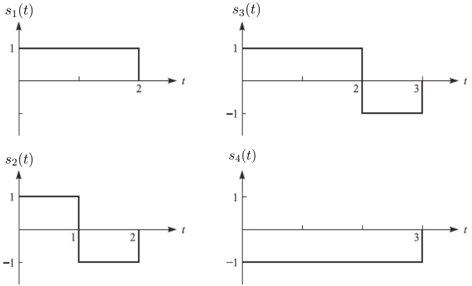
- Hence, a set of M signals $\{s_1(t), \ldots, s_M(t)\}$ can be represented by a set of N-dimensional vectors, namely $\{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$.
- This set of vectors is called the signal space representation or the constellation of the M signals $\{s_1(t), \ldots, s_M(t)\}$.

Signal Space Representation (continued)

- Energy preservation: $E_m = \int_{-\infty}^{\infty} |s_m(t)^2| \, dt = \sum_{n=1}^{N} |s_{mn}|^2 = ||\mathbf{s}_m||^2$
- Squared length of the vector $\mathbf{s}_m$ = Energy of the signal $s_m(t)$.
- Any signal can be represented geometrically as a point in the signal space spanned by the orthonormal functions $\{\phi_n(t)\}$.
- Inner product: $\langle s_k(t), s_l(t) \rangle = \langle \mathbf{s}_k, \mathbf{s}_l \rangle$
- Does the Gram-Schmidt procedure lead to unique orthonormal vectors ?
- Dimensionality of the signal space (i.e., N) remains the same.

Eg. 2: continued..

Signal Waveforms



Signal Space Representation

or Constellation

$$\mathbf{s}_1 = [1, 1, 0]^t$$
$$\mathbf{s}_2 = [1, -1, 0]^t$$
$$\mathbf{s}_3 = [1, 1, -1]^t$$
$$\mathbf{s}_4 = [-1, -1, -1]^t$$

Orthonormal Functions