ECE 5654 Lecture 10
Non Coherent Detection

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Learning Objectives

At the end of this lecture, the student should be able to:

- Differentiate between coherent and non-coherent detection.
- Understand complex random process, be able to define proper and circular random process.
- Calculate the error probabilities for non-coherent detection.

Non-coherent Detection

- Until now we had assumed that the signals $\{s_m(t)\}$ are available at the receiver
- There can be cases when this assumption may not be valid
- For example, channel may introduce random changes to the signal
  - Random attenuation
  - Random phase shift. (more about this in Fading channels)
  - Transmitter and receiver may not be perfectly synchronized.
- Receiver may know $s_m(t)$ but the received signals could be delayed, $s_m(t - t_d)$
- t_d represents the time difference between the clocks of Tx and Rx
- We usually model t_d as a random variable

Modeling Uncertainty

- To study the effect of random parameters of this type, we consider that signals $\{s_m(t)\}$ are transmitted.
- We focus on the AWGN channel and the received signal is

$$r(t) = s_m(t;\theta) + n(t)$$

- The random parameter θ is introduced to model the randomness
- θ could be a scalar or a vector
- Receiver does not know the instantaneous value of θ
- We assume that the receiver knows the statistics of θ (i.e., the PMF/PDF of θ)

Optimal Non coherent receiver

- In the vector form (by using the orthonormal basis expansion)

$$r = s_{m,\theta} + n$$

- Optimal detection rule is

$$\begin{aligned}
\hat{m} &= argmax_m \quad P_m p(r|m) \\
&= argmax_m \quad P_m \int p(r|m,\theta)p(\theta)\, d\theta \\
&= argmax_m \quad P_m \int p_n(r - s_{m,\theta})p(\theta)\, d\theta
\end{aligned}$$

- The receiver integrates over all possible realizations of θ
- From the decision rules, we can obtain the corresponding decision regions

Optimal Non coherent receiver

- This can be used to obtain the error probability:

$$
\begin{aligned}
P_m &= \sum_{m=1}^{M} P_m \int_{D_m^c} \left(\int p(r|m,\theta)p(\theta)\, d\theta \right) dr \\
&= \sum_{m=1}^{M} P_m \sum_{m' \neq m} \int_{D'_m} \left(\int p(r|m,\theta)p(\theta)\, d\theta \right) dr
\end{aligned}
$$

- These expressions are quite general and can be used for all kinds of uncertainty in channel parameters
- We will now see some applications of these to various modulation schemes with uncertainty at the receiver

Example: Binary Antipodal Signaling with Amplitude Uncertainty

- Binary antipodal signaling: two equiprobable signals $s_1(t) = s(t)$, $s_2(t) = -s(t)$ are used over an AWGN channel (with noise PSD $N_0/2$)
- Channel introduces a random gain of A

$$r(t) = As_m(t) + n(t)$$

- Suppose that the PDF of A is 0 for $A < 0$ (i.e., channel does not invert the polarity of the signal)
- Let's obtain the probability of error of the non-coherent receiver

Example: Binary Antipodal Signaling with Amplitude Uncertainty

- $p(r|m,A) = p_n(r - As_m)$, hence the optimal decision region D_1 for signal $s_1(t)$ is

$$D_1 = \left\{ r : \int_0^\infty e^{-\frac{(r-A\sqrt{E_b})^2}{N_0}} p(A)\, dA > \int_0^\infty e^{-\frac{(r+A\sqrt{E_b})^2}{N_0}} p(A)\, dA \right\}$$

- This simplifies to:

$$D_1 = \left\{ r : \int_0^\infty e^{-\frac{A^2 E_b}{N_0}} \left(e^{\frac{2rA\sqrt{E_b}}{N_0}} - e^{-\frac{2rA\sqrt{E_b}}{N_0}} \right) p(A)\, dA > 0 \right\}$$

- Expression inside ( ) is positive if and only if $r > 0$ and hence $D_1 = \{r : r > 0\}$

- Hence, we can obtain the error probability as

$$P_b = \int_{A=0}^{\infty} \left(\int_{r=0}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+A\sqrt{Eb})^2}{N_0}} \, dr \right) p(A)\, dA$$
$$= \int_{A=0}^{\infty} Q\left(A\sqrt{\frac{2E_b}{N_0}} \right) p(A)\, dA$$
$$= E\left[Q\left(A\sqrt{\frac{2E_b}{N_0}} \right) \right]$$

Complex Random Vectors

- A complex random vector is defined as $Z = X + jY$, where X and Y are real-valued random vectors of size n
- We define the following four quantities:

$$
\begin{aligned}
C_X &= E[(X - E[X])(X - E[X])^t] \\
C_Y &= E[(Y - E[Y])(Y - E[Y])^t] \\
C_{XY} &= E[(X - E[X])(Y - E[Y])^t] \\
C_{YX} &= E[(Y - E[Y])(X - E[X])^t]
\end{aligned}
$$

- C_X and C_Y are the covariance matrices of real valued random vectors X and Y
- The PDF of Z is the joint PDF of its real and imaginary parts. If we define the $2n$-dimensional real vector

$$
\tilde{Z} = \begin{bmatrix} X \\ Y \end{bmatrix}
$$

- PDF of Z is the PDF of the real vector $\tilde{Z}$

Complex Random Vectors

- The covariance matrix of $\tilde{Z}$ can be written as

$$C_{\tilde{Z}} = \begin{bmatrix} C_X & C_{XY} \\ C_{YX} & C_Y \end{bmatrix}$$

- We also define the following two (in general complex) matrices:

$$C_Z = E[(Z - E[Z])(Z - E[Z])^H]$$
$$\tilde{C}_Z = E[(Z - E[Z])(Z - E[Z])^t]$$

- A^t denotes the transpose of matrix A
- A^H denotes the Hermitian transpose of A (A is transposed and each element is conjugated)
- C_Z is called the covariance of the complex random vector Z
- $\tilde{C}_Z$ is called the pseudocovariance of the complex random vector Z

Complex Random Vectors

- From these definitions, it can be readily verified that
- $(C_Z, \tilde{C}_Z)$ in terms of $(C_X, C_Y, C_{XY}, C_{YZ})$:

$$C_Z = C_X + C_Y + j(C_{YX} - C_{XY})$$
$$\tilde{C}_Z = C_X - C_Y + j(C_{YX} + C_{XY})$$

- $(C_X, C_Y, C_{XY}, C_{YZ})$ in terms of $(C_Z, \tilde{C}_Z)$:

$$C_X = \frac{1}{2} Re[C_Z + \tilde{C}_Z]$$
$$C_Y = \frac{1}{2} Re[C_Z - \tilde{C}_Z]$$
$$C_{YX} = \frac{1}{2} Im[C_Z + \tilde{C}_Z]$$
$$C_{XY} = \frac{1}{2} Im[-C_Z + \tilde{C}_Z]$$

Proper Random Vectors

- A complex random vector Z is called proper is its pseudocovariance is zero, i.e. $\tilde{C}_Z = 0$
- Hence, for a proper random vector, we have

$$C_X = C_Y$$
$$C_{XY} = -C_{YX}$$

- Thus, for proper random vectors, we can simplify

$$C_Z = 2C_X + 2jC_{YX}$$
$$C_X = C_Y = \frac{1}{2}Re[C_Z], \quad C_{YX} = -C_{XY} = \frac{1}{2}Im[C_Z]$$

- Eg. $Z = X + jY$, where X, Y are scalar random variables, independent of each other, and each is distributed as $\mathcal{N}(\mu, \sigma^2)$
- This is a proper random variable: $C_X = C_Y = \sigma^2$ and $C_{XY} = -C_{YX} = 0$

Circular or Circularly Symmetric Random Vectors

- A complex random vector Z is called circular or circularly symmetric if rotating the vector by any angle does not change its PDF.
- Equivalently, Z is circular if Z and $e^{j\theta}Z$ have the same PDF for all θ
- Fact 1: if Z is circular, then it is zero-mean and proper (i.e., $E[Z] = 0$ and $E[ZZ^t] = 0$)
- Fact 2: if Z is a zero-mean proper Gaussian complex vector, then it is circular.
- Or, in other words, for complex Gaussian random vectors, being zero-mean and proper is equivalent to being circular.
- Fact 3: if Z is a proper complex vector, then $AZ + b$ is also a proper complex vector.
  (this is known as an affine transformation of Z)
- (verify these facts)

Complex Random Processes

- A complex random process is defined as $Z(t) = X(t) + jY(t)$, where $X(t)$ and $Y(t)$ are real-valued random processes
- In a similar manner, we can define the covariance and pseudocovariance

$$C_Z(t+\tau, t) = E[Z(t+\tau)Z^*(t)]$$
$$\tilde{C}_Z(t+\tau, t) = E[Z(t+\tau)Z(t)]$$

- We can similarly obtain

$$\begin{aligned} C_Z(t+\tau, t) &= C_X(t+\tau, t) + C_Y(t+\tau, t) \\ &\quad + j\left(C_{YX}(t+\tau, t) - C_{XY}(t+\tau, t)\right) \\ \tilde{C}_Z(t+\tau, t) &= C_X(t+\tau, t) - C_Y(t+\tau, t) \\ &\quad + j\left(C_{YX}(t+\tau, t) + C_{XY}(t+\tau, t)\right) \end{aligned}$$

Proper & Circular Random Processes

- A complex random process $Z(t)$ is proper if its pseudocovariance is zero, i.e., $\tilde{C}_Z(t+\tau, t) = 0$

$$C_X(t+\tau, t) = C_Y(t+\tau, t)$$
$$C_{YX}(t+\tau, t) = -C_{XY}(t+\tau, t)$$

- For a proper complex random process, we can then write:

$$C_Z(t+\tau, t) = 2C_X(t+\tau, t) + j2C_{YX}(t+\tau, t)$$

- A complex random process $Z(t)$ is circular if for all θ, $Z(t)$ and $e^{j\theta}Z(t)$ have the same statistical properties
- When $Z(t)$ is a WSS process, then all auto and cross-correlations are functions of τ only

Bandpass and Lowpass Random Processes

- We focus on a (real) WSS random process $X(t)$, characterized by the autocorrelation function $R_X(\tau)$
- PSD of $R_X(\tau)$ is $S_X(f)$
- $X(t)$ can be categorized as
  - Bandpass process if its PSD is located around $\pm f_0$
  - Lowpass process if its PSD is located around 0
- Hence, a random process is characterized as bandpass/lowpass if the autocorrelation function is a bandpass/lowpass signal
- We can define the lowpass equivalent process $X_l(t)$ as

$$X_l(t) = X_i(t) + jX_q(t)$$

- $X_i(t)$ and $X_q(t)$ are the in-phase and quadrature components of the random process $X(t)$ and can be obtained (recall Lecture 2) as

$$X_i(t) = X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t)$$
$$X_q(t) = \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t)$$

Bandpass and Lowpass Random Processes

- $X_i(t)$ and $X_q(t)$ are the in-phase and quadrature components of the random process $X(t)$ and can be obtained (recall Lecture 2) as

$$X_i(t) = X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t)$$
$$X_q(t) = \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t)$$

- Using these, one can obtain the auto- and cross-correlation functions of in phase and quadrature components.
- We can easily show that the following holds:

$$R_{X_i}(\tau) = R_{X_q}(\tau)$$
$$R_{X_i X_q}(\tau) = -R_{X_q X_i}(\tau)$$

- $\Rightarrow X_l(t)$ is a proper random process!
- Hence, $R_{X_l}(\tau) = 2R_{X_i}(\tau) + 2jR_{X_q X_i}(\tau)$

Bandpass and Lowpass Random Processes

- Hence, auto-correlation function of the lowpass equivalent, $X_l(t)$ of $X(t)$ is:

$$\begin{aligned} R_{X_l}(\tau) &= 2R_{X_i}(\tau) + 2jR_{X_q X_i}(\tau) \\ &= 2[R_X(\tau) + j\hat{R}_X(\tau)]e^{-j2\pi f_0 t} \\ &= 2 \times (\text{lowpass equivalent of } R_X(\tau)) \end{aligned}$$

- Taking Fourier transform of both sides,

$$S_{X_l}(f) = \begin{cases} 4S_X(f+f_0) & |f| < f_0 \\ 0 & otherwise \end{cases}$$

Bandpass and Lowpass AWGN

We also observe that if $X(t)$ is a Gaussian process, then $X_i(t)$, $X_q(t)$, and $X_l(t)$ will be jointly Gaussian processes; and since $X_l(t)$ is Gaussian, zero-mean, and proper, we conclude that $X_l(t)$ is a circular process as well. In this case if $\mathcal{S}_X(f+f_0) = \mathcal{S}_X(f-f_0)$ for $|f| < f_0$, then $X_i(t)$ and $X_q(t)$ will be independent processes.

EXAMPLE 2.9–1. White Gaussian noise with power spectral density of $\frac{N_0}{2}$ passes through an ideal bandpass filter with transfer function

$$H(f) = \begin{cases} 1 & |f - f_0| < W \\ 0 & \text{otherwise} \end{cases}$$

where $W < f_0$. The output, called *filtered white noise*, is denoted by $X(t)$. This process has a power spectral density of

$$\mathcal{S}_X(f) = \begin{cases} \frac{N_0}{2} & |f - f_0| < W \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{S}_X(f + f_0) = \mathcal{S}_X(f - f_0)$ for $|f| < f_0$, and the process is Gaussian, $X_i(f)$ and $X_q(f)$ are independent lowpass processes. Using Equation 2.9–9, we conclude that

$$\mathcal{S}_{X_i}(f) = \mathcal{S}_{X_q}(f) = \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

and from Equation 2.9–13, we conclude that

$$\mathcal{S}_{X_l}(f) = \begin{cases} 2N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

Noncoherent detection of Carrier Modulated Signals

- $\{s_m(t)\}$ are bandpass signals, with lowpass equivalents $\{s_{ml}(t)\}$, where

$$s_m(t) = Re[s_{ml}(t)e^{j2\pi f_c t}]$$

- For the AWGN channel model, the received signal is

$$r(t) = s_m(t - t_d) + n(t)$$

  where t_d is the random time asynchronism between Tx & Rx clocks
- There are three random parameters here:
  - message m (selected with probability P_m)
  - random variable t_d
  - random process $n(t)$

Time-delay to Phase-shift

- We wish to look at the low-pass equivalent of the received signal

$$\begin{aligned} r(t) &= s_m(t - t_d) + n(t) \\ &= Re[s_{ml}(t - t_d)e^{j2\pi f_c(t-t_d)}] + n(t) \\ &= Re[s_{ml}(t - t_d)e^{-j2\pi f_c t_d} e^{j2\pi f_c t}] + n(t) \end{aligned}$$

- $\Rightarrow$ low-pass equivalent of $s_m(t - t_d)$ is $s_{ml}(t - t_d)e^{-j2\pi f_c t_d}$
- In practice t_d (delay) $\ll T_s$, where T_s is the symbol duration
- $\Rightarrow$ effect of a time shift of size t_d on $s_m(t)$ is negligible
- We cannot neglect the term $e^{-j2\pi f_c t_d}$ as it can introduce a large phase shift $\phi = -2\pi f_c t_d$ if carrier frequency is large (even if t_d is small).
- Hence, we can approximate $s_{ml}(t - t_d)e^{-j2\pi f_c t_d} \approx s_{ml}(t)e^{-j2\pi f_c t_d}$

- Since t_d is random, and even small values of t_d can result in large phase shifts (modulo 2π), we usually model ϕ as a random variable uniformly distributed between 0 and 2π
- This model of the channel and detections of signals under this assumption is called non-coherent detection
- Hence, in the non coherent case, we can write

$$\begin{aligned} r(t) &= Re[r_l(t)e^{j2\pi f_c t}] \\ &= Re\left[\left(e^{j\phi}s_{ml}(t) + n_l(t)\right)e^{j2\pi f_c t}\right] \end{aligned}$$

or equivalently the baseband received signal is

$$r_l(t) = e^{j\phi}s_{ml}(t) + n_l(t)$$

- A coherent receiver can track ϕ and compensate for it.

$$r_l(t)e^{-j\phi} = s_{ml}(t) + e^{-j\phi}n_l(t)$$

- The noise process $n_l(t)$ is circular and its statistics are independent of any phase rotation.
- Hence, the coherent receiver will have the familiar form of:

$$\tilde{r}_l(t) = s_{ml}(t) + \tilde{n}_l(t)$$

- For the non-coherent case, the receiver needs to detect with:

$$r_l(t) = e^{j\phi}s_{ml}(t) + n_l(t)$$

- Optimal detection rule:

$$\hat{m} = argmax_m \quad \frac{P_m}{2\pi}\int_0^{2\pi} p_{n_l}(r_l - e^{j\phi}s_{ml})\, d\phi$$

Optimal Detection Rule (non-coherent)

$$\hat{m} = argmax_m \quad \frac{P_m}{2\pi} \int_0^{2\pi} p_{n_l}(r_l - e^{j\phi} s_{ml})\, d\phi$$

$$= argmax_m \quad \frac{P_m}{2\pi} \frac{1}{(4\pi N_0)^N} \int_0^{2\pi} e^{-\frac{||r_l - e^{j\phi} s_{ml}||^2}{4N_0}}\, d\phi$$

Simplifying and dropping the terms which do not depend on m, we obtain:

$$\hat{m} = argmax_m \quad \frac{P_m}{2\pi} e^{-\frac{E_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[r_l \cdot e^{j\phi} s_{ml}]}\, d\phi$$

$$= argmax_m \quad \frac{P_m}{2\pi} e^{-\frac{E_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[(r_l \cdot s_{ml}) e^{j\phi}]}\, d\phi$$

$$= argmax_m \quad \frac{P_m}{2\pi} e^{-\frac{E_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} Re[|r_l \cdot s_{ml}| e^{j(\phi+\theta)}]}\, d\phi$$

$$= argmax_m \quad \frac{P_m}{2\pi} e^{-\frac{E_m}{2N_0}} \int_0^{2\pi} e^{\frac{1}{2N_0} |r_l \cdot s_{ml}| \cos(\phi+\theta)}\, d\phi$$

where θ is the phase of $r_l \cdot s_{ml}$

Optimal non-coherent detection

- The integral above does not depend on θ as it is a periodic function of ϕ with period 2π, and we are integrating over a complete period.
- This is as far as we can go with the exact calculation. Beyond this, we introduce the modified Bessel function:

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \phi} \, d\phi$$

- Hence, the decision rule can be expressed as:

$$\hat{m} = argmax_m \quad P_m e^{-\frac{E_m}{2N_0}} I_0 \left(\frac{|r_l \cdot s_{ml}|}{2N_0} \right)$$

- For equiprobable and equal energy signals, decision rule simplifies to:

$$\hat{m} = argmax_m \quad I_0 \left(\frac{|r_l \cdot s_{ml}|}{2N_0} \right)$$

Envelope Detector

- $I_0(x)$ is an increasing function of x for $x > 0$
- Hence, the decision rule can be further simplified to:

$$\hat{m} = argmax_m \quad |r_l \cdot s_{ml}|$$

- This can also be equivalently written as:

$$\hat{m} = argmax_m \quad \left| \int_{-\infty}^{\infty} r_l(t) s_{ml}^*(t)\, dt \right|$$

- This is known as the Envelope Detector:
  - Rx demodulates the received signal using its non-synchronized local oscillator
  - Obtains $r_l(t)$, the lowpass equivalent of received signal
  - It then correlates $r_l(t)$ with all $s_{ml}(t)$'s and chooses the one with maximum absolute value (or envelope)
  - Hence the name envelope detector.

Difference between Coherent vs Non-Coherent Detection

- For the coherent case $r_l(t) = s_{ml}(t) + n_l(t) \Rightarrow r_l = s_{ml} + n_l$
- Optimal Coherent detector:

$$\hat{m} = argmax_m \quad P_m p_{n_l}(r_l - s_{ml})$$

- If equiprobable and equal energy signals, MAP is optimal.
- Optimal MAP detector:

$$\hat{m} = argmax_m Re[r_l \cdot s_{ml}]$$

Comparison

Coherent MAP detection: $\hat{m} = argmax_m \quad Re[r_l \cdot s_{ml}]$

Noncoherent MAP detection: $\hat{m} = argmax_m \quad |r_l \cdot s_{ml}|$

- Note that the above two r_l's are different. For a coherent system, r_l is obtained from a synchronized local carrier
- For a non coherent system, r_l is obtained from a non-synchronized local carrier

Noncoherent Detection of FSK Modulated Signals

- FSK modulated signals with frequency separation Δf have the form

$$\begin{aligned} s_m(t) &= g(t)\cos(2\pi f_c t + 2\pi(m-1)\Delta f t) \\ &= Re\left[g(t)e^{2j\pi(m-1)\Delta f t} e^{j2\pi f_c t}\right] \end{aligned}$$

- Hence, low-pass equivalent of $s_m(t)$ is $s_{ml}(t) = g(t)e^{2j\pi(m-1)\Delta f t}$
- Suppose $g(t)$ is a rectangular pulse of duration T.
- Given $s_{ml}(t)$ is transmitted, the received signal is

$$r_l(t) = s_{ml}(t)e^{j\phi} + n_l(t)$$

- Non-coherent detector finds the envelope of $r_l \cdot s_{m' l}(t)$

$$\left| \int_{-\infty}^{\infty} r_l(t) s_{m' l}^*(t)\, dt \right| = \left| \int_{-\infty}^{\infty} (s_{ml}(t) e^{j\phi} + n_l(t)) s_{m' l}^*(t)\, dt \right|$$
$$= \left| e^{j\phi} \int_{-\infty}^{\infty} s_{ml}(t) s_{m' l}^*(t)\, dt + \int_{-\infty}^{\infty} n_l(t) s_{m' l}^*(t)\, dt \right|$$

- If $g(t)$ is a square pulse of duration T, we can obtain

$$\int_{-\infty}^{\infty} s_{ml}(t) s_{m' l}^*(t)\, dt = \frac{E_g}{T} \int_0^T e^{j2\pi(m-m^{'})\Delta f t}\, dt$$
$$= \frac{E_g}{T} \left(\frac{e^{j2\pi(m-m^{'})\Delta f T} - 1}{j2\pi(m-m^{'})\Delta f} \right)$$
$$= E_g e^{j2\pi(m-m^{'})\Delta f T} sinc\left[(m-m^{'})\Delta f T \right]$$

- Hence, for $m \neq m^{'}$, if $\Delta f = \frac{k}{T}$, then $|r_l \cdot s_{ml}| \xrightarrow{\mathbb{E}} 0$ (in expectation)

Comparison with Coherent FSK

- $r_l(t) = s_{ml}(t) + n_l(t)$
- Coherent detector compares $Re[r_l \cdot s_{m'l}]$
- If s_{ml} is sent, then

$$
\begin{aligned}
& Re\left[\int_{-\infty}^{\infty} s_{ml}(t) s^*_{m'l}(t)\, dt\right] \\
&= E_g \cos(\pi(m-m^{'})\Delta f T) sinc\left[(m-m^{'})\Delta f T\right] \\
&= E_g sinc\left[2(m-m^{'})\Delta f T\right]
\end{aligned}
$$

- Condition for orthogonality is $\Delta f = \frac{k}{2T}$
- Orthogonality of non-coherent ($\Delta f = \frac{k}{T}$) is more stringent than the coherent case ($\Delta f = \frac{k}{2T}$)

Probability of Error for Orthogonal Signaling (Non-coherent Rx)

- M equiprobable, equal energy, orthogonal signals:

$$
\begin{aligned}
s_{1l} &= (\sqrt{2E}, 0, \ldots, 0) \\
&\vdots \\
s_{Ml} &= (0, 0, \ldots, \sqrt{2E})
\end{aligned}
$$

- Due to symmetry, assume that s_{1l} was transmitted
- Received signal:

$$r_l = e^{j\phi} s_{1l} + n_l$$

- n_l is a complex circular zero-mean Gaussian random vector, with variance of each complex component $2N_0$ (see Section 2.9 in the Book).

Probability of Error for Orthogonal Signaling

- The receiver computes:

$$
\begin{aligned}
|r_l \cdot s_{1l}| &= |2Ee^{j\phi} + n_l \cdot s_{1l}| \\
|r_l \cdot s_{ml}| &= |n_l \cdot s_{ml}|, \quad 2 \le m \le M
\end{aligned}
$$

- $n_l \cdot s_{ml}$ is a circular zero-mean complex Gaussian r.v. with variance $4EN_0$ ($2EN_0$ per real and imaginary parts)
- Denote $R_m = |r_l \cdot s_{ml}|$
- How are the random variables R_m distributed ?

Rayleigh Random Variable

- If X_1 and X_2 are two iid Gaussian random variables distributed as $\mathcal{N}(0, \sigma^2)$, (both with zero-mean and same variance) then

$$X = \sqrt{X_1^2 + X_2^2}$$

  is a Rayleigh random variable
- PDF of Rayleigh random variable:

$$p(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Ricean Random Variable

- If X_1 and X_2 are two iid Gaussian random variables distributed as $\mathcal{N}(m_1, \sigma^2)$ and $\mathcal{N}(m_2, \sigma^2)$, (different means, same variance) then

$$X = \sqrt{X_1^2 + X_2^2}$$

  is a Ricean random variable

- PDF of Ricean random variable:

$$p(x) = \begin{cases} \frac{x}{\sigma^2} I_0\left(\frac{sx}{\sigma^2}\right) e^{-\frac{x^2+s^2}{2\sigma^2}}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

  where $s = \sqrt{m_1^2 + m_2^2}$ and $I_0(x)$ is the modified Bessel function of the first kind and order 0. (see page 46 in Book for more details including their CDFs and other properties)

Probability of Error for Orthogonal Signaling

- The receiver computes:

$$|r_l \cdot s_{1l}| = |2Ee^{j\phi} + n_l \cdot s_{1l}|$$
$$|r_l \cdot s_{ml}| = |n_l \cdot s_{ml}|, \quad 2 \leq m \leq M$$

- $n_l \cdot s_{ml}$ is a circular zero-mean complex Gaussian r.v. with variance $4EN_0$ ($2EN_0$ per real and imaginary parts)
- Denote $R_m = |r_l \cdot s_{ml}|$
  - R_1 has a Ricean distribution with parameters $s = 2E$ and $\sigma^2 = 2EN_0$
  - R_m, $2 \leq m \leq M$ are Rayleigh r.v.'s with parameter $\sigma^2 = 2EN_0$
- Correct decision occurs if $R_1 > R_2, R_1 > R_3, \ldots, R_1 > R_M$

Probability of Error for Orthogonal Signaling

- Probability of correct decision:

$$
\begin{aligned}
P_c &= P[R_2 < R_1, R_3 < R_1, \ldots, R_M < R_1] \\
&= \int_0^\infty P[R_2 < r_1, R_3 < r_1, \ldots, R_M < r_1] p_{R_1}(r_1)\, dr_1 \\
&= \int_0^\infty (P[R_2 < r_1])^{M-1} p_{R_1}(r_1)\, dr_1
\end{aligned}
$$

- $P[R_2 < r_1]) = \int_0^{r_1} p_{R_2}(r_2)\, dr_2 = 1 - e^{-\frac{r_1^2}{2\sigma^2}}$
  (this follows from CDF of Rayleigh r.v. R_2)
- Using Binomial expansion:

$$
\left(1 - e^{-\frac{r_1^2}{2\sigma^2}}\right)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} e^{-\frac{n r_1^2}{2\sigma^2}}
$$

Probability of Error for Orthogonal Signaling

- We thus obtain

$$
\begin{aligned}
P_c &= \int_0^{\infty} \left(P[R_2 < r_1]\right)^{M-1} p_{R_1}(r_1)\, dr_1 \\
&= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^{\infty} e^{-\frac{n r_1^2}{2\sigma^2}} p_{R_1}(r_1)\, dr_1
\end{aligned}
$$

- Using the PDF of R_1 (recall R_1 is a Ricean r.v.) and some simple manipulations, we finally arrive at

$$
P_e = \sum_{n=1}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} e^{-\frac{n \log_2(M)}{n+1} \frac{E_b}{N_0}}
$$

Probability of Error for Binary Orthogonal Signaling

- For $M = 2$ (binary orthogonal signaling),

$$P_e^{non-coherent} = \frac{1}{2} e^{-\frac{E_b}{2N_0}}$$

- For the coherent receiver,

$$P_e^{coherent} = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- Using the inequality, $Q(x) \leq \frac{1}{2} e^{-x^2/2}$, we conclude that

$$P_e^{coherent} \leq P_e^{non-coherent}$$