ECE 5654 – Advanced Digital Communications

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Lecture #2.5 – Spectrum, Bandwidth and Pulse Shaping

Spring 2014

Learning Objectives

- After this lecture the student should be able to
  - Find the spectrum of any linearly modulated digital communication signal
  - Describe the impact of pulse shaping on linear modulation
  - Find the signal space diagram for a modulation scheme that includes pulse shaping

Linear modulation: ASK

- *M*-ASK can be written as

$$v(t)=\sum_{n=-\infty}^{\infty} s_m\left(t-nT_s;I_n\right)$$

$$m=0,1,\ldots M-1$$

$$I_n=data$$

where

$$s_m(t)=\mathrm{Re}\left\{A_m g(t)e^{j2\pi f_c t}\right\}$$

$$=A_m g(t)\cos\left(2\pi f_c t\right)$$

$$A_m=f\left(I_n\right)$$

- In complex baseband

$$s_m(t)=A_m g(t)$$

- *g(t)* can be any general pulse shape, although we typically assume square pulses of duration T_s

Linear Modulation: PSK

- *M*-PSK

$$s_m(t) = \operatorname{Re}\left\{ g(t) e^{j2\pi m/M} e^{j2\pi f_c t} \right\} \qquad m = 0,1,\ldots M-1$$

$$= g(t)\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right)$$

$$s_m(t) = g(t) e^{j2\pi m/M}$$

$$= g(t)\left(\cos\left(\frac{2\pi m}{M}\right) + j\sin\left(\frac{2\pi m}{M}\right)\right)$$

Linear Modulation: QAM

- QAM

$$s_m(t) = \mathrm{Re}\left\{ g(t)\left[a_m + jb_m\right] e^{j2\pi f_c t} \right\} \qquad m = 0,1,\ldots M-1$$

$$= g(t) a_m \cos(2\pi f_c t) + g(t) b_m \sin(2\pi f_c t)$$

$$s_m(t) = g(t)\left[a_m + jb_m\right]$$

$$a_m \in \left\{ -\sqrt{M}+1, -\sqrt{M}+3 \ldots \sqrt{M}-1, \right\}$$

$$b_m \in \left\{ -\sqrt{M}+1, -\sqrt{M}+3 \ldots \sqrt{M}-1, \right\}$$

Linearly modulated signals

- The complex baseband of each of these modulation techniques can be represented as a special case of Pulse Amplitude Modulation:

$$v_l(t) = \sum_n I_n g(t - nT_s)$$

- Where I_n is the data sequence and *g(t)* is the pulse shape used.
- The pulse is a deterministic quantity, but the data sequence is random. Thus, we rely on the power spectral density for spectral information

Power Spectral Density

- Wiener-Khinchine Theorem (for WSS process)

$$\boxed{S_v(f) = F\left\{R_v(\tau)\right\}}$$

Auto-correlation of *v(t)*
Fourier Transform
Power Spectral Density

- Bandpass vs. Baseband

$$\boxed{S_v(f) = \frac{1}{4}\left(S_{v_l}(f - f_c) + S_{v_l}(-f - f_c)\right)}$$

Complex baseband
Bandpass

Spectrum of Linearly Modulated Signals

- As shown in Section 3.4-2, the power spectral density of a digital waveform depends on the pulse spectrum *G(f)* and the autocorrelation of the data $R_I(k)$.

$$S_{v_l}(f) = \frac{|G(f)|^2}{T_s} \sum_{k=-\infty}^{\infty} R_I(k) e^{-j2\pi k f T}$$

- If data are independent, only the data variance σ_I^2 and mean m_I have an impact:

$$S_{v_l}(f) = \underbrace{\frac{\sigma_I^2}{T_s} |G(f)|^2}_{continuous} + \underbrace{\frac{m_I^2}{T_s^2} \sum_{k=-\infty}^{\infty} \left| G\left(\frac{k}{T_s}\right) \right|^2 \delta\left(f - \frac{k}{T_s}\right)}_{discrete}$$

Ex: BPSK

- For BPSK modulation $\sigma_I^2=1$ and $m_I=0$:

$$\boxed{S_{bpsk}(f) = \frac{1}{T}\left|G(f)\right|^2}$$

- And thus the spectrum for BPSK depends entirely on the baseband pulse shape.
- This is also true for MPSK and QAM

Definitions of Bandwidth for Baseband Signals

- Absolute Bandwidth

$$S(f) = 0, \text{ for } |f| > B$$

- *X* dB Bandwidth

$$10\log_{10}\left[\left|\frac{\max\{S(f)\}}{S(f)}\right|^2\right] > X \text{ dB}, |f| > B$$

- *Y* % Power Bandwidth

$$\frac{\int_{-B}^{B} |S(f)|^2\, df}{\int_{-\infty}^{\infty} |S(f)|^2\, df} \geq \frac{Y}{100}$$

Note that usually BW is defined over *positive* frequencies

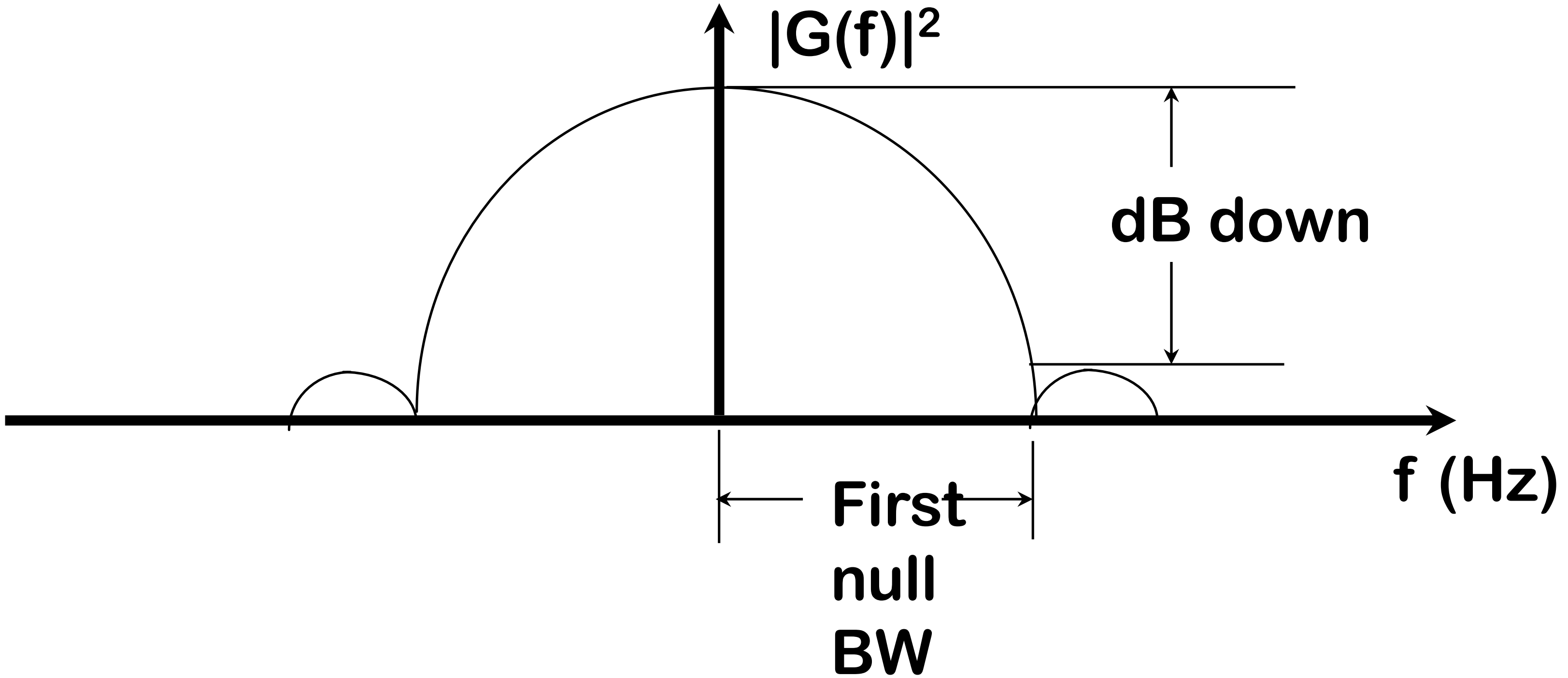
- First Null Bandwidth

What impacts bandwidth?

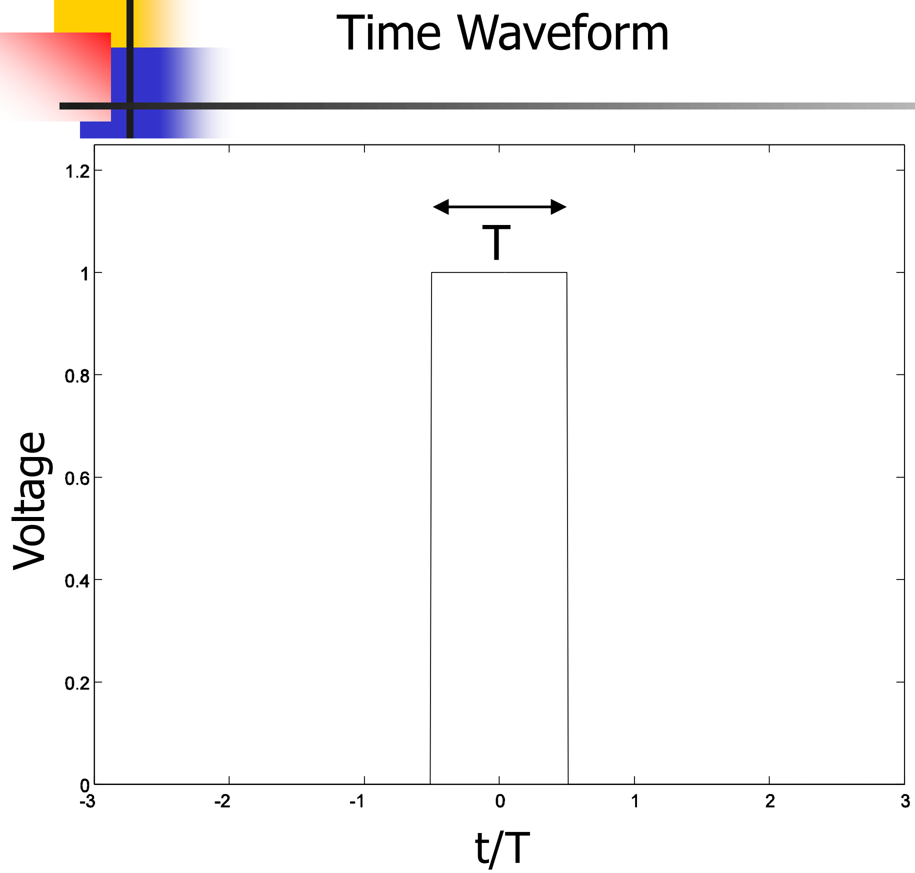
- From our previous discussion, if the data is independent, the pulse shape dominates the spectral characteristics
- Bandwidth determined by
  - Pulse duration (i.e., the symbol rate)
    - Bit rate
    - Modulation scheme
  - Pulse shape

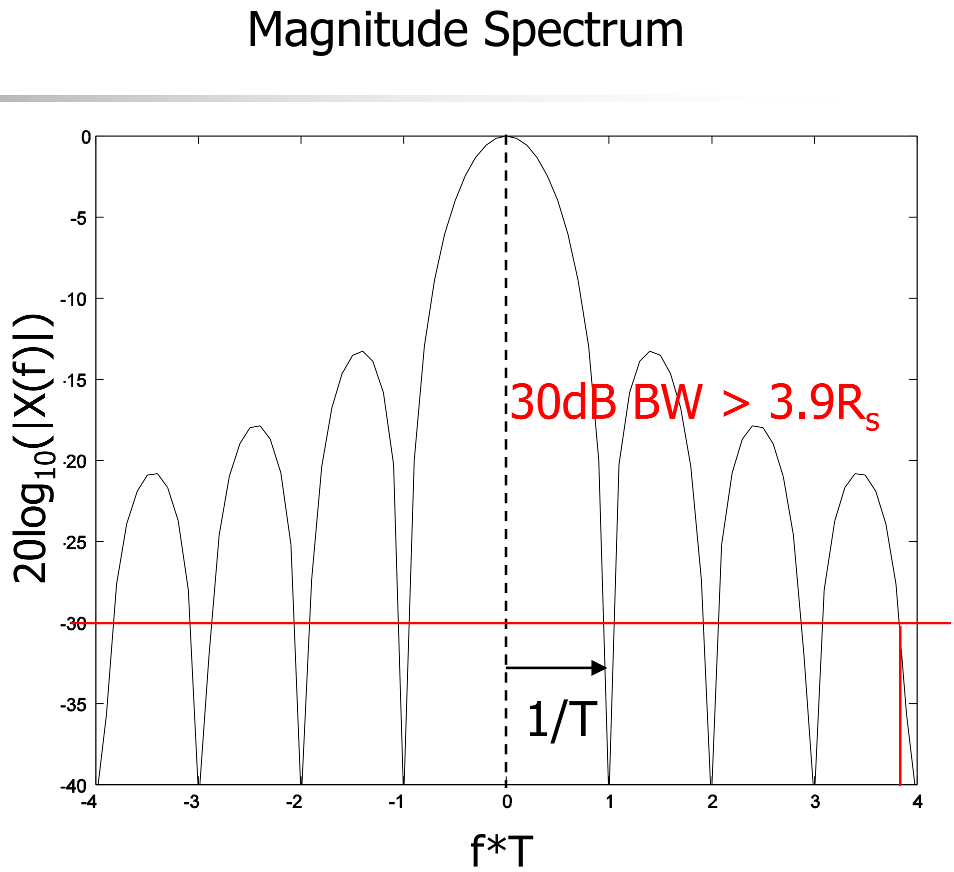
Design Criteria for Pulse Spectrum

- Two important spectral characteristics
  - First null bandwidth
  - Size of sidelobes (and rate of decay)



Rectangular Pulse





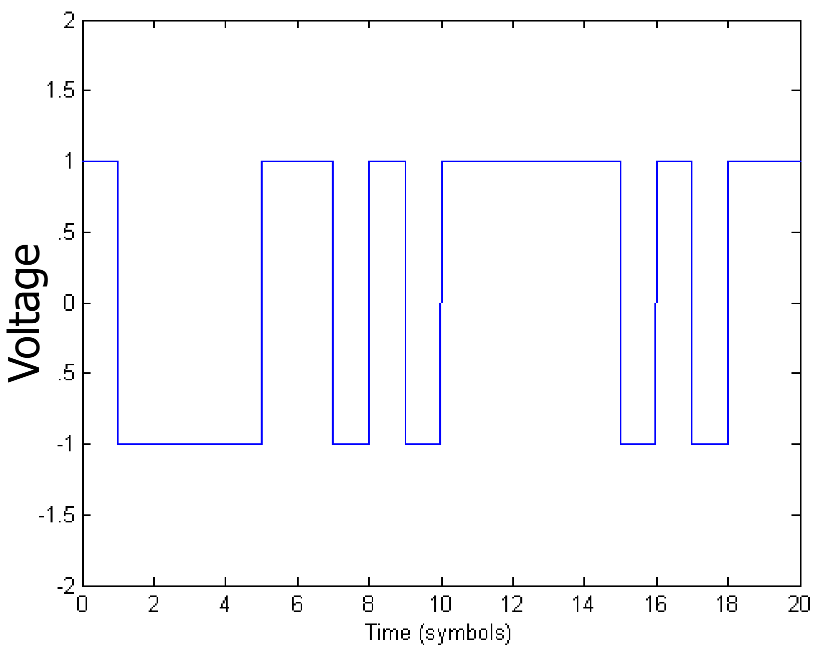
- **First Null BW: 1/T = R_s (symbol rate)**
- **First Sidelobe: 13.6 dB down (~3dB decay rate)**

Peaks equal $\frac{2}{\pi(2n+1)}$ $n = 1, 2, 3, \ldots$

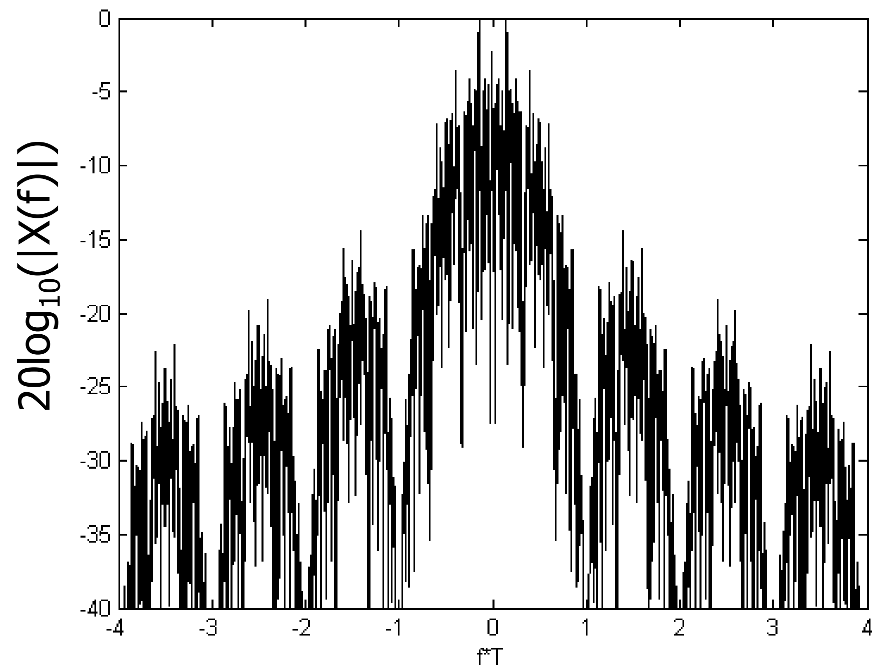
Located at $fT = \frac{2n+1}{2}$

Example Signal

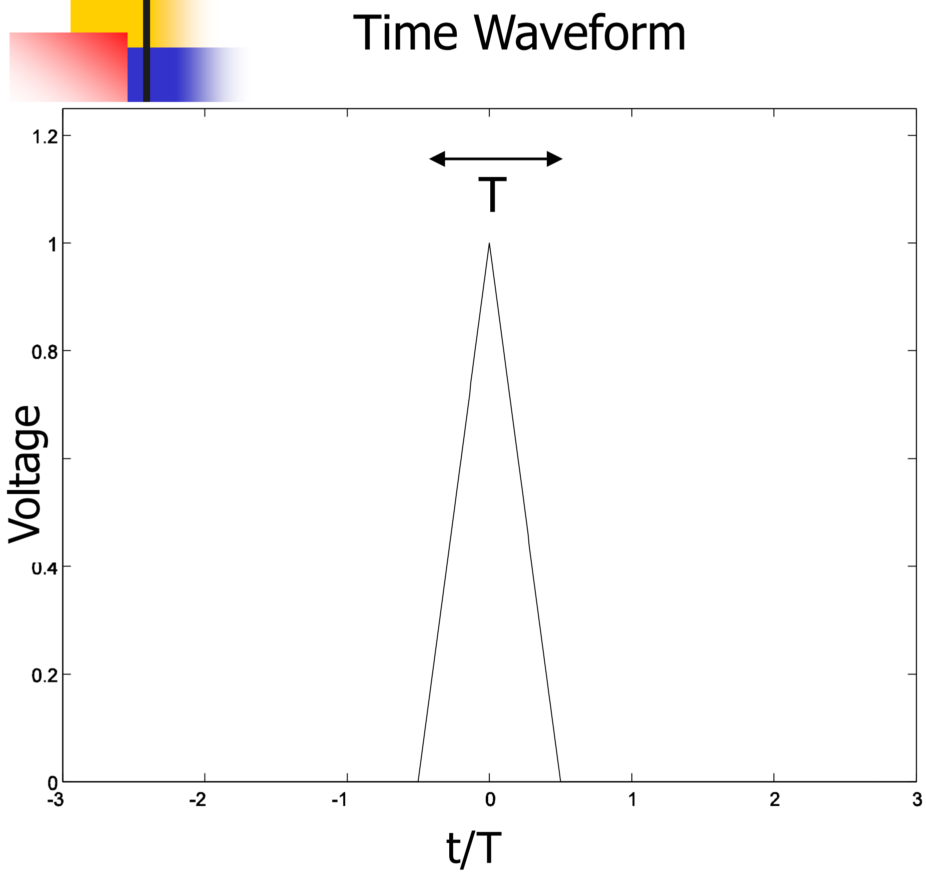
Example signal

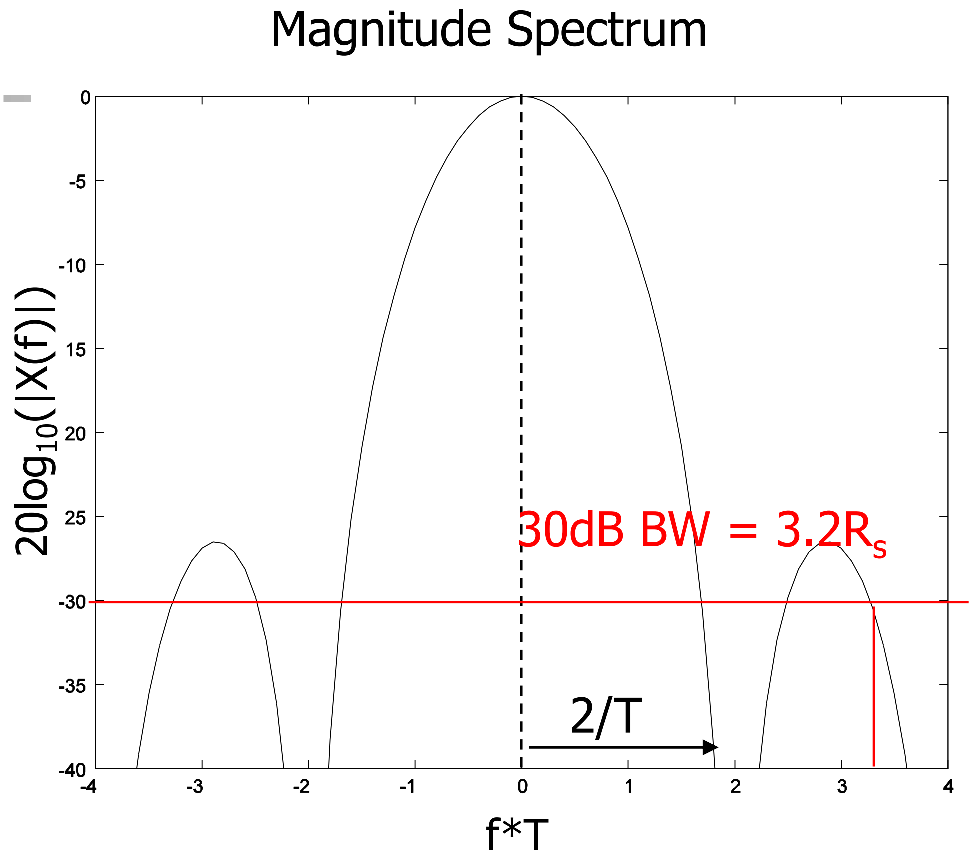


Example Spectrum



Triangular Pulse

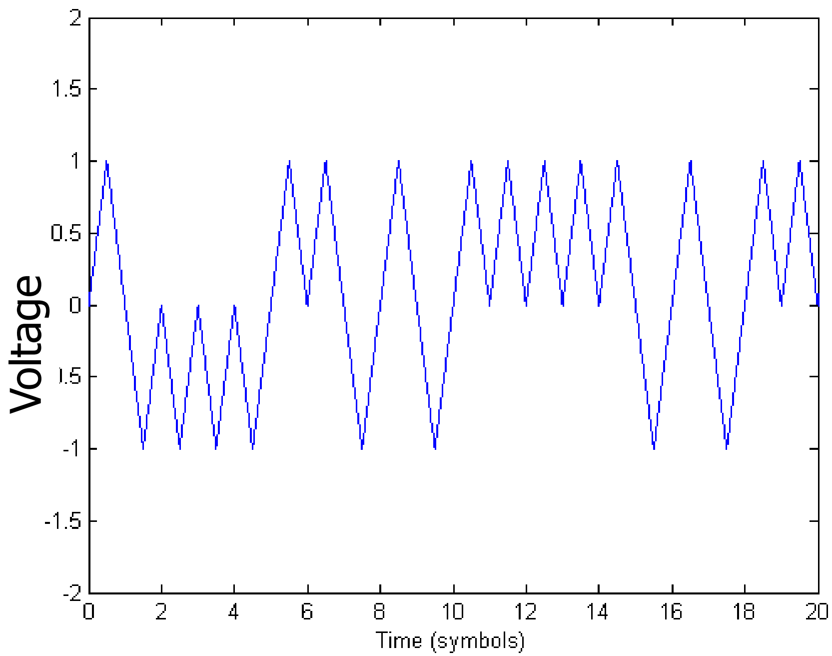




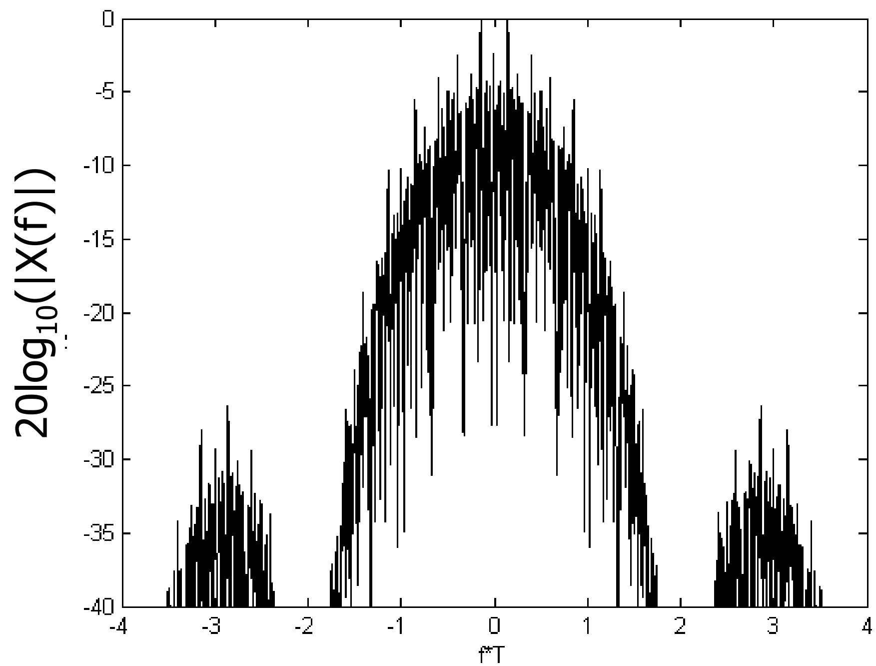
- **First Null BW: 2/T = 2R_s**
- **First Sidelobe: 26 dB down**

Example Signal

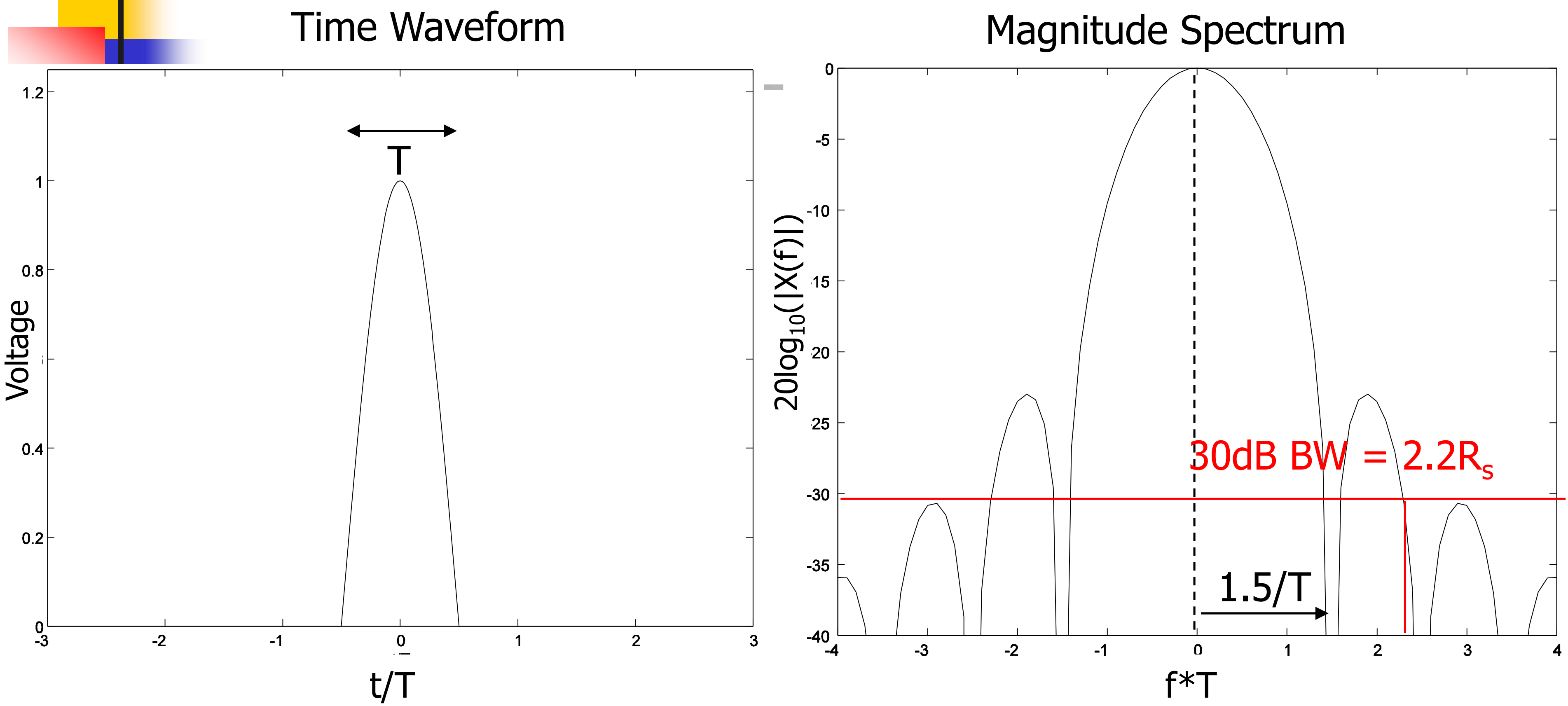
Example signal



Example Spectrum



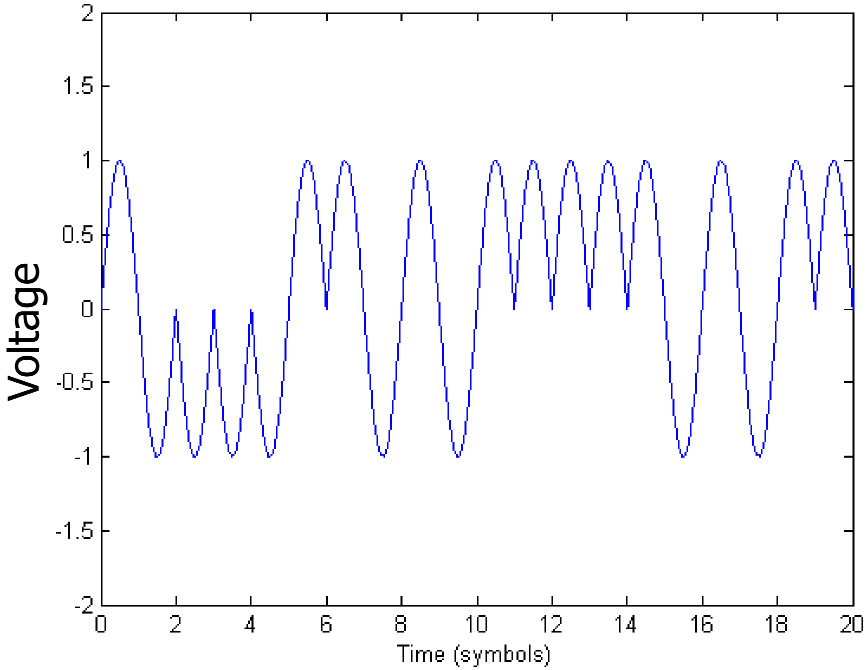
Sinusoidal Pulse Shape



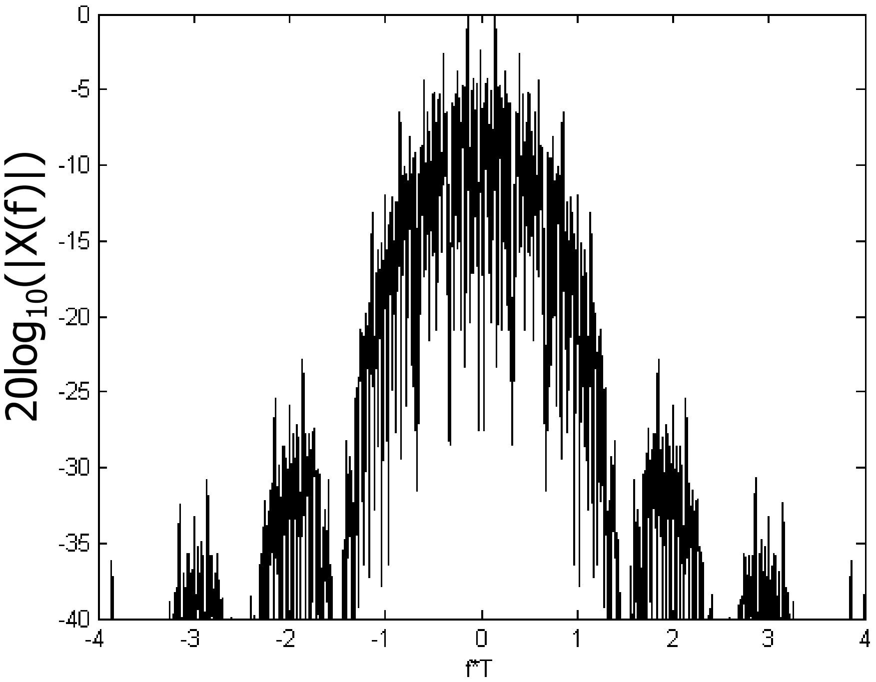
- **First Null BW: 1.5/T = 1.5R$_s$**
- **First Sidelobe: 22 dB down**

Example Signal

Example signal

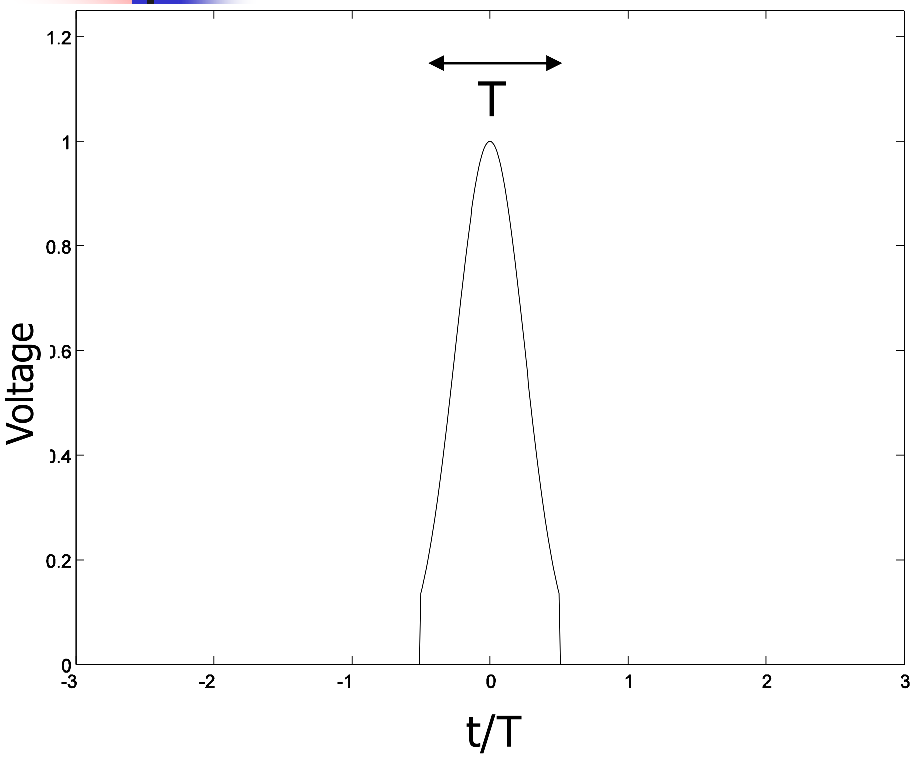


Example Spectrum

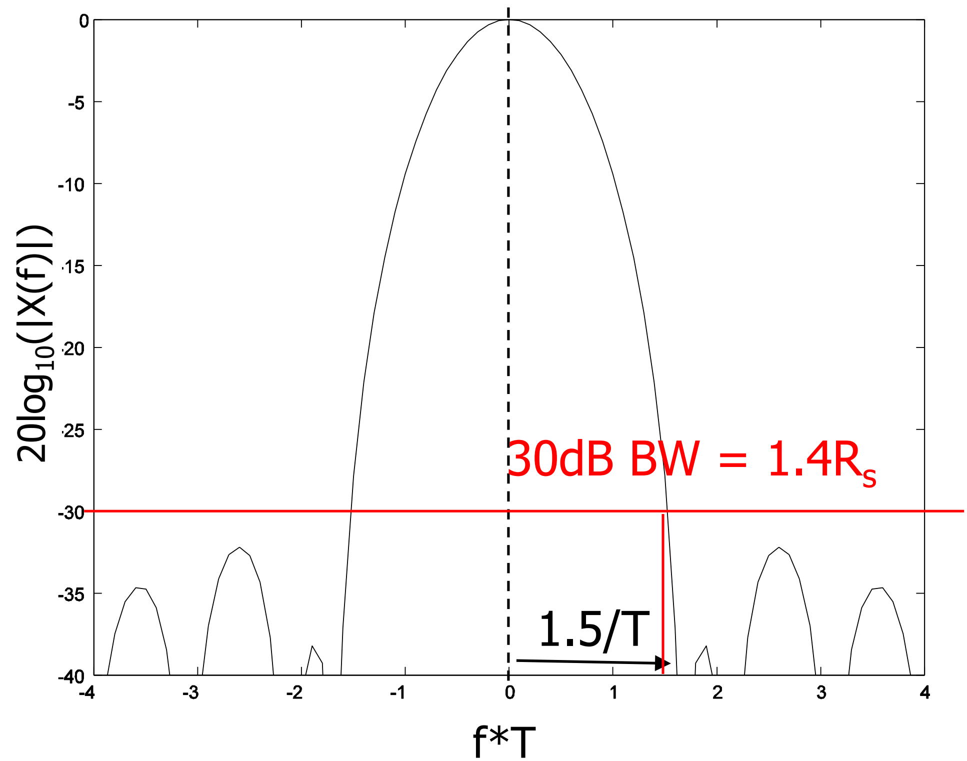


Truncated Gaussian Pulse

Time Waveform



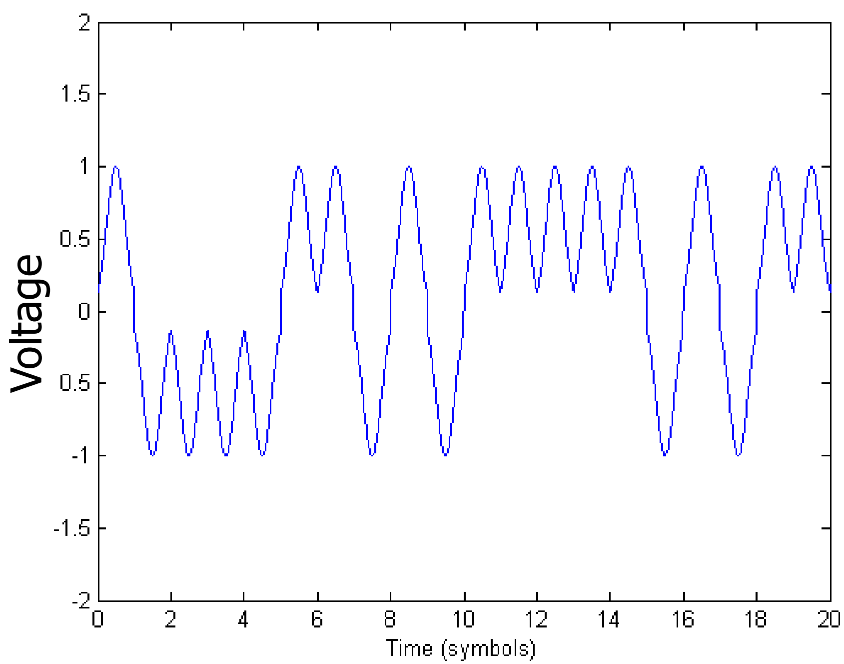
Magnitude Spectrum



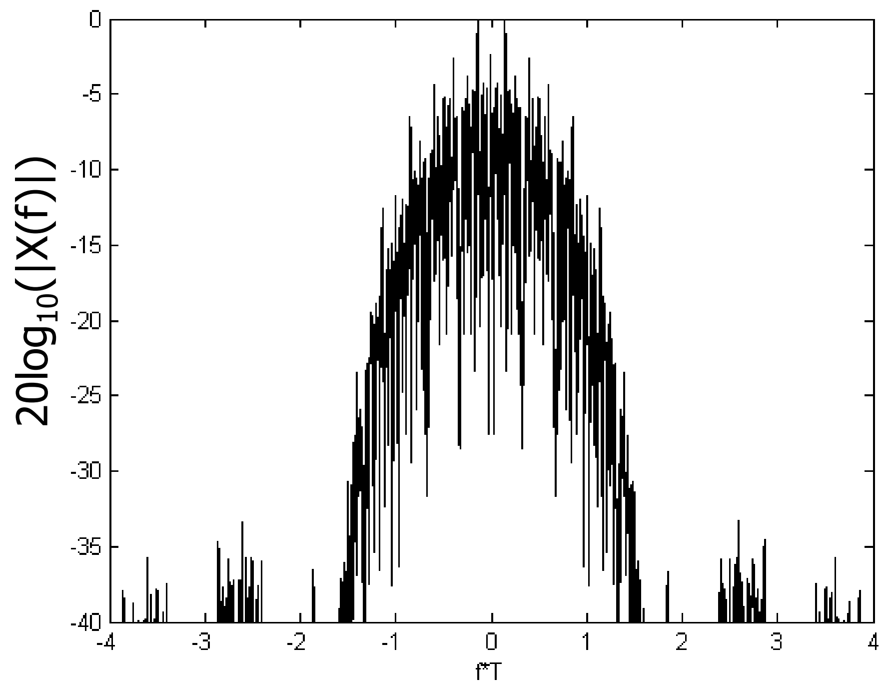
- **First Null BW: 1.5/T = 1.5Rs**
- **Largest Sidelobe: 31 dB down**

Example Signal

Example signal

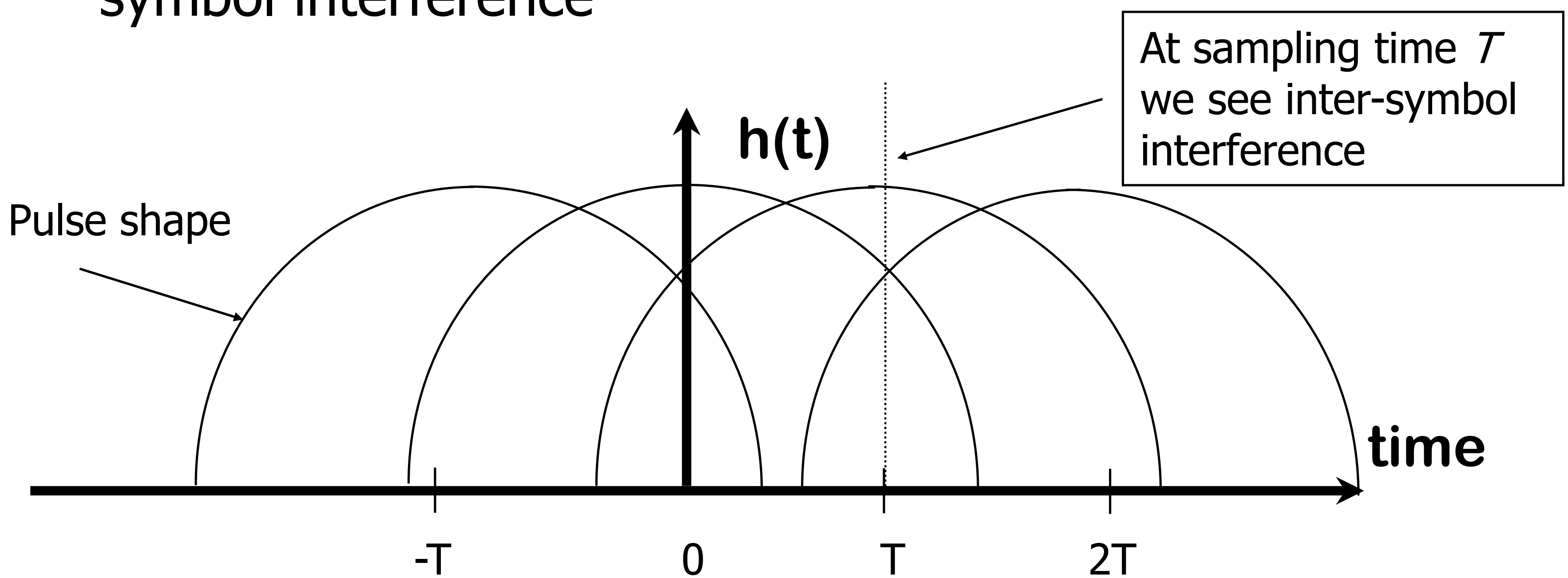


Example Spectrum

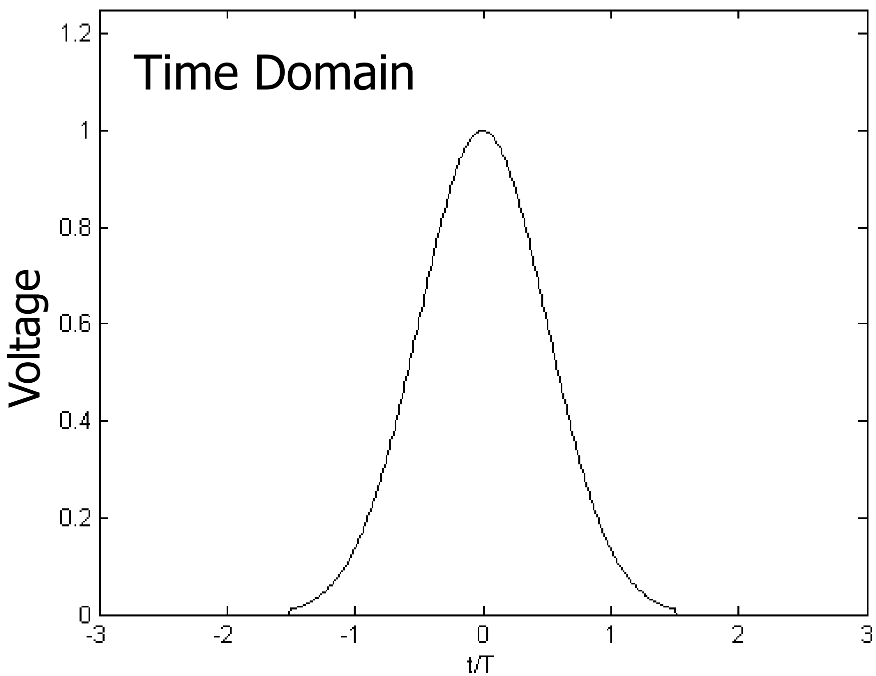


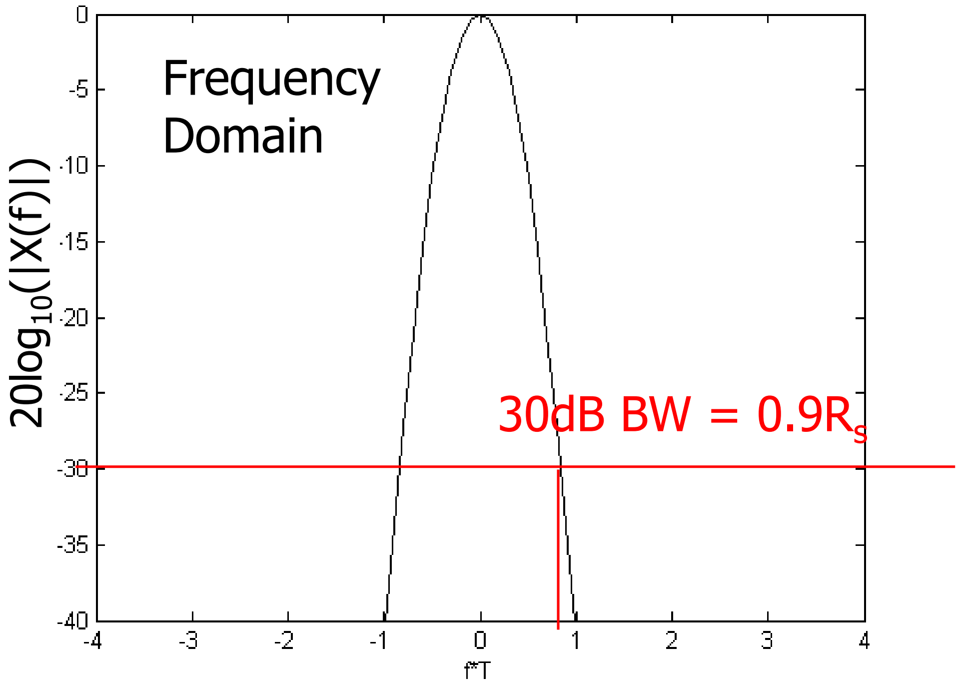
Elongating the pulse

- In order to reduce the bandwidth further, we must elongate the symbols to beyond one symbol duration.
- However, if pulses overlap they may produce inter-symbol interference



Example 1 – Exponential pulse

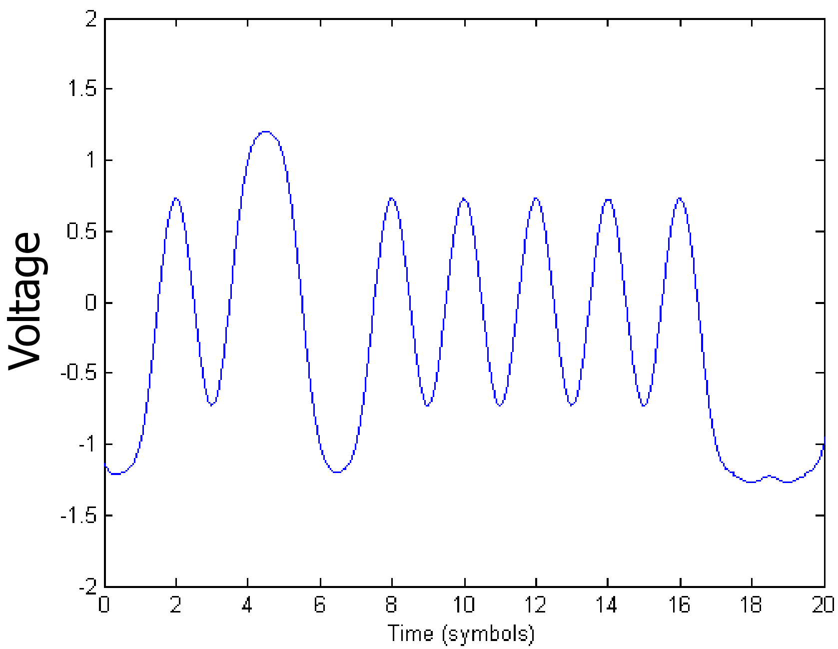


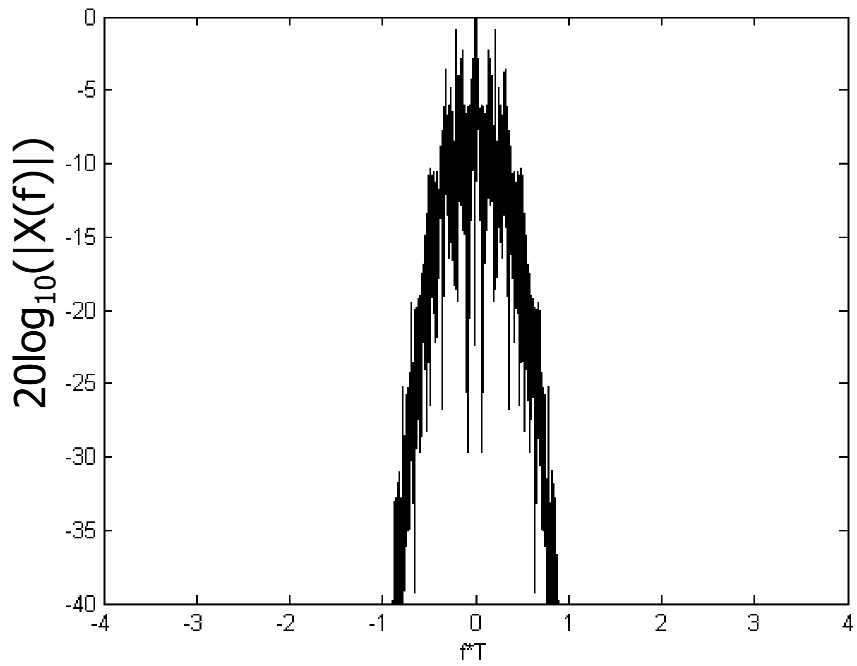


- **First Null BW: 1/T = 1R_s**
- **First Sidelobe: > 40 dB down**

Example - continued

- Unfortunately, this leads to inter-symbol interference





Nyquist's First Criteria for Zero ISI

- Overlapping pulses will not cause inter-symbol interference if they have zero amplitude at the time we sample the signal.
- Mathematically:

$$p(kT_s) = \begin{cases} C, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

  - where k is an integer and T_s is one symbol duration

- One function which can do this is the *sinc* function

Using the *sinc* function as a pulse shape

- The pulse shape is: $$g(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$
- The baseband transmit signal is

$$s(t) = \sum_n I_n g(t - nT_s)$$
$$= \sum_n I_n \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

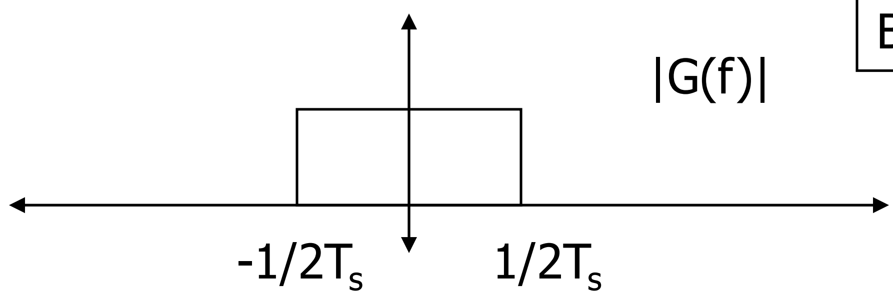
- And the resulting PSD is (for PSK modulation)

$$\boxed{S_{bpsk}(f) = \frac{1}{T_s}\left|G(f)\right|^2 = \frac{1}{T_s}\left|\mathcal{F}\{\mathrm{sinc}(t/T)\}\right|^2 = \frac{1}{T_s}\left|\Pi\left(\frac{f}{T_s}\right)\right|^2}$$

Using the *sinc* function as a pulse shape

- Satisfies Nyquist criterion (no ISI)
- Great BW properties

Baseband BW = $R_s/2$
Bandpass BW = R_s



- However, it requires a non-causal signal since the pulse must start before *t=0.* We can accommodate some non-causality through delays, however, we must truncate the pulse somewhere to avoid infinite delay.

Raised Cosine Pulse Family - Satisfies the Nyquist Criteria

- Frequency Domain:

$$G(f) = \begin{cases} \frac{\sqrt{E}}{2B_o} & 0 \le |f| < f_1 \\ \frac{1}{2}\frac{\sqrt{E}}{2B_o}\left[1+\cos\left(\frac{\pi}{2}\frac{(|f|-f_1)}{(B_o-f_1)}\right)\right] & f_1 \le |f| \le 2B_o - f_1 \\ 0 & |f| > 2B_o - f_1 \end{cases}$$

E = energy of the pulse

  - $B = 2B_o - f_1$ is the absolute bandwidth of the filter
  - B_o and f_1 are related through α which is termed the roll-off factor

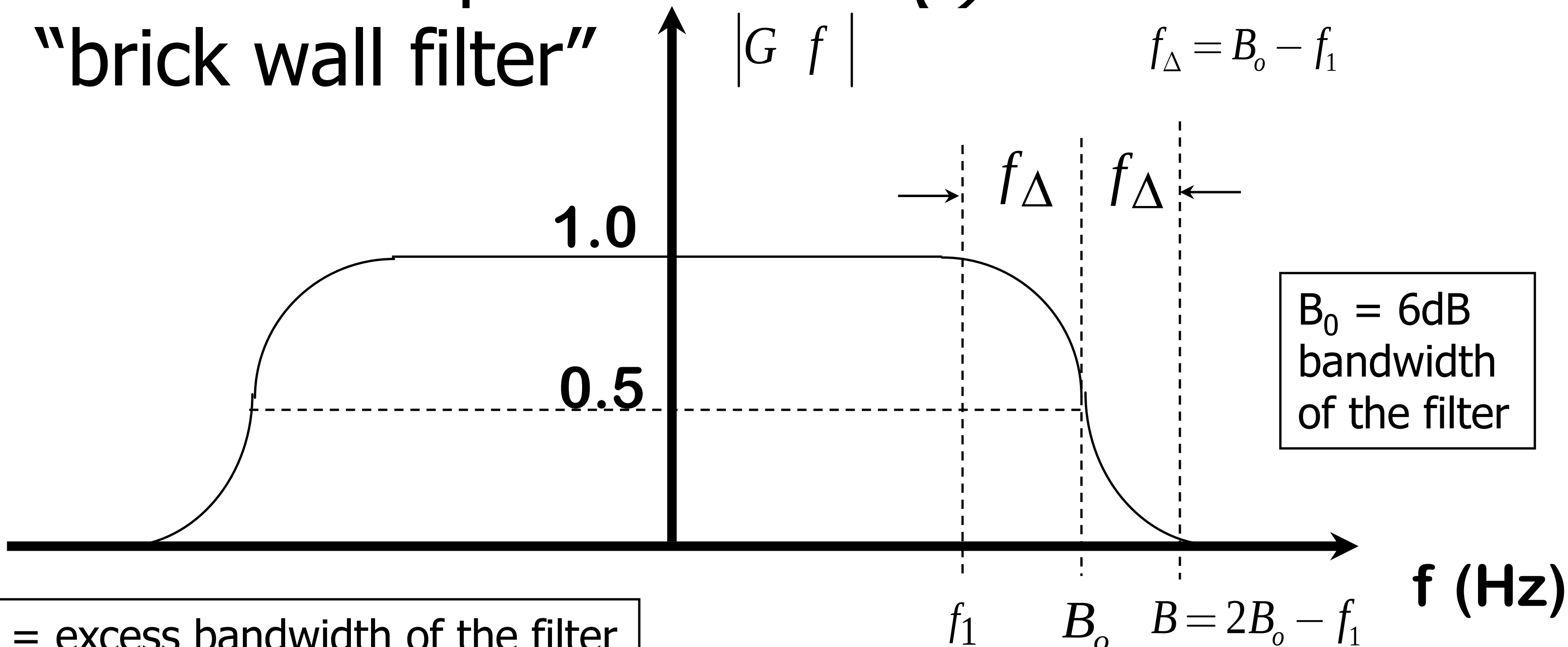
$$\alpha = 1 - \frac{f_1}{B_o}$$

- Time Domain: $g(t) = F^{-1}\left[H(f)\right] = \sqrt{E}\left(\frac{\sin(2\pi B_0 t)}{2\pi B_0 t}\right)\cdot\left[\frac{\cos(2\pi\alpha B_o t)}{1-(4\alpha B_o t)^2}\right]$

$$= \sqrt{E}\,\mathrm{sinc}(2B_0 t)\cdot\left[\frac{\cos(2\pi\alpha B_o t)}{1-(4\alpha B_o t)^2}\right]$$

Spectrum of Raised Cosine Pulse

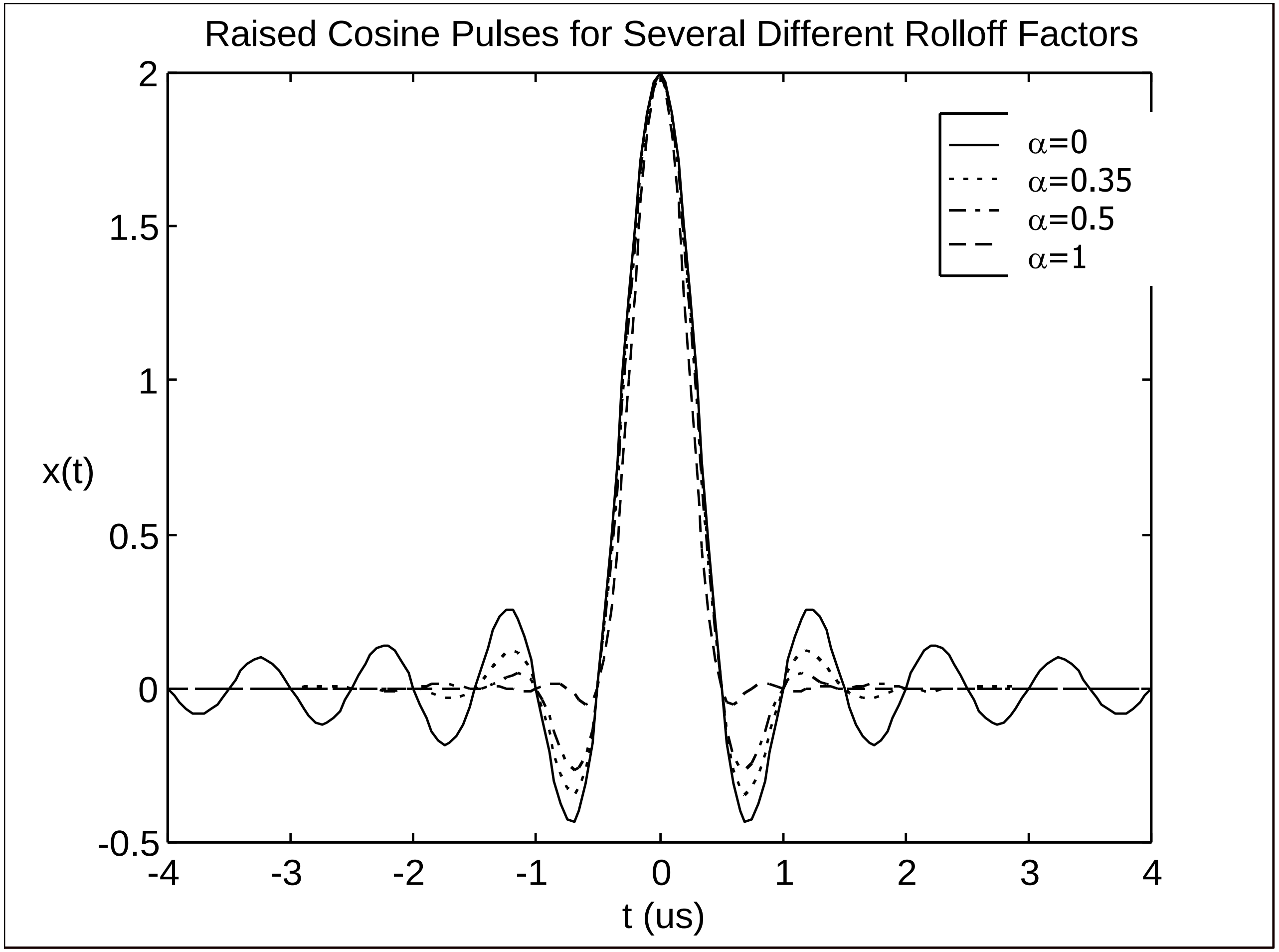
- $\alpha = 0$ corresponds to *sinc( )* function and "brick wall filter"



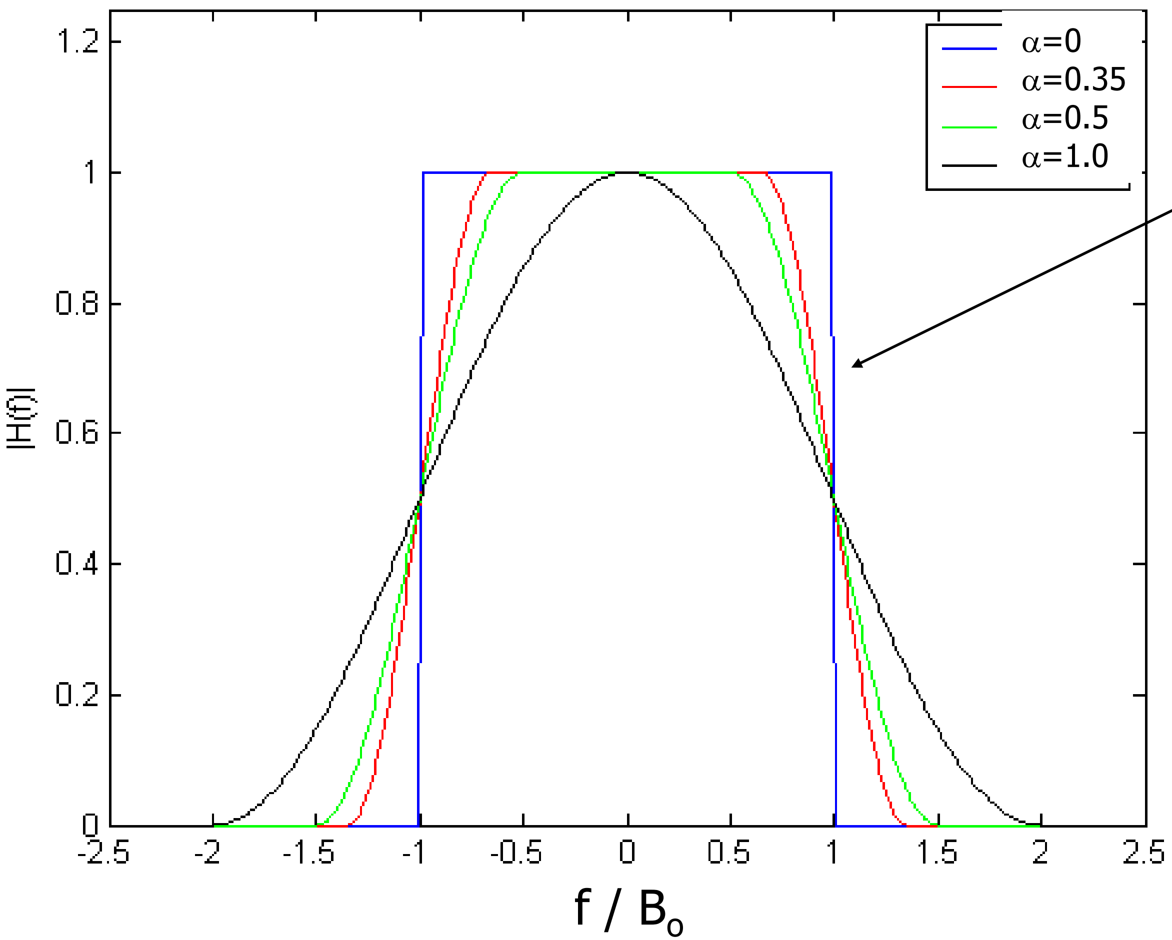
- f_Δ = excess bandwidth of the filter since it represents the bandwidth beyond the minimum.
- As α increases, f_Δ increases

$$\alpha = 0 \longrightarrow B = B_o = R_s / 2$$
$$\alpha = 1 \longrightarrow B = 2B_o = R_s$$

Raised Cosine Pulse - Time Domain



Raised Cosine Pulse - Frequency Domain



Increasing α increases bandwidth

Note: $B_o = R_s/2$

$B_{min} = B_o = R_s/2$
$B_{max} = 2B_o = R_s$

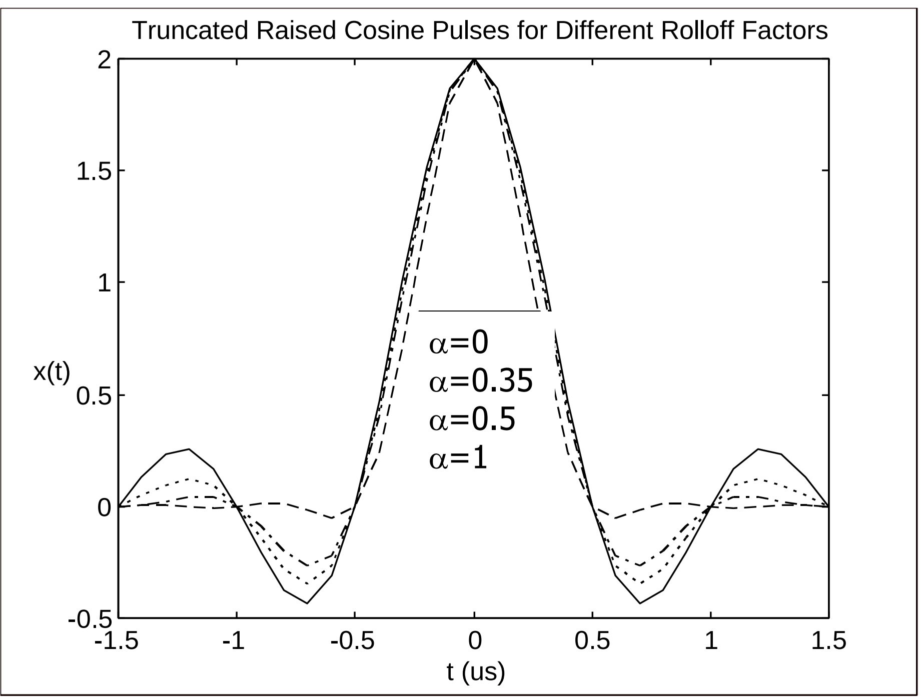
Implementation of Raised Cosine Pulse

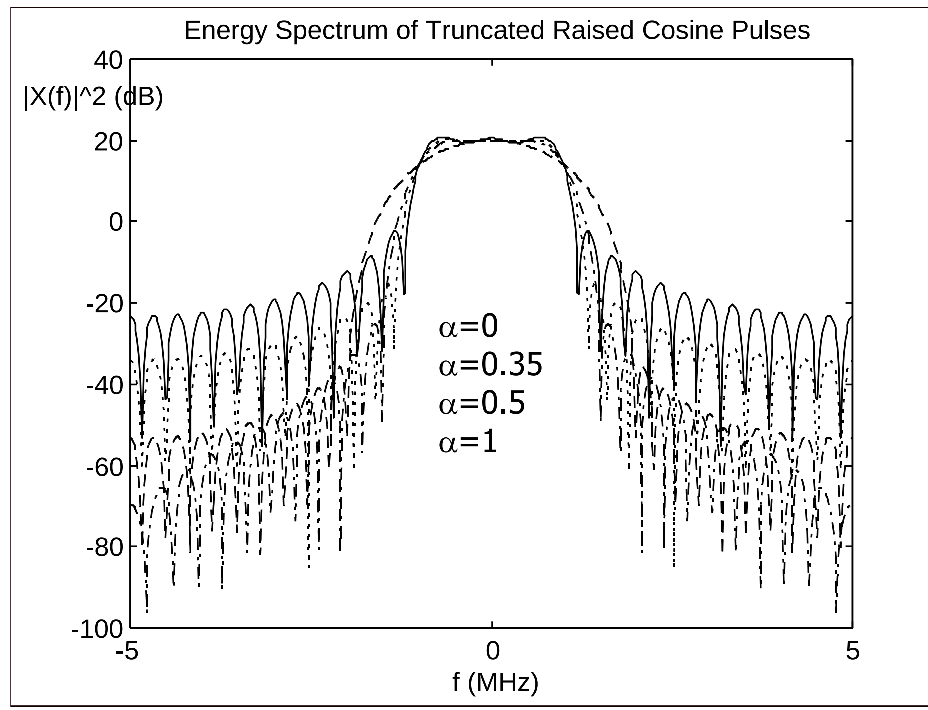
- Can be digitally implemented with an FIR filter
- Analog filters such as Butterworth filters may approximate the tight shape of this spectrum
- Practical pulses must be truncated in time
  - Truncation leads to sidelobes - even in RC pulses

- Typically a "square-root" raised cosine spectrum is used since a matched filter receiver requires identical filters by implemented at transmitter and receiver
  - We will discuss this more when we talk about "matched filtering"

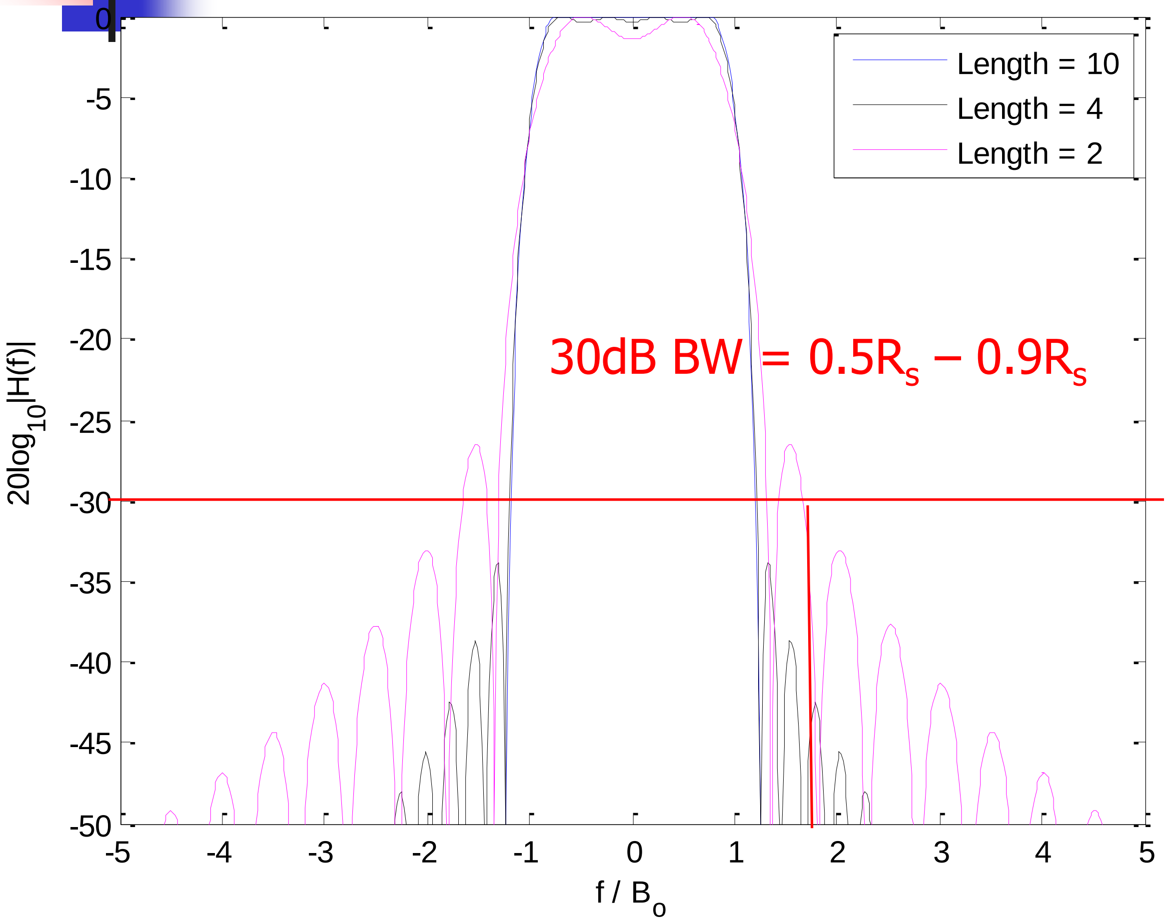
Truncated Raised Cosine Pulses

- Truncating raised cosine pulse to finite duration results in some sidelobes





Example



Note: $B_o = R_s/2$

- $\alpha = 0.25$
- Truncated to 10, 4, and 2 symbols
- For 10-symbol long approximation, we see no side-lobes within 50dB
- As truncation length gets smaller, side-lobes rise
- Larger truncation length requires larger delay to make pulse causal.

Bandwidth Requirements for **Bandpass** Modulation

- Optimum Pulse Shaping:

$$BW = R_s = \frac{R_b}{\log_2 M}$$

- Rectangular Pulse Shaping (rule of thumb):

$$BW = 4R_s = \frac{4R_b}{\log_2 M}$$

- Raised Cosine Pulse Shaping:

$$BW = R_s \cdot (1+\alpha) = \frac{R_b}{\log_2 M} \cdot (1+\alpha)$$

Bandwidth Efficiency:

- **Definition:** $\eta_B = R_b \,/\, \mathrm{BW}$ (bits/sec)/Hz
- Measures how efficiently a modulation type uses bandwidth
- Typical Values (assuming optimum pulse shaping):
  - BPSK: 1 bits/sec/Hz
  - QPSK: 2 bits/sec/Hz
  - 8-ary PSK: 3 bits/sec/Hz
  - 16 QAM: 4 bits/sec/Hz
  - 2-ary FSK: 0.5 bits/sec/Hz
  - 8-ary FSK: 3/8 bits/sec/Hz

Signal Space Representation

- How does pulse shaping impact signal space representation?
- We include the pulse in the basis function
  - Constellation doesn't change!

Example: M-PSK

- Assuming square pulses:

$$s_i(t) = \cos\left(2\pi f_c t + \frac{2\pi}{M} i\right) \quad 0 \le t \le T_s \quad i = 0, \ldots, M-1$$

- With general pulse shaping

$$s_i(t) = g(t)\cos\left(2\pi f_c t + \frac{2\pi}{M} i\right) \qquad i = 0, \ldots, M-1$$

Example: M–PSK (cont.)

$$s_i(t) = \cos\left(2\pi f_c t + \frac{2\pi}{M} i\right) \quad 0 \le t \le T_s \quad i = 0, \ldots, M-1$$

$$f_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\Big|_0^T \qquad f_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)\Big|_0^T$$

$$c_{1i} = \sqrt{\frac{T}{2}} \cos\left(\frac{2\pi}{M} i\right) \qquad c_{2i} = \sqrt{\frac{T}{2}} \sin\left(\frac{2\pi}{M} i\right)$$

$$s_i(t) = c_{1i} f_1(t) + c_{2i} f_2(t)$$

Example: M–PSK (cont.)

$$s_i(t) = g(t)\cos\left(2\pi f_c t + \frac{2\pi}{M} i\right) \quad 0 \le t \le T_s \quad i = 0, \ldots, M-1$$

$$f_1(t) = \sqrt{\frac{2}{T}\frac{1}{E_g}} g(t)\cos(2\pi f_c t)\Big|_0^T \qquad f_2(t) = \sqrt{\frac{2}{T}\frac{1}{E_g}} g(t)\sin(2\pi f_c t)\Big|_0^T$$

$$c_{1i} = \sqrt{\frac{T}{2}E_g}\cos\left(\frac{2\pi}{M} i\right) \quad c_{2i} = \sqrt{\frac{T}{2}E_g}\sin\left(\frac{2\pi}{M} i\right)$$

$$s_i(t) = c_{1i} f_1(t) + c_{2i} f_2(t)$$

Average Symbol Energy

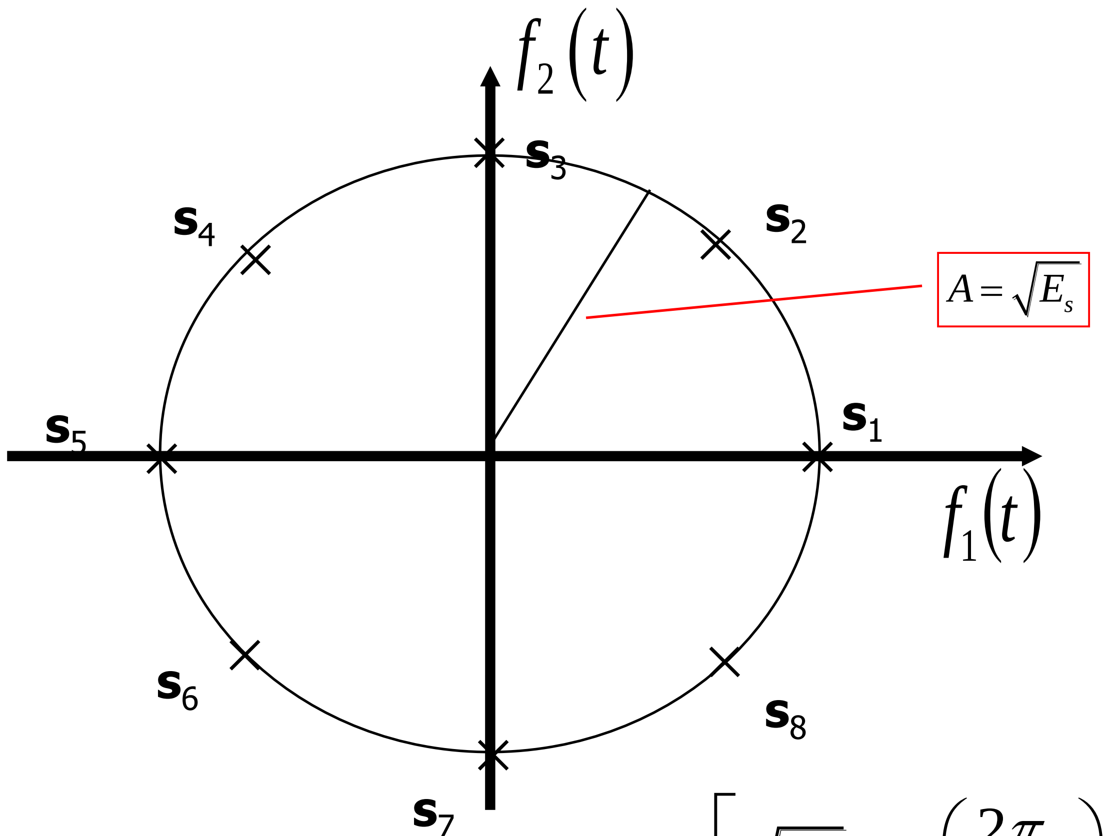
- Without Pulse shaping

$$E_s = \frac{T_s}{2}\cos^2\left(\frac{2\pi}{M}i\right) + \frac{T_s}{2}\sin^2\left(\frac{2\pi}{M}i\right)$$
$$= \frac{T_s}{2}$$
$$\mathbf{s}_i = \left[\sqrt{E_s}\cos\left(\frac{2\pi}{M}i\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi}{M}i\right)\right]$$

- With pulse shaping

$$E_s = \frac{T_s}{2}E_g\cos^2\left(\frac{2\pi}{M}i\right) + \frac{T_s}{2}E_g\sin^2\left(\frac{2\pi}{M}i\right)$$
$$= \frac{T_s}{2}E_g$$
$$\mathbf{s}_i = \left[\sqrt{E_s}\cos\left(\frac{2\pi}{M}i\right) \quad \sqrt{E_s}\sin\left(\frac{2\pi}{M}i\right)\right]$$

Ex: 8-ary PSK (either case)



$$\mathbf{s}_i = \left[\sqrt{E_s} \cos\left(\frac{2\pi}{M} i \right) \quad \sqrt{E_s} \sin\left(\frac{2\pi}{M} i \right) \right]$$

Conclusions

- Today we have examined the spectral properties of linearly modulated digital signals through the power spectral density (PSD)
- We have also shown the impact of pulse shaping with linearly modulated signals
- The PSD for other modulation schemes (e.g., FSK) can be found similarly