

ECE 5654 – Advanced Digital Communications



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Lecture #3.1 -

Optimal Detection in the AWGN Channel





Learning Objectives

- After completing this lecture the students should be able to
 - Describe the AWGN channel in both waveform and vector forms
 - Formulate the optimal receiver for an AWGN channel
 - Describe the functional blocks for implementing the optimal receiver in an AWGN channel



Modulation

- We want to modulate signals with digital data using signal sets which are:
 - bandwidth efficient – high bits/sec/Hz
 - energy efficient – low SNR needed for BER requirement
- A signal space representation is a convenient form for viewing modulation which allows us to:
 - design energy and bandwidth efficient signal constellations
 - **determine the form of the optimal receiver for a given constellation**
 - evaluate the performance of a modulation type



AWGN Channel

- Additive White Gaussian Noise
- “Waveform channel”

$$r(t) = s(t) + n(t)$$

$r(t)$ – received waveform

$s(t)$ – transmitted waveform

$n(t)$ – noise waveform –Gaussian random process

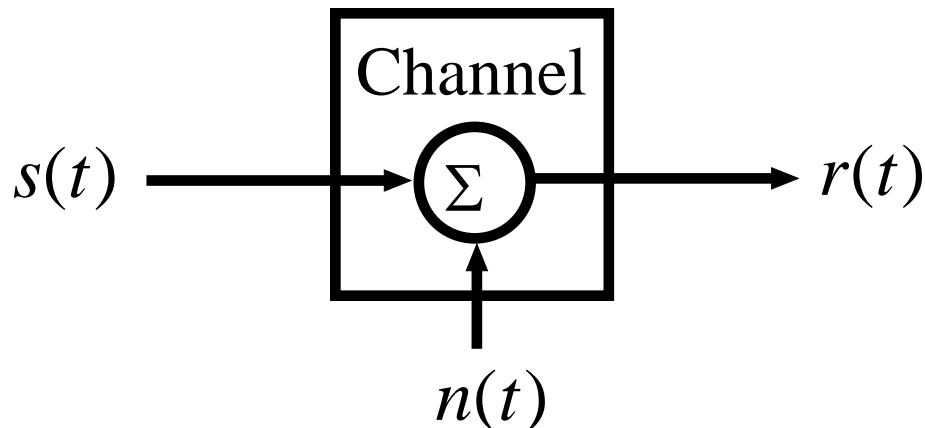
- NOTE: We ignore the absolute power/amplitude of the two signals and only concern ourselves with the *relative* powers

Noise Model

- The signal is corrupted only by Additive White Gaussian Noise (AWGN) $n(t)$ (Gaussian RP)
- White: The noise $n(t)$ has autocorrelation

$$\phi_{nn}(\tau) = \frac{N_0}{2} \delta(\tau)$$

and power spectral density $\Phi_{nn}(f) = N_0/2$





Properties of $n(t)$

- Any linear function of $n(t)$ will be a Gaussian random process
- Samples of $n(t)$ will be (complex) Gaussian random variables
- For complex AWGN, the real and imaginary components are independent
- Theoretically, AWGN has infinite power (due to infinite bandwidth) – however, we will always assume that we filter AWGN which results in finite power consistent with the filter bandwidth



Problem Statement

- We transmit a signal $s(t) \in \{s_1(t), s_2(t), \dots, s_M(t)\}$, where $s(t)$ is nonzero only on $t \in [0, T]$.

- Let the M signals be transmitted with probability:

$$p_1 = \Pr[s_1(t)], \dots, p_M = \Pr[s_M(t)]$$

- The received signal is corrupted by AWGN:

$$r(t) = s(t) + n(t)$$

- Given $r(t)$, the receiver forms an estimate $\hat{s}(t)$ of the signal $s(t)$ with the goal of minimizing *symbol error probability*

$$P_s = \Pr[\hat{s}(t) \neq s(t)]$$



Signal Space Representation

- The transmitted signal can be represented as:

$$s_m(t) = \sum_{k=1}^K s_{m,k} f_k(t)$$

where

$$s_{m,k} = \int_0^T s_m(t) f_k(t) dt$$

In terms of K *basis functions*

- The noise can be represented as:

where

$$n_k = \int_0^T n(t) f_k(t) dt$$

and

$$n'(t) = n(t) - \sum_{k=1}^K n_k f_k(t)$$

$$n(t) = n'(t) + \sum_{k=1}^K n_k f_k(t)$$

Note that the noise has higher dimensionality than K



Signal Space Representation (continued)

- The received signal can be represented as:

$$r(t) = \sum_{k=1}^K s_{m,k} f_k(t) + \sum_{k=1}^K n_k f_k(t) + n'(t) = \sum_{k=1}^K r_k f_k(t) + n'(t)$$

where $r_k = s_{m,k} + n_k$

Note: We can represent the coefficients r_k by a K -dimensional vector \mathbf{r}

$$\mathbf{r} = \mathbf{s}_m + \mathbf{n} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_K \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_K \end{bmatrix}$$



The Orthogonal Noise: $n'(t)$

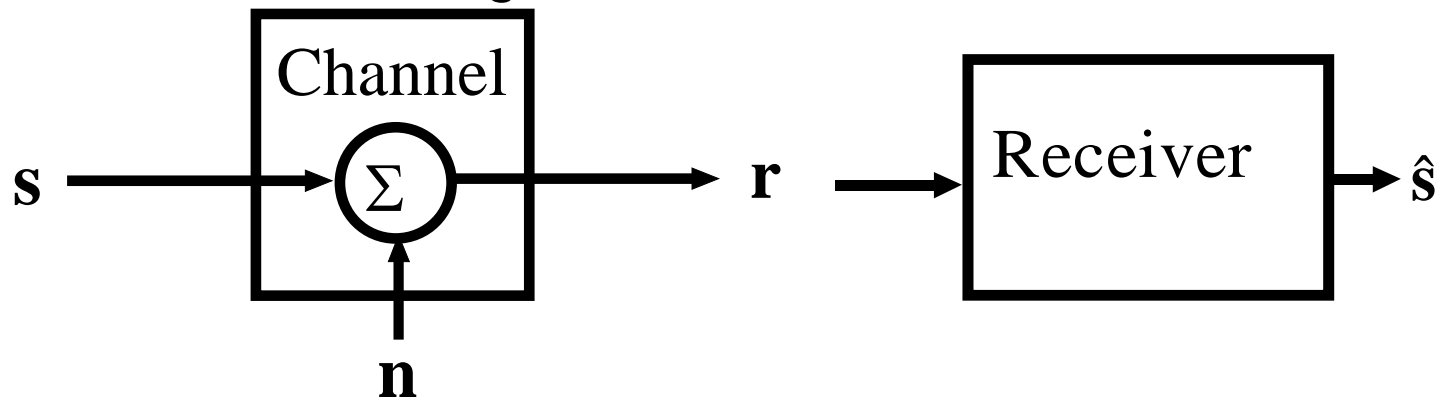
- The noise $n'(t)$ can be disregarded by the receiver

$$\begin{aligned} E[n'(t)r_k] &= E\left[n'(t)(s_{m,k} + n_k)\right] \\ &= E[n'(t)s_{m,k}] + E[n'(t)n_k] \\ &= E[n'(t)n_k] \\ &= E\left[\left(n(t) - \sum_{i=1}^K n_i f_i(t)\right)n_k\right] \\ &= E\left[\left(n(t) \int_0^T n(\tau) f_k(\tau) d\tau - \sum_{i=1}^K n_k n_i f_i(t)\right)\right] \\ &= \int_0^T E[n(t)n(\tau)] f_k(\tau) d\tau - \sum_{i=1}^K E[n_k n_i] f_i(t) \\ &= \frac{1}{2} N_o f_k(\tau) - \frac{1}{2} N_o f_k(\tau) = 0 \end{aligned}$$

Since $n'(t)$ and r_k are uncorrelated they are also independent since they are GRV. Thus, we can ignore $n'(t)$ since it has no bearing on the decision.

AWGN vector channel

- We transmit a K dimensional signal vector:
$$\mathbf{s} = [s_1, s_2, \dots, s_K] \in \{\mathbf{s}_1, \dots, \mathbf{s}_M\}$$
- We receive a vector $\mathbf{r} = [r_1, \dots, r_K] = \mathbf{s} + \mathbf{n}$ which is the sum of the signal vector and AWGN noise vector
$$\mathbf{n} = [n_1, \dots, n_K]$$
- Given \mathbf{r} , we wish to form an estimate $\hat{\mathbf{s}}$ of the transmitted signal vector which minimizes $P_S = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}]$





AWGN Vector Channel (cont.)

- $n(t)$ is a Gaussian random process
- Thus,

$$n_k = \int_0^{T_s} n(t) f_k(t) dt$$

is a Gaussian random variable and

$$p(n_1, n_2, \dots, n_K)$$

is a multivariate Gaussian pdf



The Noise p.d.f

$$\begin{aligned} E[n_i \cdot n_k] &= E \left[\int_0^T n(t) f_i(t) dt \cdot \int_0^T n(s) f_k(s) ds \right] \\ &= E \left[\int_0^T \int_0^T n(t) n(s) f_i(t) f_k(s) ds dt \right] = \int_0^T \int_0^T E[n(t) n(s)] f_i(t) f_k(s) ds dt \\ &= \int_0^T \int_0^T \phi_{nn}(t-s) f_i(t) f_k(s) ds dt = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-s) f_i(t) f_k(s) ds dt \\ &= \int_0^T \frac{N_0}{2} f_i(t) f_k(t) dt = \begin{cases} N_0/2, & i = k \\ 0, & i \neq k \end{cases} \end{aligned}$$



The Noise p.d.f (continued)

- Since $E[n_i \cdot n_k] = 0, \forall i \neq k$, individual noise components are uncorrelated (and therefore independent)
- Since $E[n_k^2] = N_0/2$, each noise component has a variance of $N_0/2$.

$$\begin{aligned} p(n_1, \dots, n_K) &= p(n_1) \cdots p(n_K) \\ &= \prod_{k=1}^K \frac{1}{\sqrt{\pi N_0}} \exp(-n_k^2 / N_0) \\ &= (\pi N_0)^{-K/2} \exp\left(-\frac{\sum_{k=1}^K n_k^2}{N_0}\right) \end{aligned}$$



MAP (Maximum a posteriori probability) Decision Rule

- Suppose that signal vectors $\{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_M\}$ are transmitted with probabilities $\{p_1, p_2, \dots, p_M\}$ respectively, and the signal vector \mathbf{r} is received
- We minimize symbol error probability by choosing the signal \mathbf{s}_m which satisfies: $\Pr(\mathbf{s}_m|\mathbf{r}) \geq \Pr(\mathbf{s}_i|\mathbf{r}), \forall m \neq i$
- Equivalently (using Bayes' rule):

$$\frac{p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m)}{p(\mathbf{r})} \geq \frac{p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i)}{p(\mathbf{r})}, \forall m \neq i$$

or

$$p(\mathbf{r}|\mathbf{s}_m)\Pr(\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i)\Pr(\mathbf{s}_i), \forall m \neq i$$



Maximum Likelihood (ML) Decision Rule

- If $p_1 = \dots = p_m$ or the a priori probabilities are unknown, then the MAP rule simplifies to the ML Rule
- We minimize symbol error probability by choosing the signal \mathbf{s}_m which satisfies

$$p(\mathbf{r}|\mathbf{s}_m) \geq p(\mathbf{r}|\mathbf{s}_i), \forall m \neq i$$



Conditional pdf of Received Signal

- In order to evaluate either the MAP or ML decision rules we must evaluate

$$p(\mathbf{r}|\mathbf{s}_m)$$

- This is found from $p(n_1, n_2 \dots n_K)$

$$p(n_1, \dots, n_K) = (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^K n_k^2 / N_0\right)$$

- Transmitted signal values in each dimension represent the mean values for each signal

$$p(\mathbf{r}|\mathbf{s}_m) = (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^K (r_k - s_{m,k})^2 / N_0\right)$$



Structure of Optimum Receiver

- MAP rule: $\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} p_m \cdot p(\mathbf{r}|\mathbf{s}_m)$

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} p_m \cdot (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^K (r_k - s_{m,k})^2 / N_0\right)$$

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \ln\left[p_m \cdot (\pi N_0)^{-K/2} \exp\left(-\sum_{k=1}^K (r_k - s_{m,k})^2 / N_0\right)\right]$$

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \ln[p_m] - \frac{K}{2} \ln[\pi N_0] - \frac{1}{N_0} \sum_{k=1}^K (r_k - s_{m,k})^2$$



Structure of Optimum Receiver (continued)

- $\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \ln[p_m] - \frac{K}{2} \ln[\pi N_0]$
$$- \frac{1}{N_0} \left(\sum_{k=1}^K r_k^2 - 2 \sum_{k=1}^K r_k s_{m,k} + \sum_{k=1}^K s_{m,k}^2 \right)$$
- Eliminating terms which are identical for all choices:

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \ln[p_m] + \frac{2}{N_0} \sum_{k=1}^K r_k s_{m,k} - \frac{1}{N_0} \sum_{k=1}^K s_{m,k}^2$$



Final Form of MAP Receiver

- Multiplying through by the constant $N_0/2$:

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \frac{N_0}{2} \ln[p_m] + \sum_{k=1}^K r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^K s_{m,k}^2$$



Interpreting This Result

- $\frac{N_0}{2} \ln[p_m]$ weights the a priori probabilities
 - If the noise is large, p_m counts a lot
 - If the noise is small, our received signal will be an accurate estimate and p_m counts less
- $\sum_{k=1}^K r_k s_{m,k} = \int_0^T s_m(t) r(t) dt$ represents the correlation with the received signal
- $\frac{1}{2} \sum_{k=1}^K s_{m,k}^2 = \frac{1}{2} \int_0^T s_m^2(t) dt = \frac{E_m}{2}$ represents signal energy



Interpreting This Result: Another View

$$\hat{s} = \operatorname{argmax}_{\{s_1, s_2, \dots, s_m\}} \ln[p_m] - \frac{1}{N_0} \sum_{k=1}^K (r_k - s_{m,k})^2$$

If equally likely:

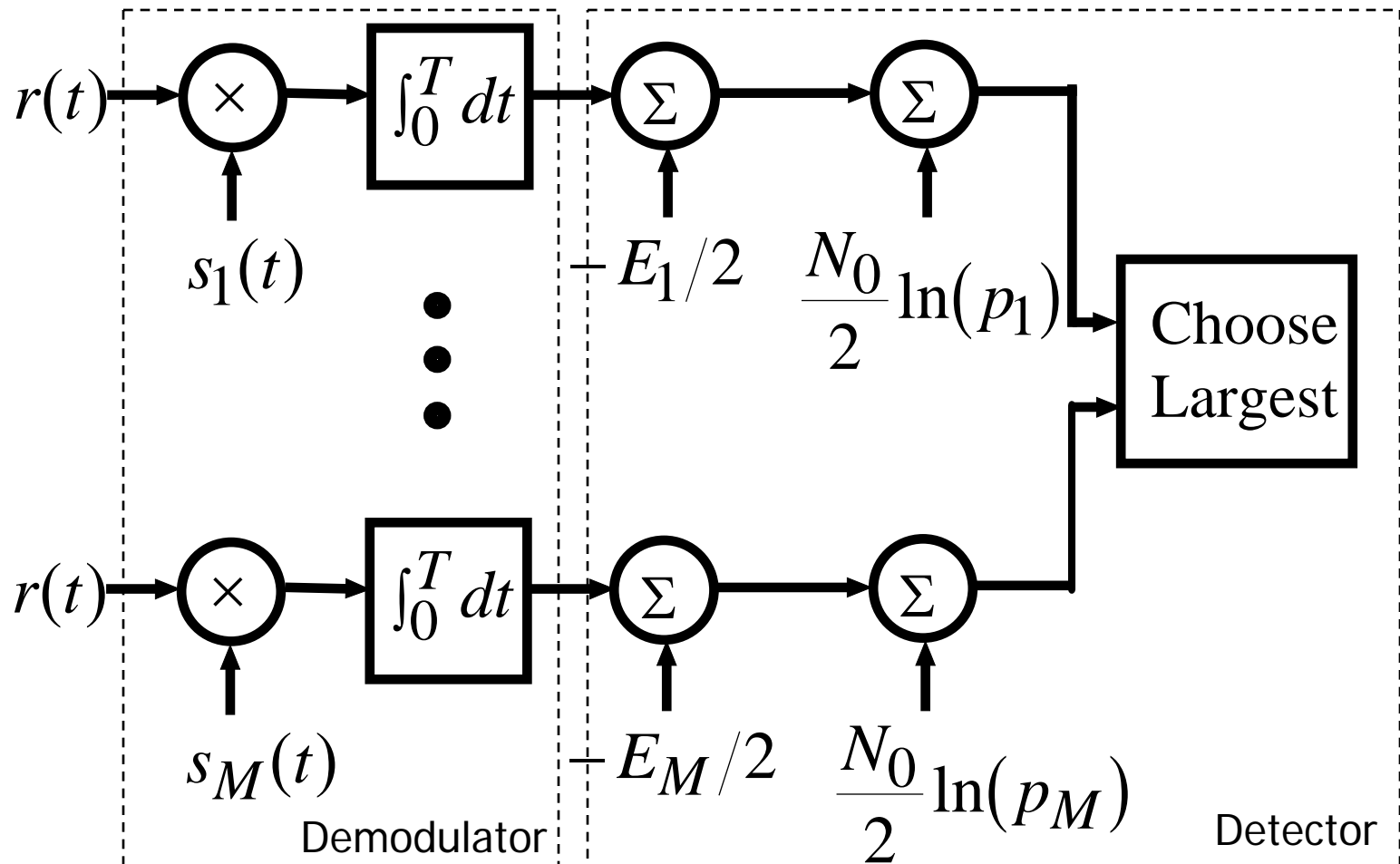
$$\hat{s} = \operatorname{argmax}_{\{s_1, s_2, \dots, s_m\}} - \frac{1}{N_0} \sum_{k=1}^K (r_k - s_{m,k})^2$$

In order to maximize the argument, we need to minimize the sum:

$$\hat{s} = \operatorname{argmin}_{\{s_1, s_2, \dots, s_m\}} \frac{1}{N_0} \sum_{k=1}^K (r_k - s_{m,k})^2$$

Thus, we wish to find the symbol which is closest in Euclidean distance

An Implementation of the Optimal Receiver - Correlation Receiver





Optimal Receiver Functionality

- Demodulator
 - The demodulator converts the received signal into a vector of *decision statistics* which are passed to the detector
- Detector
 - The detector uses the vector of decision statistics to create an estimate of the transmitted signal using either the Maximum A Posteriori or Maximum Likelihood decision rules

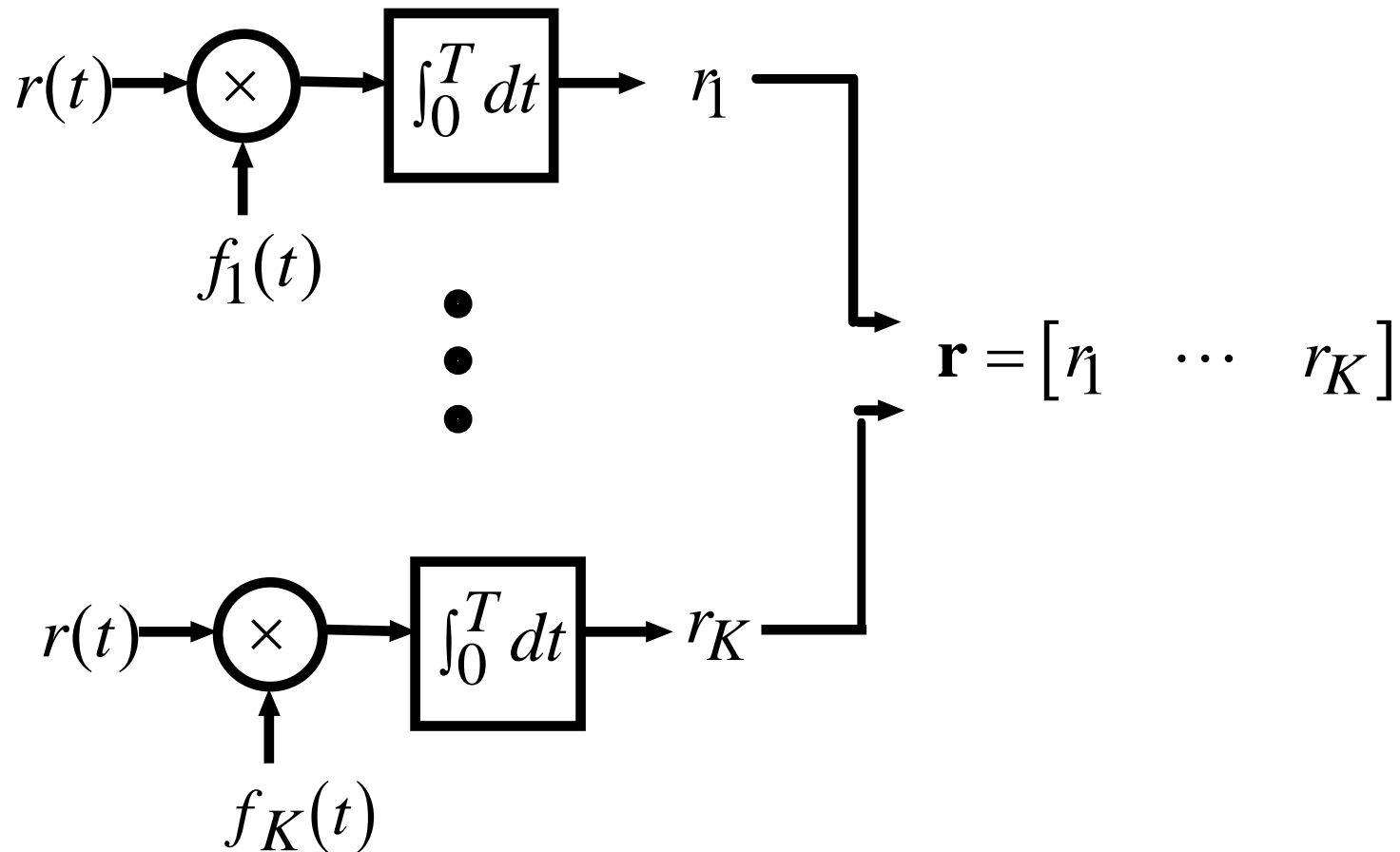


Simplifications for Special Cases

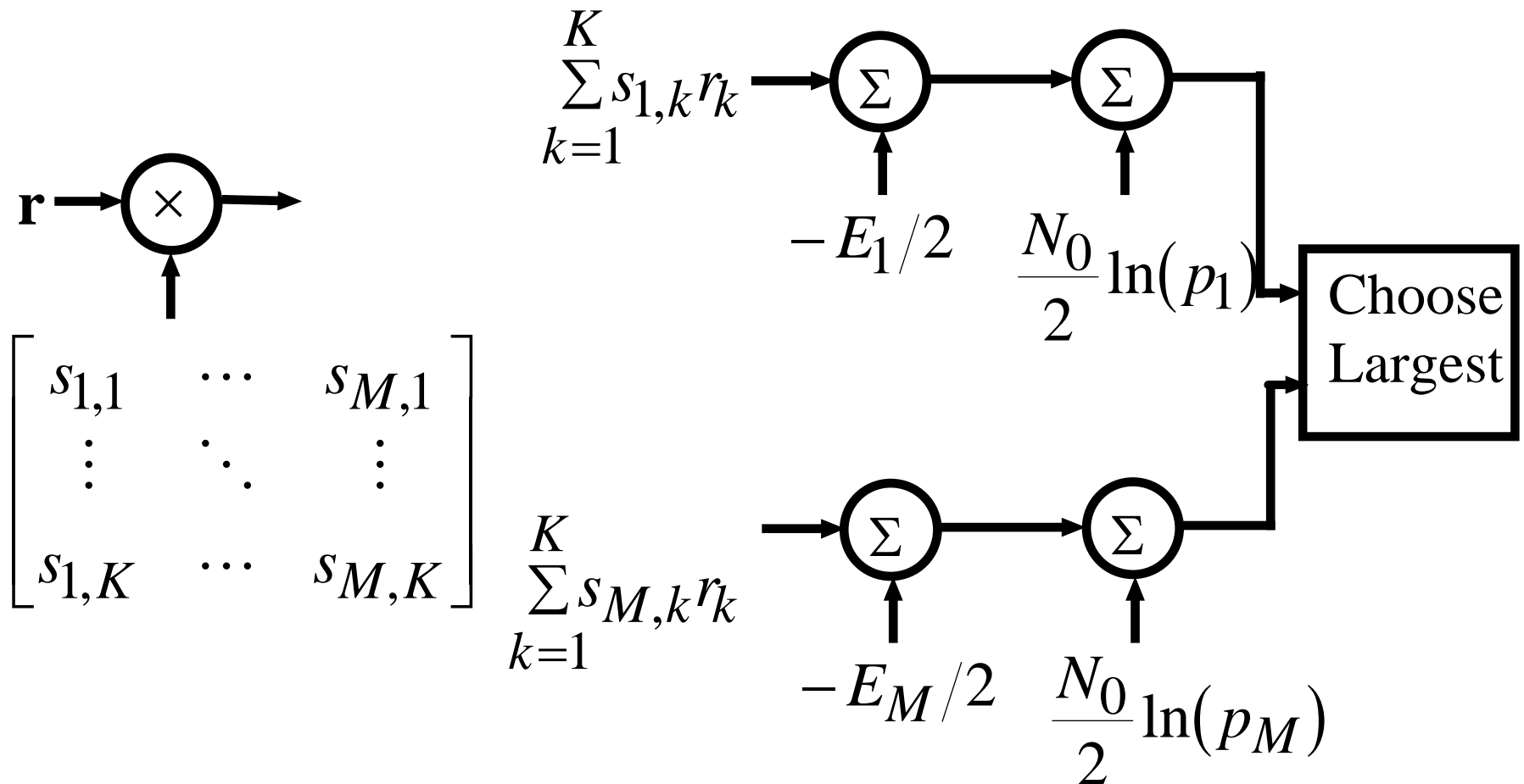
- ML case: All signals are equally likely ($p_1 = \dots = p_M$). A priori probabilities can be ignored.
- All signals have equal energy ($E_1 = \dots = E_M$). Energy terms can be ignored.
- We can reduce the number of correlations by directly implementing:

$$\hat{\mathbf{s}} = \arg \max_{\{\mathbf{s}_1, \dots, \mathbf{s}_M\}} \frac{N_0}{2} \ln[p_m] + \sum_{k=1}^K r_k s_{m,k} - \frac{1}{2} \sum_{k=1}^K s_{m,k}^2$$

Reduced Complexity Implementation: Correlation Stage



Reduced Complexity Implementation - Processing Stage





Matched Filter Implementation

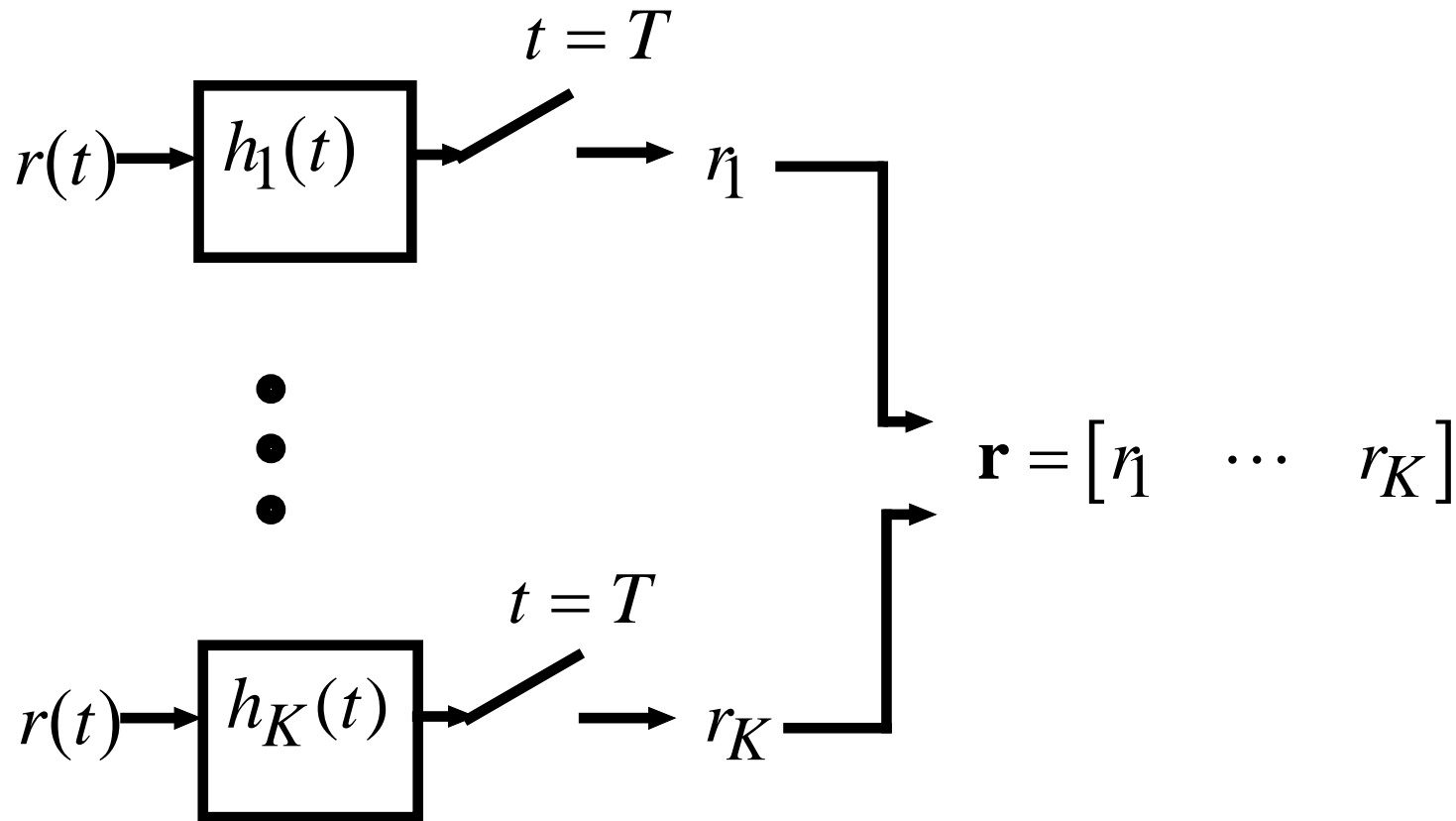
- Assume $f_k(t)$ is time-limited to $t \in [0, T]$, and let $h_k(t) = f_k(T - t)$

- Then
$$r_k = \int_0^T r(t) f_k(t) dt = \int_0^T r(t) f_k(T - (T - t)) dt$$
$$= \int_0^T r(t) h_k(T - t) dt = r(t) \otimes h_k(t) \Big|_{t=T}$$

where $r(t) \otimes h_k(t) \Big|_{t=T}$ denotes the convolution of the signals $r(t)$ and $h_k(t)$ evaluated at time T

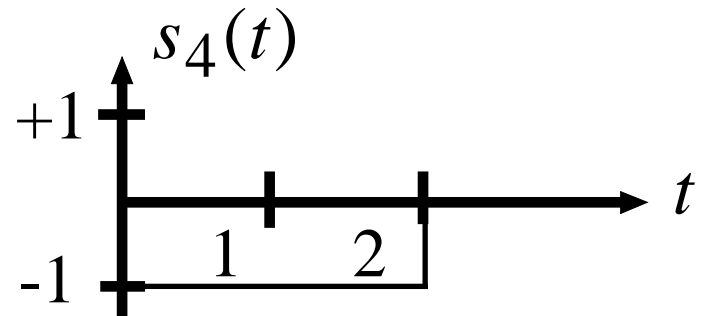
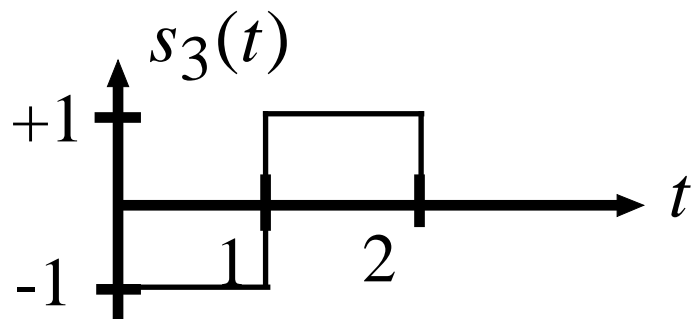
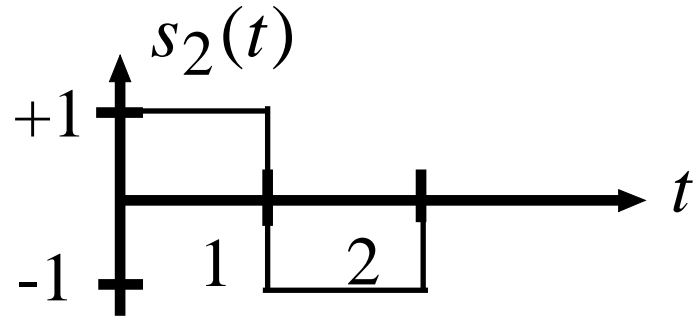
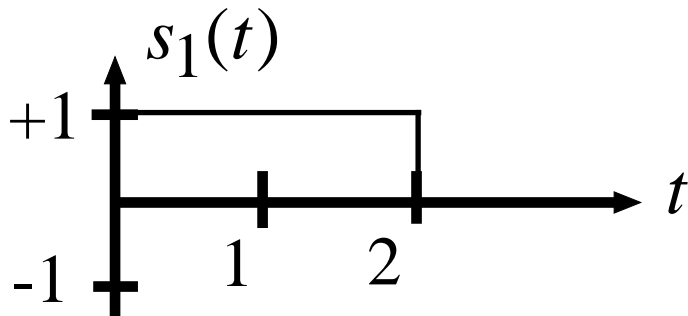
- We can implement each correlation by passing $r(t)$ through a filter with impulse response $h_k(t)$

Matched Filter Implementation of Correlation



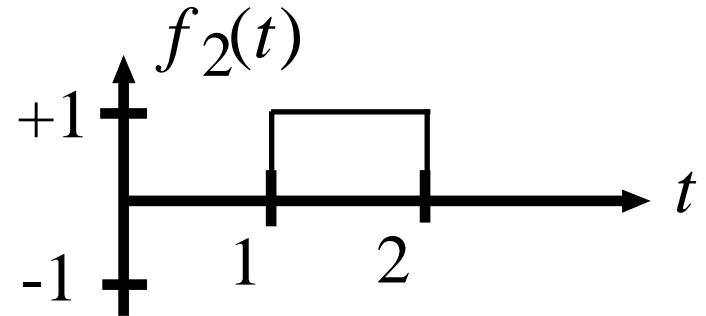
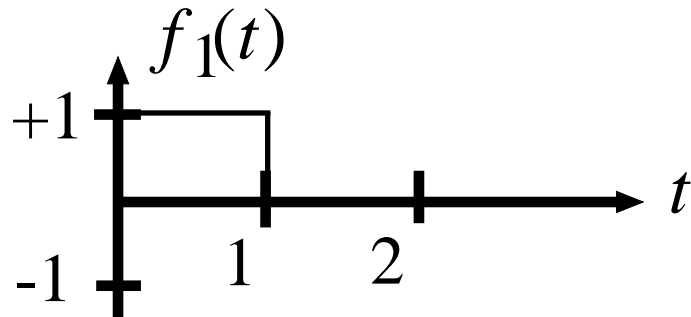
Example

- Consider the signal set:



Example (continued)

- Suppose we use the basis functions:



$$s_1(t) = 1 \cdot f_1(t) + 1 \cdot f_2(t)$$

$$s_2(t) = 1 \cdot f_1(t) - 1 \cdot f_2(t)$$

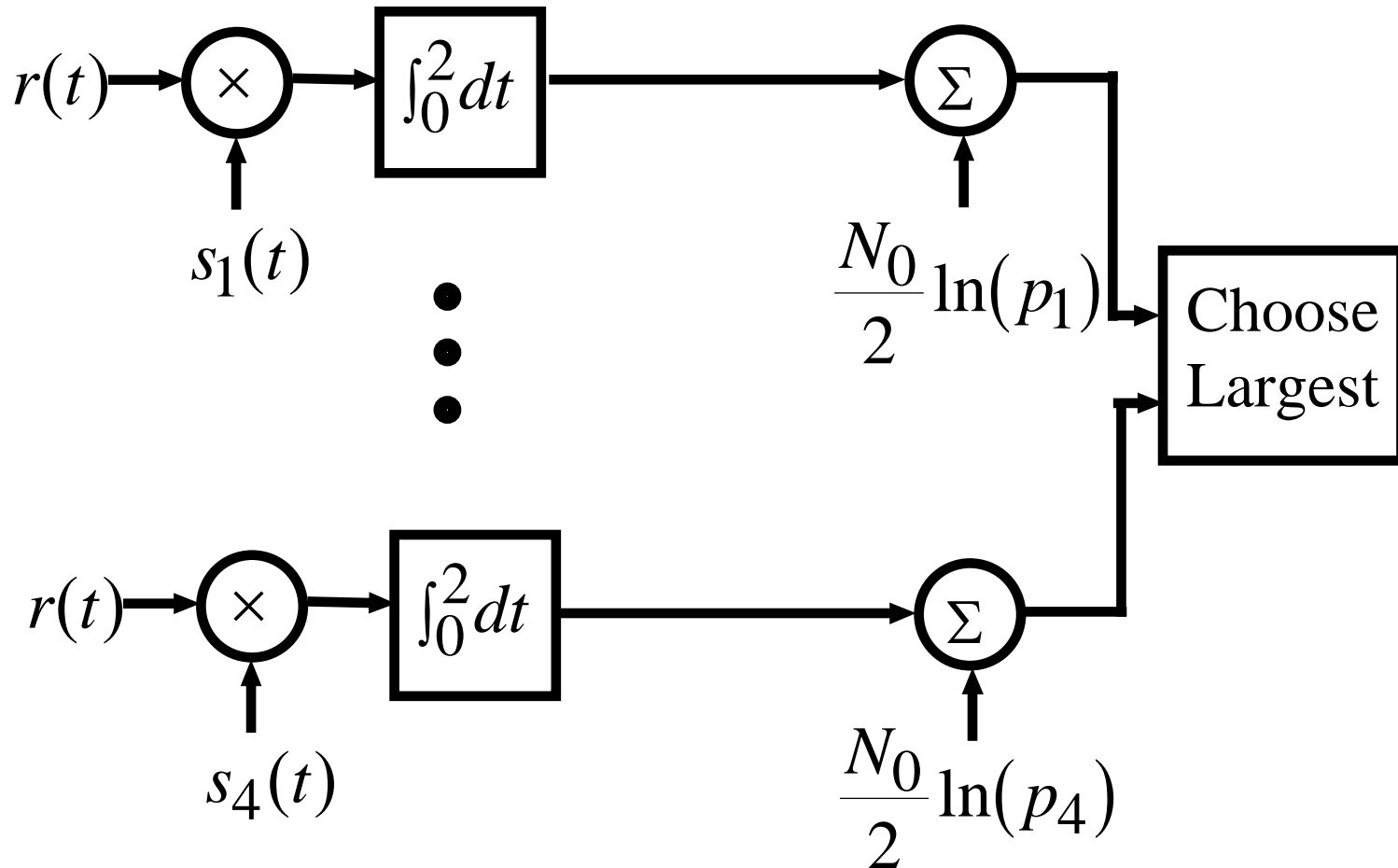
$$s_3(t) = -1 \cdot f_1(t) + 1 \cdot f_2(t)$$

$$s_4(t) = -1 \cdot f_1(t) - 1 \cdot f_2(t)$$

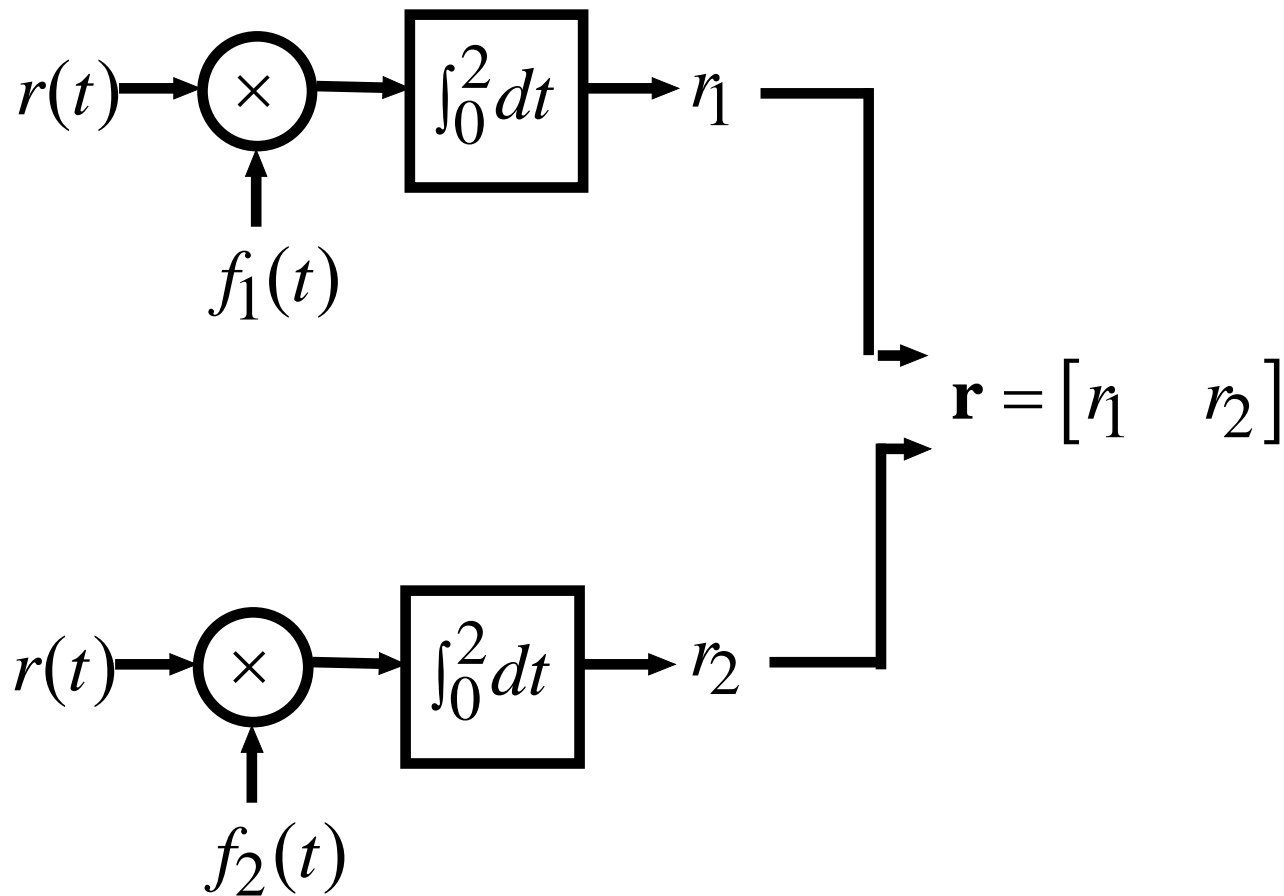
$$T = 2$$

$$E_1 = E_2 = E_3 = E_4 = 2$$

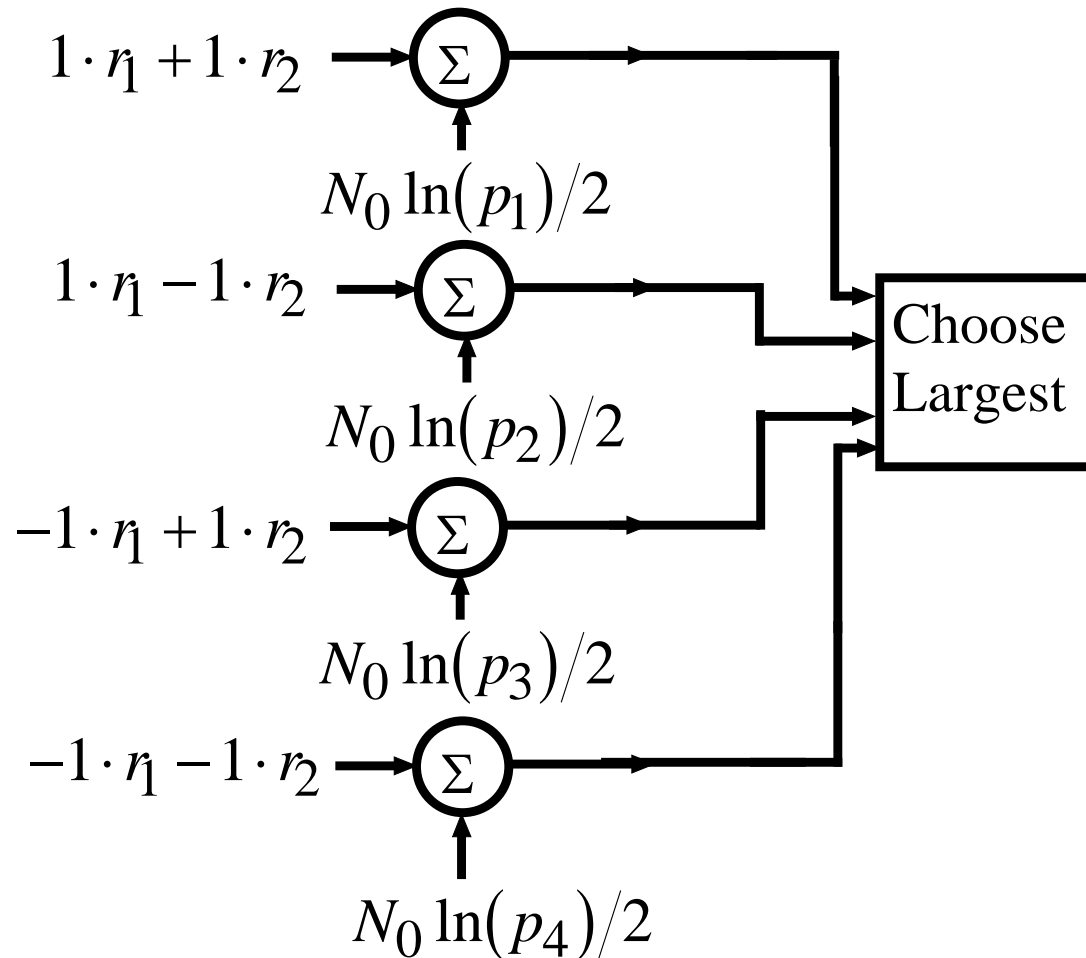
1st Implementation of Correlation Receiver



Reduced Complexity Correlation Receiver - Correlation Stage

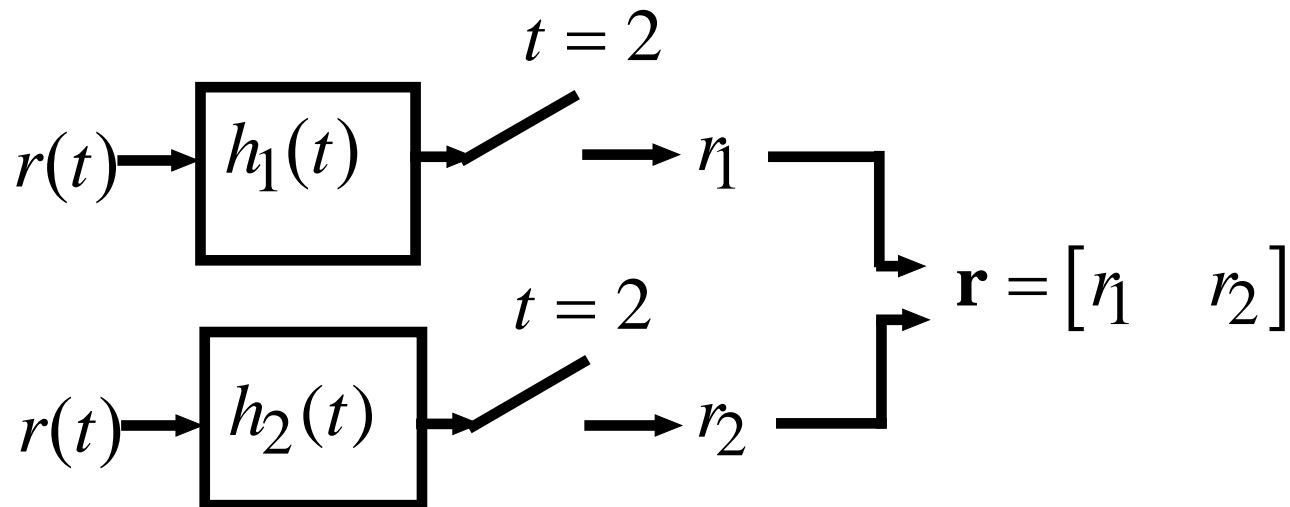
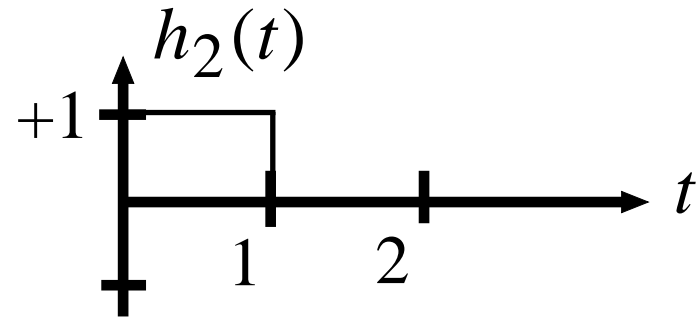
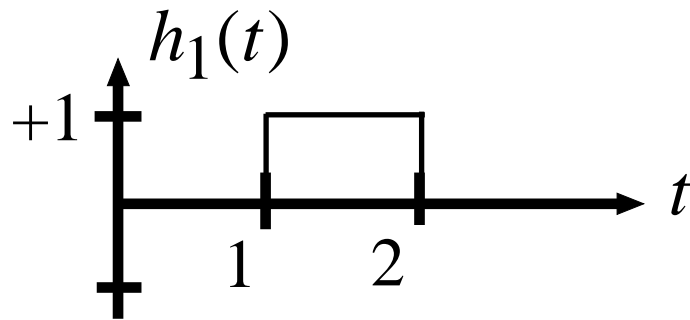


Reduced Complexity Correlation Receiver - Processing Stage



Matched Filter Implementation of Correlations

$$h_k(t) = f_k(2 - t)$$





Summary of Optimal Receiver Design in AWGN Channels

- Optimal coherent receiver for AWGN has three parts:
 - Correlates the received signal with each possible transmitted signal (*We will show later that this is optimal demodulator*)
 - Normalizes the correlation to account for energy
 - Weights the a priori probabilities according to noise power
- This receiver is completely general for any signal set
- Simplifications are possible under many circumstances (e.g., only one basis function, equal symbol energies, equal probabilities)