ECE 5654: Homework 1, Due: Saturday, Feb. 8, 2014

1. Derive the probability density function (PDF) of the sum of two independent Gaussian random variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

2. For a random variable S, its characteristic function is defined by $\Phi_S(\omega) \triangleq E[e^{j\omega S}]$. The random variable Y is defined as $Y = \sum_{i=1}^{n} X_i$, where $X_i = 1$ with probability p and $X_i = 0$ with probability $1-p$. In addition, the random variables X_is are independent.

   (a) Determine the characteristic function of Y.

   (b) Using the characteristic function, find the first and second order moments of Y, i.e., $E[Y]$ and $E[Y^2]$.

3. For a $\mathcal{N}(\mu, \sigma^2)$ random variable X, show that

$$E\left[(X-\mu)^n\right] = \begin{cases} 1 \times 3 \times 5 \times \cdots \times (2k-1)\sigma^{2k} = \frac{(2k)!\sigma^{2k}}{2^k k!}, & \text{for } n = 2k \\ 0, & \text{for } n = 2k+1 \end{cases}$$

   [Hint: first determine the characteristic function of X]

4. Suppose X is a random variable which is uniformly distributed on the interval $[0,1]$, i.e., $f_X(x)$ is 1 on the interval $[0,1]$ and it is 0 elsewhere.

   (a) Suppose that a random process (which is only defined for $t > 0$) is given by $Y(t) = e^{-tX}$. Find the CDF and PDF of the random variable $Y(t_0)$, where t_0 is a fixed positive number.

   (b) Find the mean and autocorrelation function of the random process $\{Y(t)\}$.

   (c) Is $Y(t)$ stationary ?

5. An exponential random variable X with mean λ has the following PDF

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed (i.i.d.) exponential random variables, each with mean λ.

   (a) Find the PDF of the random variable $Y = \min(X_1, X_2, \ldots, X_n)$.

   (b) Find the PDF of the random variable $Z = \max(X_1, X_2, \ldots, X_n)$.

6. Consider a sinusoidal process with *random phase*, defined by

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

   where A and f_c are constants and Θ is a random variable with the PDF

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine the autocorrelation function of $X(t)$.

(b) Determine the power spectral density of $X(t)$.

7. A random process $X(t)$ is defined by

$$X(t) = A\cos(2\pi f_c t)$$

where f_c is a constant and A is a Gaussian random variable with zero mean and variance σ^2. This random process is applied to an ideal integrator, which produces the output

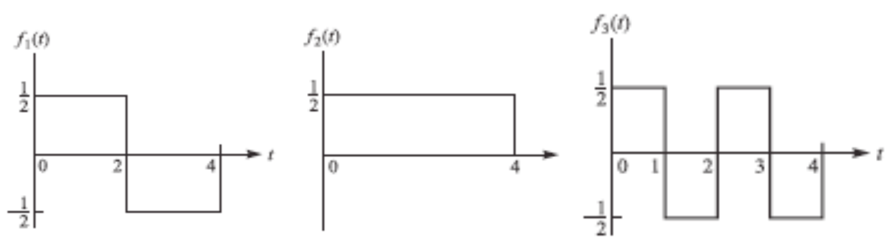
$$Y(t) = \int_0^t X(\tau)\, d\tau$$

(a) Determine the probability density function of the output $Y(t)$ at time t.

(b) Determine whether or not $Y(t)$ is stationary.

(c) Determine whether or not $Y(t)$ is ergodic.

8. Consider the three waveforms $f_1(t)$, $f_2(t)$ and $f_3(t)$ shown in the figure below



(a) Show that these waveforms are orthonormal.

(b) Can you express the following waveform $x(t)$ as a linear combination of $f_1(t)$, $f_2(t)$ and $f_3(t)$?

$$x(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & 1 \le t \le 3 \\ -1, & 3 \le t < 4 \\ 0, & \text{elsewhere} \end{cases}$$

If yes, determine the weighting coefficients.

(c) Repeat part (b) for the following waveform $y(t)$:

$$y(t) = \begin{cases} 2, & 1 \le t < 2 \\ -1, & 2 \le t \le 3 \\ 1, & 3 \le t < 4 \\ 0, & \text{elsewhere} \end{cases}$$