ECE 5654 Lecture 11
DPSK, Spectral Efficiency

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Learning Objectives

At the end of this lecture, the student should be able to:

- Understand differential encoding and obtain probability error of Differential PSK
- Define spectral efficiency and compare various digital modulation schemes

Noncoherent detection

- For FSK with sufficiently spaces tones, demodulation can be accomplished with no phase information using a non-coherent receiver
- PSK and QAM signals represent information with the phase of the signal
- In many cases, even though it is not possible to detect the absolute phase of a signal, it is possible to detect the different in phase from one symbol to the next
- Thus a form of PSK can be implemented via Differential Encoding/Detection

Differential Encoding of Data

- Consider BPSK modulation:

$$0 \Rightarrow s(t) = g(t)\cos(2\pi f_c t)$$
$$1 \Rightarrow s(t) = g(t)\cos(2\pi f_c t + \pi) = -g(t)\cos(2\pi f_c t)$$

- Differential Encoding Transforms the raw data bits:
  - Raw data bits: $b_1, b_2, b_3, \ldots$
  - $d_i = b_i \oplus d_{i-1}$
  - $\oplus$ - modulo-2 operation
  - d_i is either 0 or 1
  - Use BPSK modulation on d_i

Example of Differential Encoding

b_i :	0	1	0	1	1	1	0
d_{i-1} :	0	0	1	1	0	1	0
d_i :	0	1	1	0	1	0	0
$Phase$:	0	π	π	0	π	0	0

- $d_i = b_i \oplus d_{i-1}$
- If $b_i = 1$, then change the phase ($d_i = \overline{d_{i-1}}$)
- If $b_i = 0$, then keep same phase ($d_i = d_{i-1}$)
- 0 is the initial condition; initial phase $= 0$

Does Initial Condition Matter ?

b_i :	0	1	0	1	1	1	0
d_{i-1} :	1	1	0	0	1	0	1
d_i :	1	0	0	1	0	1	1
$Phase$:	π	0	0	π	0	π	π

- Absolute values change
- But we still follow the same rule
  - If $b_i = 1$, then change the phase ($d_i = \overline{d_{i-1}}$)
  - If $b_i = 0$, then keep same phase ($d_i = d_{i-1}$)

Differential Decoding

- If we coherently detect the signal, we follow it with differential decoding
- $d_i = b_i \oplus d_{i-1}$
- Hence b_i can be obtained as $b_i = d_i \oplus d_{i-1}$

$$
\begin{array}{rcccccccc}
Rx\ Phase: & 0 & 0 & \pi & \pi & 0 & \pi & 0 & 0 \\
\hat{d_i}: & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\hat{d_{i-1}}: & & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hat{b_i}: & & 0 & 1 & 0 & 1 & 1 & 1 & 0
\end{array}
$$

QDPSK

- Recall, QPSK,
  - If input $= 00$ - phase 0
  - If input $= 01$ - phase $\pi/2$
  - If input $= 11$ - phase π
  - If input $= 10$ - phase $3\pi/2$
- Differentially encoded QPSK (or QDPSK)
  - If input 00 - shift the phase of previous symbol by 0
  - If input 01 - shift the phase of previous symbol by $\pi/2$
  - If input 11 - shift the phase of previous symbol by π
  - If input 10 - shift the phase of previous symbol by $3\pi/2$

- Over two symbols, the lowpass equivalent of the mth signal is

$$s_l^{(i-1)} = \sqrt{2E_s}e^{j\phi_0} \quad s_{ml}^{(i)} = \sqrt{2E_s}e^{j(\theta_m+\phi_0)}, \quad 1 \le m \le M$$

  where $\theta_m = 2\pi(m-1)/M$
- The received signal (over two symbols) is then:

$$\begin{aligned}\vec{r}_l = \begin{bmatrix} r_l^{(i-1)} \\ r_l^{(i)} \end{bmatrix} &= e^{j\phi}\begin{bmatrix} s_l^{(i-1)} \\ s_{ml}^{(i)} \end{bmatrix} + \begin{bmatrix} n_l^{(i-1)} \\ n_l^{(i)} \end{bmatrix} \\ &= e^{j\phi}\vec{s}_{m,l} + \vec{n}_l\end{aligned}$$

Envelope Detector

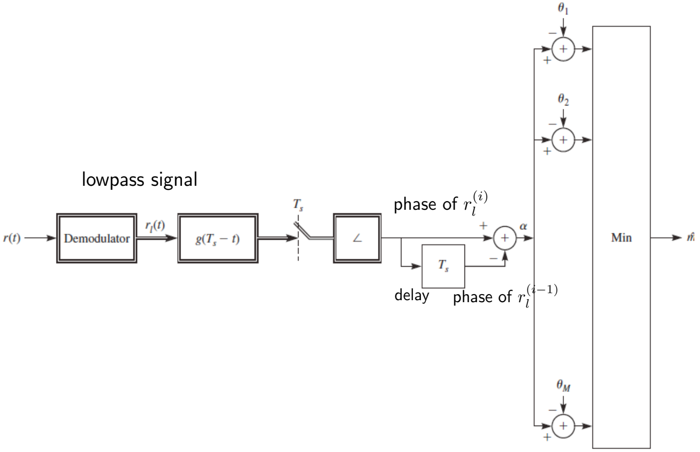
$$
\begin{aligned}
\hat{m} &= arg\max_m |\vec{r}_l \cdot \vec{s}_{ml}| \\
&= arg\max_m \sqrt{2E_s} \left| \begin{bmatrix} e^{-j\phi_0} & e^{-j(\theta_m+\phi_0)} \end{bmatrix} \begin{bmatrix} r^{(i-1)} \\ r^{(i)} \end{bmatrix} \right| \\
&= arg\max_m \left| r^{(i-1)} + r^{(i)} e^{-j\theta_m} \right| \\
&= arg\max_m \left| r^{(i-1)} + r^{(i)} e^{-j\theta_m} \right|^2 \\
&= arg\max_m \left(|r^{(i-1)}|^2 + |r^{(i)}|^2 + 2Re[(r^{(i-1)})^* r^{(i)} e^{-j\theta_m}] \right) \\
&= arg\max_m Re[(r^{(i-1)})^* r^{(i)} e^{-j\theta_m}] \\
&= arg\max_m |r^{(i-1)} r^i| \cos\left(\angle r^{(i)} - \angle r^{(i-1)} - \angle\theta_m \right) \\
&= arg\min_m \left(\angle r^{(i)} - \angle r^{(i-1)} - \angle\theta_m \right)
\end{aligned}
$$

Envelope Detector for DPSK

$$\hat{m} = arg \min_{m} \left(\angle r^{(i)} - \angle r^{(i-1)} - \angle \theta_m \right)$$

- $\alpha = \angle r^{(i)} - \angle r^{(i-1)}$ is the phase difference of the received signals in two adjacent intervals
- The receiver computes this difference and compares it with $\theta_m = \frac{2\pi}{M}(m-1)$ for all $1 \leq m \leq M$
- Selects the m for which θ_m is closest to α
- Receiver does not track ϕ (non-coherent detection)
- The key assumption here is that ϕ remains the same over two symbols.

DPSK Receiver



Probability of error for Binary DPSK

- In binary DPSK, the phase difference between adjacent symbols is either 0 or π (corresponding to bit 0 or bit 1)
- The corresponding lowpass equivalent signals (over a two symbol duration) are

$$\vec{s}_{1l} = \sqrt{2E_s} e^{j\phi_0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{s}_{2l} = \sqrt{2E_s} e^{j\phi_0} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- These signals are orthogonal
- Hence, we can use the probability of error calculation (see previous lecture)
- $P_e^{non-coherent} = \frac{1}{2} e^{-\frac{2E_s}{2N_0}} = \frac{1}{2} e^{-\frac{E_s}{N_0}}$

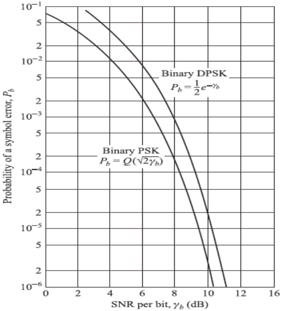


FIGURE 4.5–5
Probability of error for binary PSK and DPSK.

- $P_e^{BDPSK} = \frac{1}{2}e^{-\frac{2E_s}{2N_0}} = \frac{1}{2}e^{-\frac{E_s}{N_0}}$
- $P_e^{BPSK} = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$
- $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$
- $\Rightarrow Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \leq \frac{1}{2}e^{-\frac{E_s}{N_0}}$
- $\Rightarrow P_e^{BPSK} \leq P_e^{BDPSK}$

Comparison of Digital Signaling Methods

- Digital signaling methods (or modulation schemes) can be compared in a variety of ways
- One way to compare is on the bases of required SNR to achieve a specific probability of error
- This metric alone may not be good enough as it does not take into account the data rate or required bandwidth
- Recall that we have studied two broad classes of signaling methods: bandwidth-efficient and power-efficient signaling methods

Bandwidth Efficiency

- Power Efficiency: the criterion for power efficiency of a signaling scheme is the SNR per bit required to achieve a certain probability of error. For e.g., if $P_e = 10^{-5}$, we can compare power efficiency for different schemes by comparing $\gamma_b = \frac{E_b}{N_0}$ which achieves this
- Bandwidth efficiency:

$$r = \frac{R}{W} \quad b/s/Hz$$

- For a given P_e, the parameters r and γ_b are the criteria used for comparison

Bandwidth and Dimensionality

- Sampling Theorem states that to reconstruct a signal with bandwidth W, we need to sample the signal at a rate of at least $2W$ samples per second
- Or, the signal has $2W$ degrees of freedom (dimensions) per second
- For a signal of bandwidth W and duration T, the dimensionality is $N = 2WT$
- Signals which are both bandwidth limited and time-limited do not exist
- However, all practical signals are approximately bandwidth- and time-limited
- We can make this precise.

- Energy of a signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

- We focus on time-limited signals which are also nearly bandwidth-limited.
  - $x(t)$ is non zero in $[-T/2, T/2]$ and
  - $x(t)$ is $\eta-$bandwidth-limited to W

$$\frac{1}{E_x} \int_{-W}^{W} |X(f)|^2\, df \geq 1 - \eta$$

  - η-bandwidth-limited means that at most η fraction of the energy in $x(t)$ is outside $[-W, W]$

- Dimensionality Theorem: For any signal $x(t)$ with support $[-T/2, T/2]$ which is η-bandwidth-limited to W, there exists a set of N orthonormal signals $\{\phi_j(t)\}_{j=1}^{N}$ with support $[-T/2, T/2]$ such that $x(t)$ can be ϵ-approximated by this set of signals.

$$\frac{1}{E_x}\int_{-\infty}^{\infty}\left(x(t) - \sum_{j=1}^{N} < x(t), \phi_j(t) > \phi_j(t)\right)^2 dt \leq \epsilon$$

where $\epsilon = 12\eta$ and $N = \lfloor 2WT + 1 \rfloor$

- Hence, we see that the relationship

$$N \approx 2WT$$

is a good approximation for signals which are approximately bandwidth- and time-limited

Bandwidth Efficiency

- $N = 2WT \Rightarrow \frac{1}{W} = \frac{2T}{N}$
- Since rate $R = \frac{\log_2 M}{T}$, we have for M-ary signaling

$$\frac{R}{W} = \left(\frac{\log_2 M}{T}\right)\left(\frac{2T}{N}\right) = 2\frac{\log_2 M}{N}$$

- Hence, R/W can also be regarded as bit/dimension

Power-limited vs Band-limited

Consider modulations satisfying $N = 2WT$, or $W = N/2T$

- For M-ary FSK, $N = M$; hence

$$\left(\frac{R}{W}\right)_{FSK} = \frac{2\log_2(M)}{M} \leq 1$$

- FSK improves the performance (i.e., reduces the required SNR for a given P_e) by increasing M; hence it is good for channels with power constraint
- This improvement is achieved at a price of increasing bandwidth; hence it is bad for channels with bandwidth constraint
- We refer to $R/W \leq 1$ as the power-limited regime

- For PAM, $N = 1$, hence

$$\left(\frac{R}{W}\right)_{PAM} = \frac{2\log_2(M)}{1} > 1$$

- For PSK, QAM, $N = 2$, hence

$$\left(\frac{R}{W}\right)_{PSK} = \left(\frac{R}{W}\right)_{QAM} = \log_2(M) > 1$$

- We refer to $R/W \geq 1$ as the bandwidth-limited regime

Shannon's Channel Coding Theorem

- Given max power constraint P over bandwidth W, the maximum number of bits per channel use, which can be sent over the channel reliably, is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{WN_0}\right) \quad bits/channel\ use$$

- One channel use = one discrete sample of the AWGN channel
- In duration T, total number of samples $= 2WT$
- Hence, Capacity in bits-per-second:

$$\begin{aligned} C &= (2WT) \times \left(\frac{1}{T}\right) \times \frac{1}{2}\log_2\left(1 + \frac{P}{WN_0}\right) \ bits/per\ second \\ &= W\log_2\left(1 + \frac{P}{WN_0}\right) \ bits/per\ second \end{aligned}$$

- For any reliable scheme of rate R, R bits/second $<$ C bits/second

$$R < W\log_2\left(1 + \frac{P}{WN_0}\right)$$

or

$$\frac{R}{W} < \log_2\left(1 + \frac{P}{WN_0}\right)$$

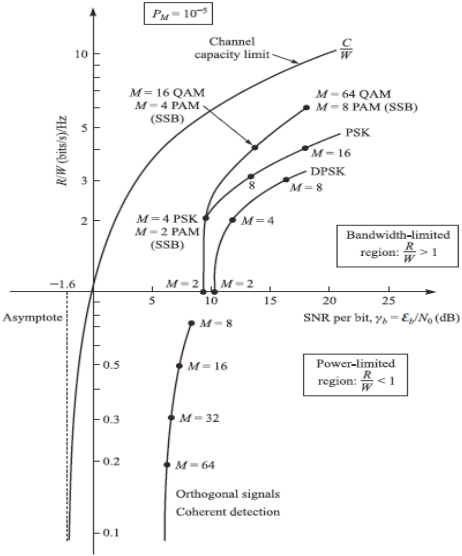
- Also, recall that

$$E_b = \frac{E}{\log_2(M)} = \frac{PT}{\log_2(M)} = \frac{P}{R}$$

which gives

$$\frac{R}{W} < \log_2\left(1 + \frac{E_b}{N_0} \times \frac{R}{W}\right)$$

- This gives the tradeoff between power-efficiency and bandwidth efficiency


$P_M = 10^{-5}$
Channel capacity limit
$\frac{C}{W}$
$M = 16$ QAM
$M = 4$ PAM (SSB)
$M = 64$ QAM
$M = 8$ PAM (SSB)
PSK
$M = 16$
8
DPSK
$M = 8$
$M = 4$ PSK
$M = 2$ PAM (SSB)
$M = 4$
Bandwidth-limited region: $\frac{R}{W} > 1$
R/W (bits/s)/Hz
10
5
3
2
−1.6
$M = 2$
$M = 2$
5
10
15
20
25
SNR per bit, $\gamma_b = \mathcal{E}_b/N_0$ (dB)
Asymptote
$M = 8$
$M = 16$
$M = 32$
$M = 64$
0.5
0.3
0.2
0.1
Power-limited region: $\frac{R}{W} < 1$
Orthogonal signals
Coherent detection

Modulation Schemes in LTE

- LTE devices use primarily
  - QPSK (2 bits per symbol)
  - 16-QAM (4 bits per symbol)
  - 64-QAM (6 bits per symbol)
- The selection of modulation scheme and the data rate usually is dictated by CQI (or Channel Quality Indicator).
- CQI is usually a 4-bit integer which the receiver (UE) sends back to the transmitter (eNodeB)
- It primarily depends on the received Signal-to-interference-plus-noise ratio (SINR)
- CQI is used by the transmitter (eNodeB) to make decision about scheduling the transmission to the receiver and selecting modulation and coding scheme

4-bit CQI Table

CQI Index	Modulation	Code-rate $\times 1024$	Efficiency
0			
1	QPSK	78	0.1523
$\vdots$	$\vdots$	$\vdots$	$\vdots$
6	QPSK	602	1.1758
7	16-QAM	378	1.4766
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	16-QAM	616	2.4063
11 $\vdots$	64-QAM $\vdots$	466 $\vdots$	2.7305 $\vdots$
15	64-QAM	948	5.5547

- Efficiency $=$ (#of bits in constellation) $\times$ *coderate*
- E.g., for CQI $= 1$, QPSK, efficiency $= 2 \times \frac{78}{1024} = 0.1523$