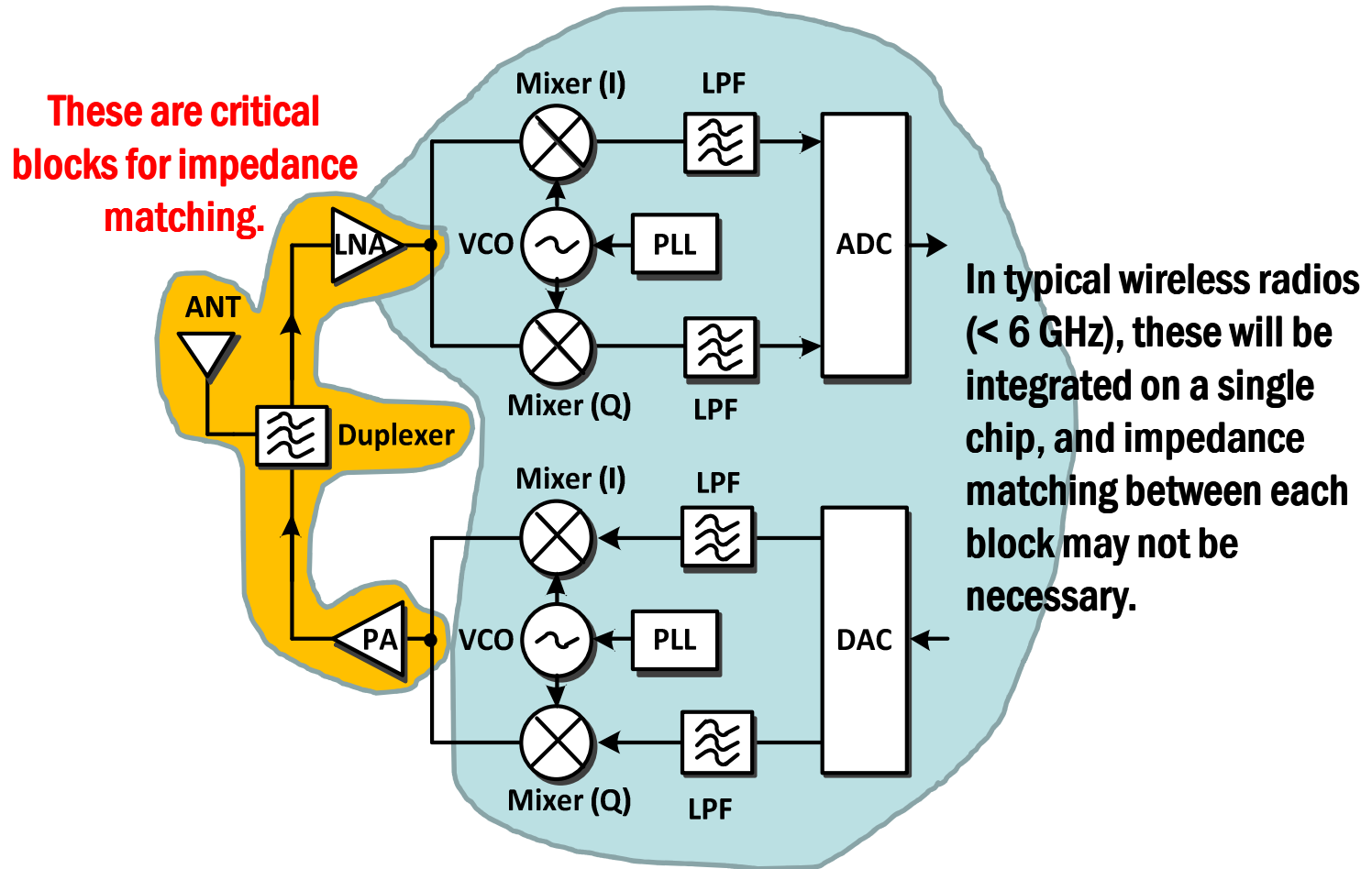
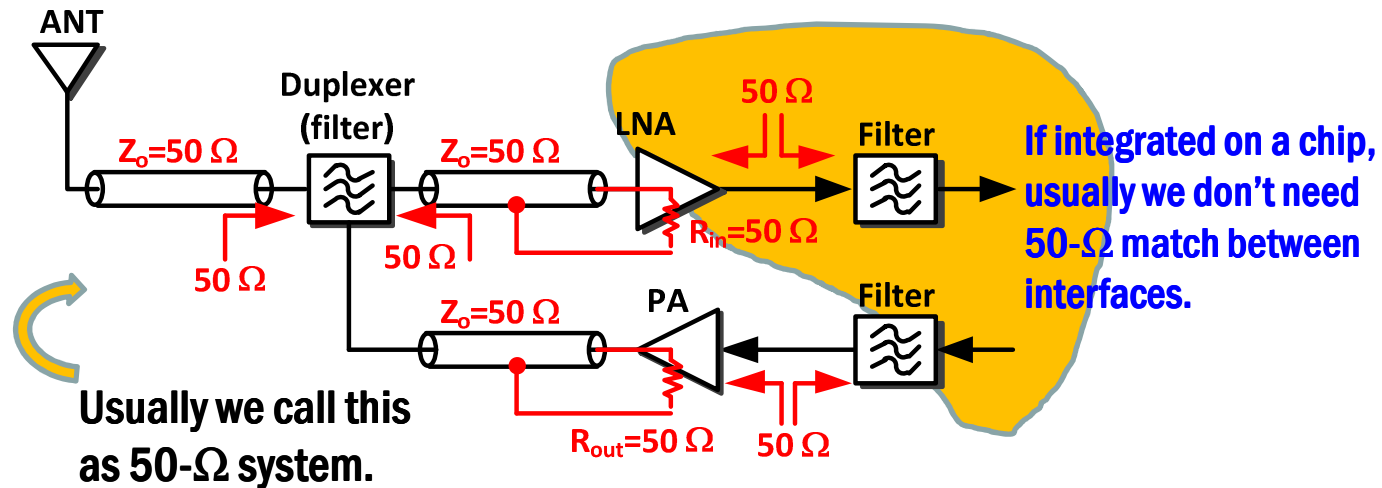

ECE 5220 RFIC Technology & Design (Impedance Matching)

Introduction to Impedance Matching



Introduction to Impedance Matching

Q) Where do we need impedance matching ?



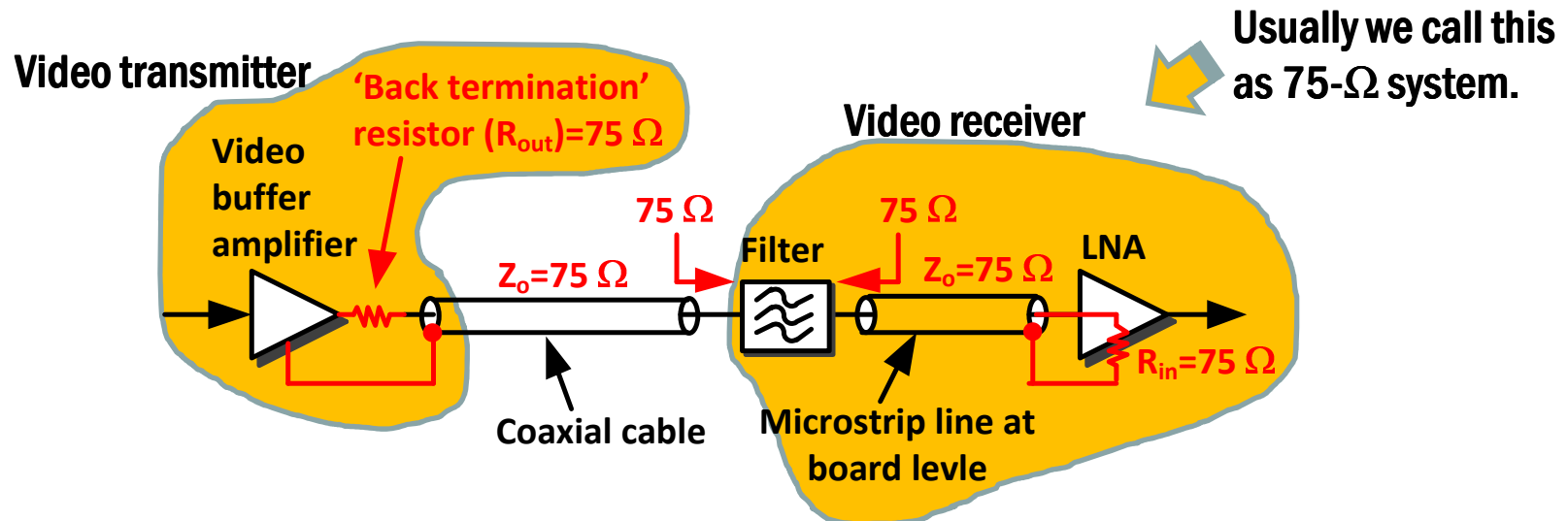
Usually we call this as 50-Ω system.

Ex: Wireless radio: Any interface with antenna needs to be matched with $50\ \Omega$. Any RFIC driving **external RF component (e.g. SAW filter, image rejection filter)** needs to be matched with $50\ \Omega$. Usually PA is external component and needs 50-Ω interface.

Q) Why $50\ \Omega$ (not $500\ \Omega$ for instance)?

Introduction to Impedance Matching

Q) Where do we need impedance matching ?



Ex) Video receiver (TV) system: Any interface with antenna (for terrestrial TV) and coaxial cable (for cable TV) needs to be matched with 75 Ω .

Q) Again, why 75 Ω ?

RF Energy Flow from Antenna to Radio

Air: RF energy exists in the form of E- & H-field energy in the air, and it propagate by alternating its form between E- & H- field. The ratio of strength of E- & H-field is characteristic impedance, Z_{air} (also called as “wave impedance”)

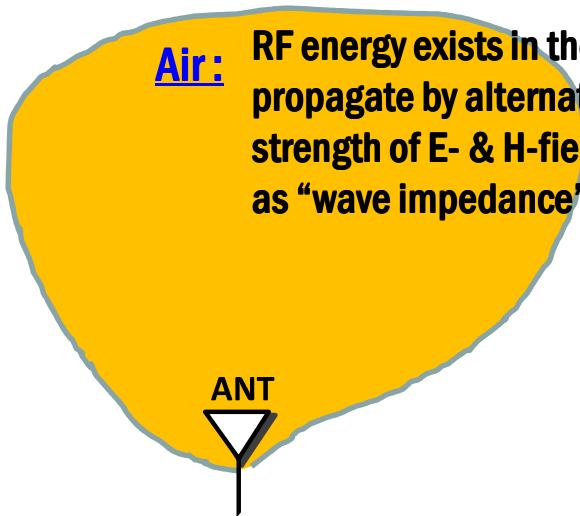
$$\text{characteristic impedance, } Z_{\text{air}} = \frac{E_{\text{Max}}}{H_{\text{Max}}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = 377 \, \Omega$$

μ_0 = permeability of air = $4\pi \times 10^{-7} \, \text{H/m}$

ϵ_0 = permittivity (dielectric constant) of air = $8.854 \times 10^{-12} \, \text{F/m}$

μ_r = relative permeability, air: $\mu_r = 1$

ϵ_r = relative permittivity, air: $\epsilon_r = 1$



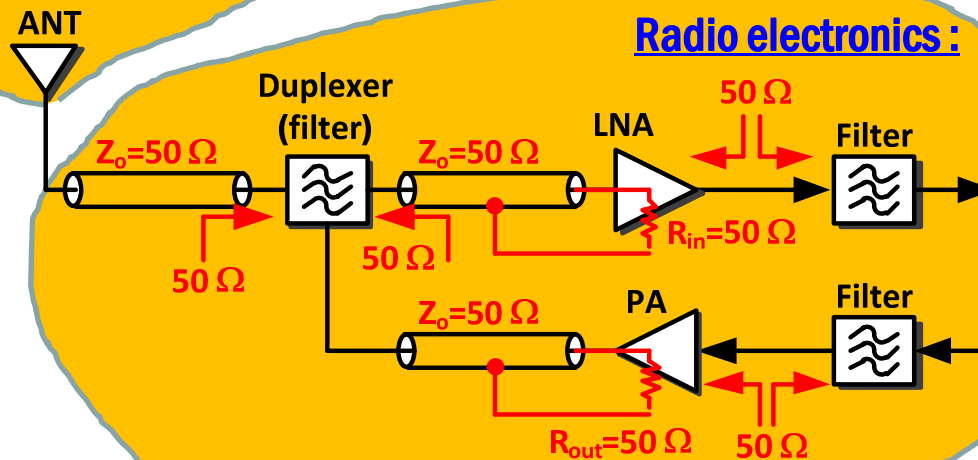
Example numbers,

- If electrical field strength (E) is measured at a receiving antenna is $1 \, \mu\text{V}_{\text{rms}}/\text{m}$.
Then, magnetic field strength (H) = $1 \, (\mu\text{V}_{\text{rms}}/\text{m}) / 377 \, \Omega = 2.65 \, \text{nA}_{\text{rms}}/\text{m}$.
Power density (P/m^2) = $E \times H = 1 \times 10^{-6} \times 2.65 \times 10^{-9} = 2.65 \times 10^{-15} \, \text{W}/\text{m}^2$.
If receiving antenna effective area is $5 \, \text{m}^2$, the overall incident power into antenna is $P = 2.65 \times 10^{-15} \, \text{W}/\text{m}^2 \times 5 \, \text{m}^2 = 13.25 \, \text{fW} = -108.77 \, \text{dBm}$

RF Energy Flow from Antenna to Radio

Air: RF energy exists in the form of E- & H-field energy in the air, and it propagate by alternating its form between E- & H- field. The ratio of strength of E- & H-field is characteristic impedance, Z_{air} (also called as “wave impedance”)

$$\text{characteristic impedance, } Z_{air} = \frac{E_{Max}}{H_{Max}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = 377 \, \Omega$$



Radio electronics: RF energy exists in the form of V & I. And characteristic impedance will be expressed by the ratio of V/I.

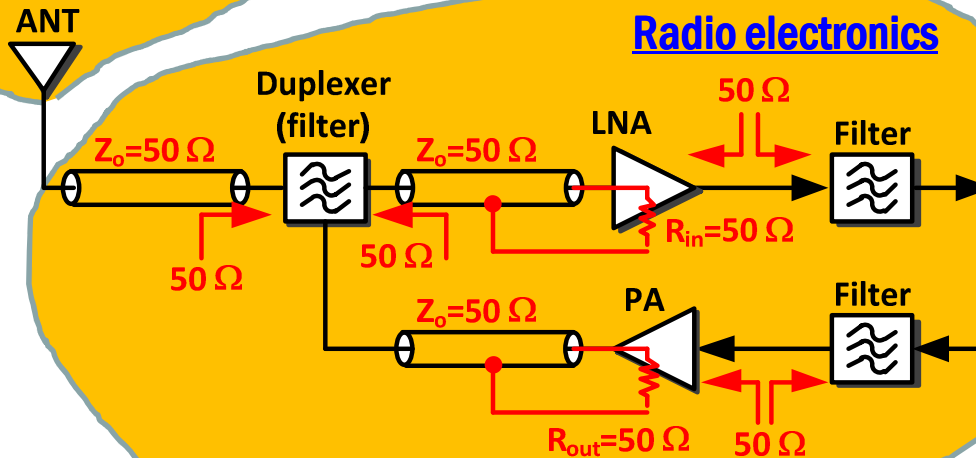
RF Energy Flow from Antenna to Radio

Air

characteristic impedance,

$$Z_{\text{air}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = 377 \, \Omega$$

So, antenna can be regarded as a “sensor” or “transducer” which transforms the RF energy from wave energy (in E- & H-fields) to electrical energy (in V and I forms).



RF Energy Flow from Antenna to Radio

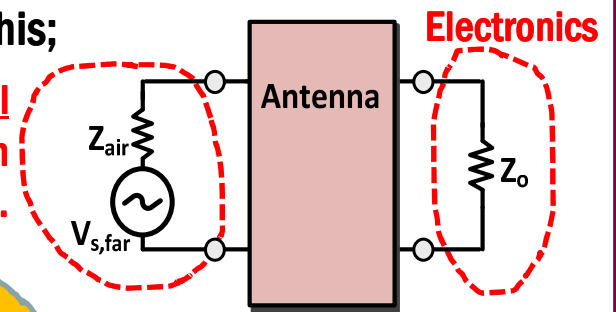
Air

characteristic impedance,

$$Z_{\text{air}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = 377 \, \Omega$$

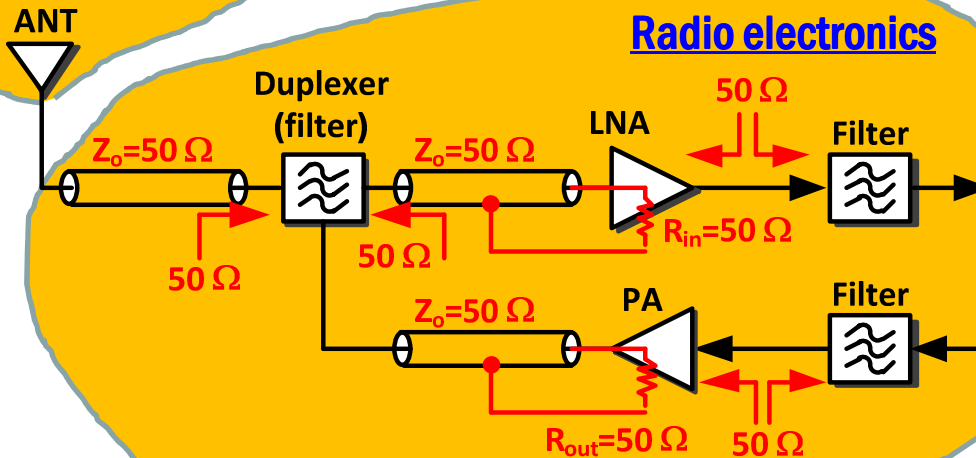
Now we have two different impedance systems, and we can approximately model them as this;

Mental model
of far-field air medium
under wave propagation.



$V_{s, \text{far}}$ is an equivalent source that generate E-field in far-field air medium.

Radio electronics



RF Energy Flow from Antenna to Radio

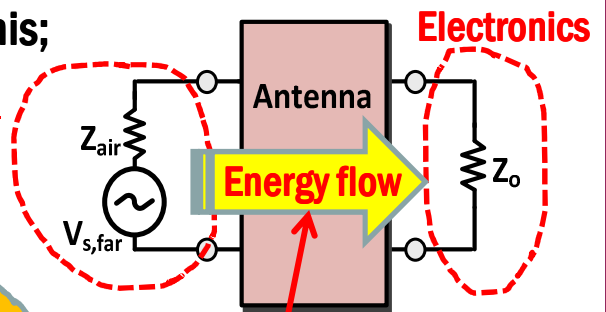
Air

characteristic impedance,

$$Z_{\text{air}} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = 377 \Omega$$

Now we have two different impedance systems, and we can approximately model them as this;

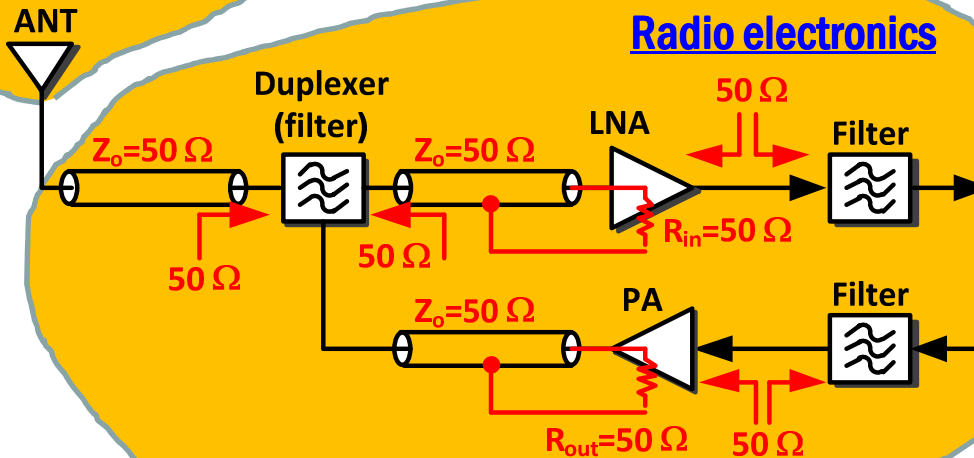
Mental model
of far-field air medium
under wave propagation.



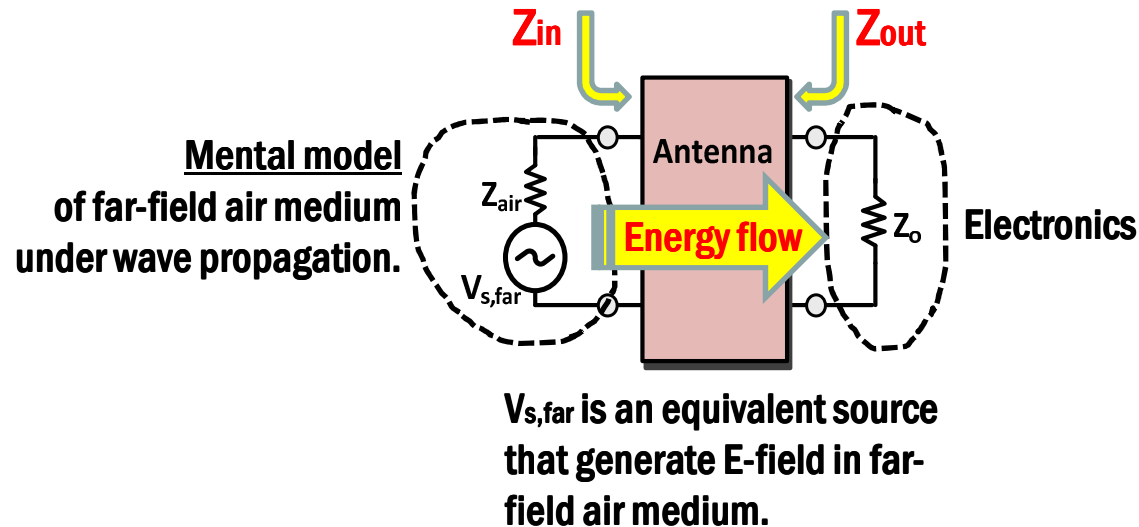
$V_{s, \text{far}}$ is an equivalent source that generate E-field in far-field air medium.

Antenna is the device to maximize this energy flow.

Radio electronics



Maximum Energy (Power) Transfer



Energy

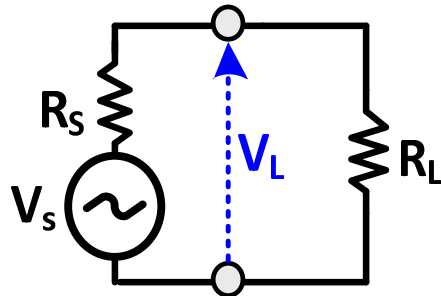
$$= \int \text{Power} \times dt$$

Q) To maximize the **energy** flow from air medium to electronics, what should be Z_{in} and Z_{out} ?

Q) To maximize the **power** flow from air medium to electronics, what should be Z_{in} and Z_{out} ?

Maximum Power Transfer Theory

To answer the question, let's review "Maximum Power Transfer Theory"



For arbitrary load resistance, R_L ,

$$V_L = \frac{R_L}{R_s + R_L} \times V_s$$

Power delivered to the load R_L

$$P_L = \frac{V_L^2}{R_L} = \frac{1}{R_L} \left(\frac{R_L}{R_s + R_L} \times V_s \right)^2$$

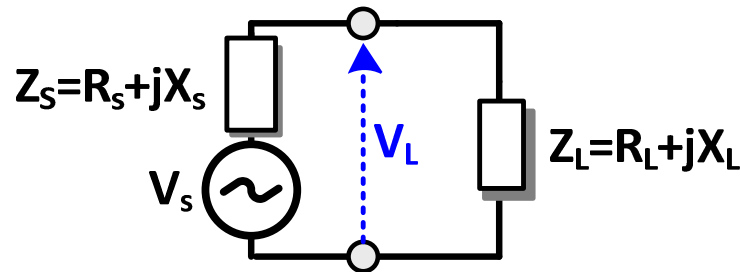
Note, V_L is just $\frac{1}{2}V_s$ at $R_L = R_s$; i.e. voltage efficiency is only 50%. This is just what we can do best when maximizing power delivery.

P_L will be maximum at $R_L = R_s$,

$$\therefore \frac{\partial P_L}{\partial R_L} = 0, @ R_L = R_s.$$

$$P_{L,Max} = \frac{V_L^2}{4R_s} = \frac{V_L^2}{4R_L}$$

Maximum Power Transfer Theory



For arbitrary source impedance of Z_s and load impedance of Z_L ,

$$P_{L,Max} = \frac{V_L^2}{4R_s} = \frac{V_L^2}{4R_L}$$

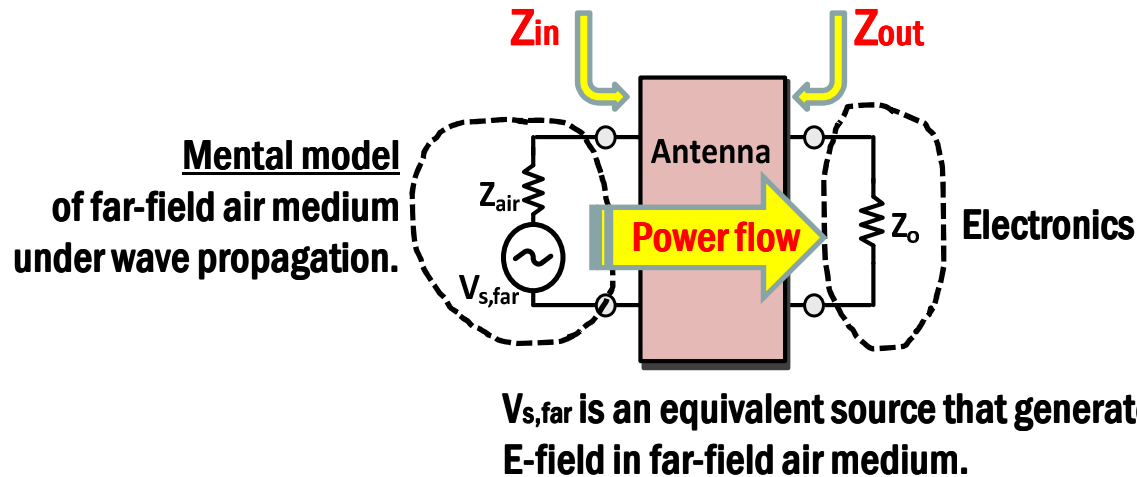
when $Z_L = Z_s^*$ (you can verify this easily).

Conclusion: For maximum power transfer to the load, we need to match the load impedance Z_L to the conjugate of source impedance Z_s . This is called as “**conjugate matching**”, and when we say “impedance matching” usually it means for the conjugate matching for maximum power transfer .

Caution: there is also “impedance matching for minimum noise figure”, called “noise matching”. We will discuss it later in noise lecture.

Maximum Energy (Power) Transfer

Now let's come back to this issue:



Q) To maximize the **power** flow from air medium to electronics, what should be Z_{out} ?

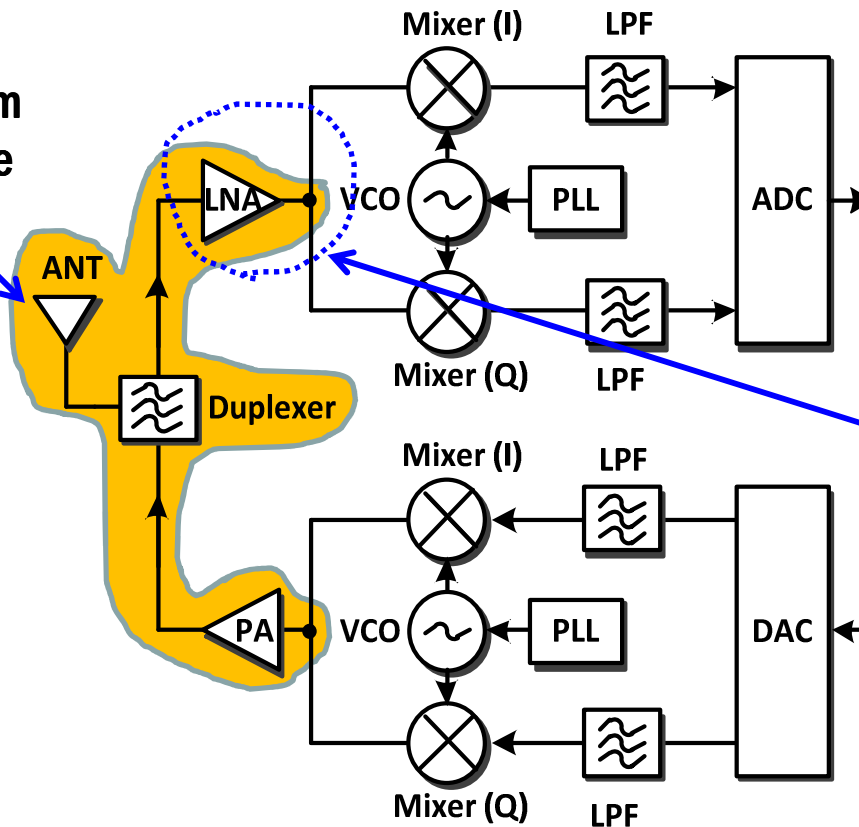
Ans) Z_{in} should be $Z_{in}=Z_{air}=377\ \Omega$, and Z_{out} should be $Z_{out}=Z_o=50\ \Omega$.

*In a strict sense, it's not quite true, and it's hard to define exact impedance at near-field region of antenna.

This discussion provides another point of view of the antenna; i.e., it is impedance transformer from air characteristic impedance to electronic characteristic impedance.

Objective Of Impedance Matching

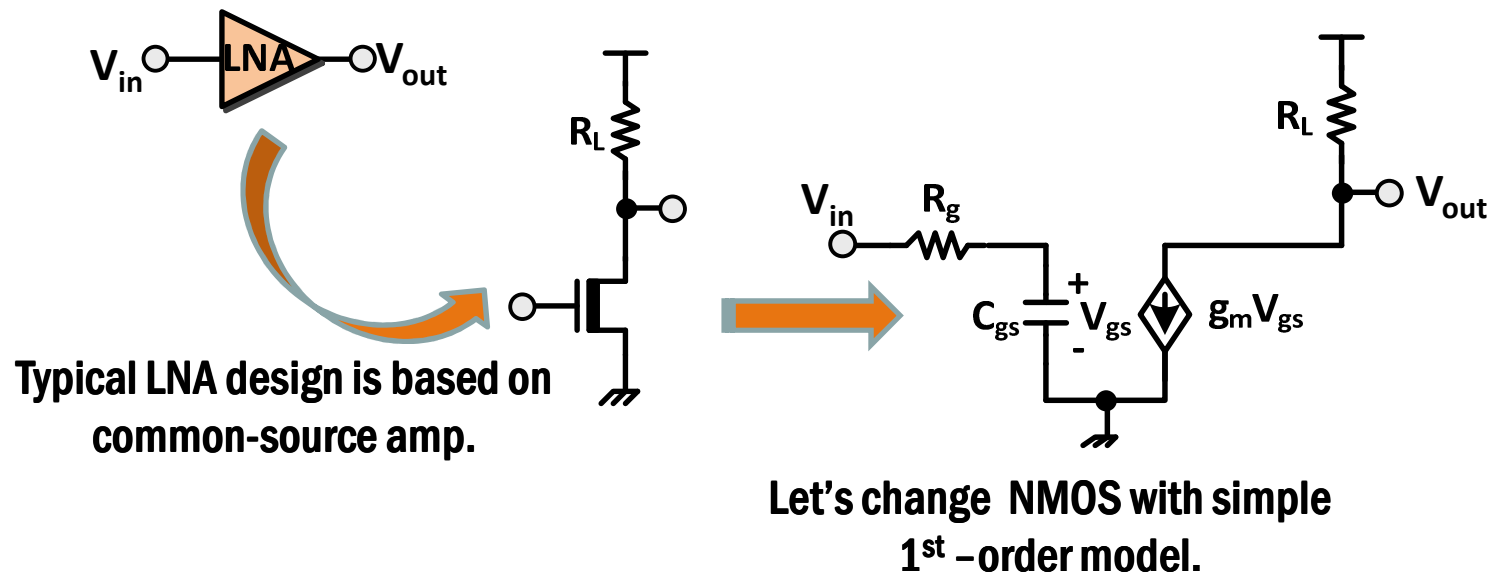
Until now, we have considered maximum power transfer in the antenna



Now, let's examine how we can make maximum power transfer inside RF IC.

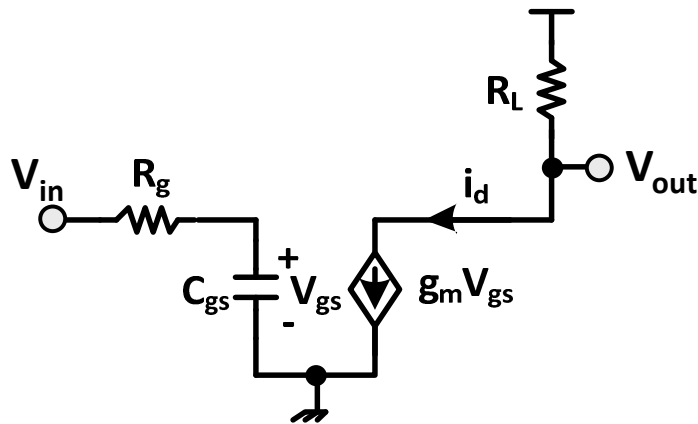
To do this, let's think about this LNA design.

Objective Of Impedance Matching



Objective) To maximize the signal power delivered to load, R_L , from input V_{in} .

Maximize Power Delivered to Load



First, let's calculate power delivered to the load, R_L .

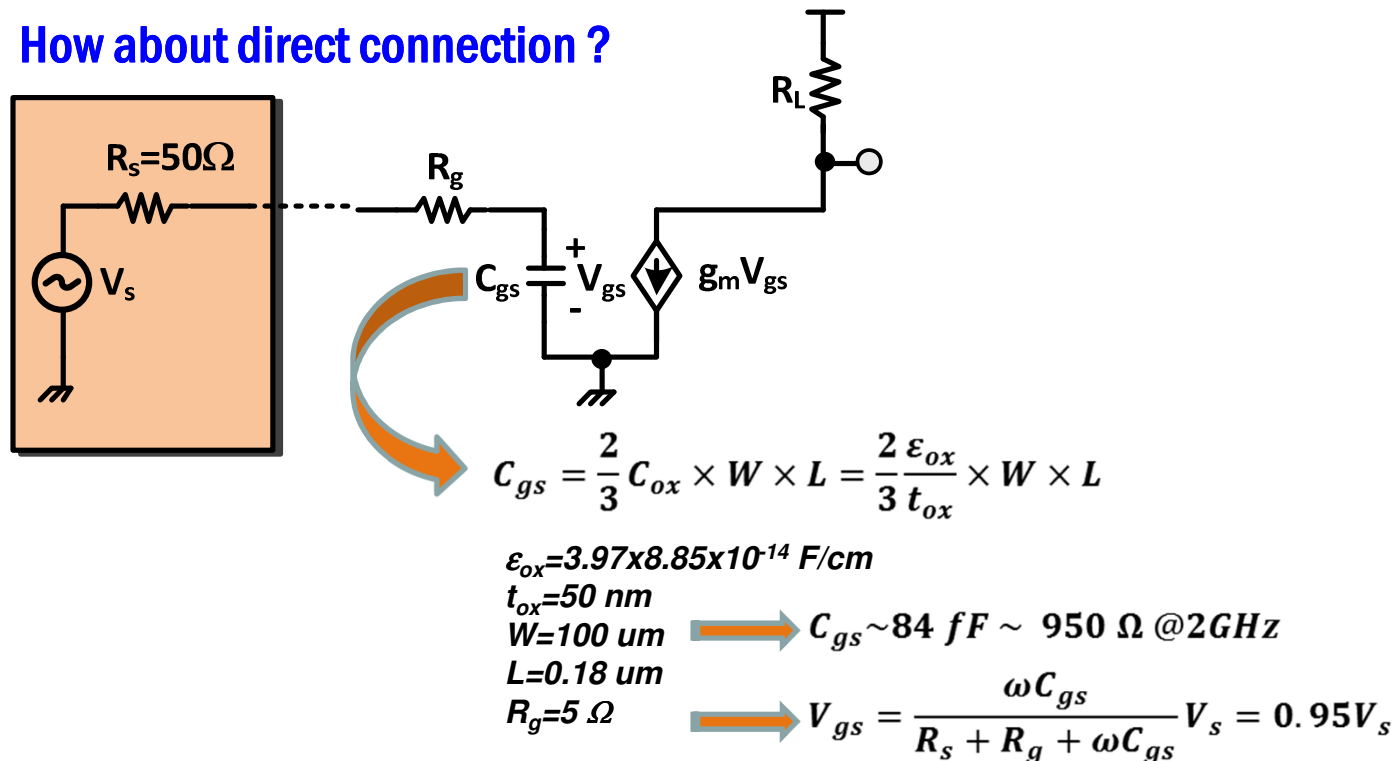
$$\begin{aligned}
 P_{out} &= |i_d|^2 \times R_L \\
 &= |g_m V_{gs}|^2 \times R_L
 \end{aligned}$$

Apparently, to maximize P_{out} , voltage across C_{gs} needs to be maximized. (since, g_m is constant for a given bias condition of the NMOS)

Q) How can we maximize voltage across C_{gs} ?

Power Transfer Through Direct Connection

How about direct connection ?



Virtually all of V_s appears across C_{gs} . Not bad !!

Q) Can we do better ?

Improved Power Transfer Technique

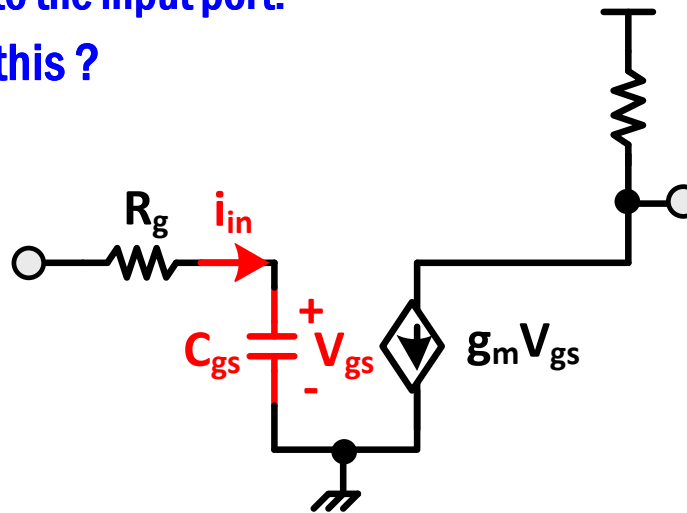
Think like this !

Voltage across C_{gs} will be maximized by maximizing current through C_{gs} (i_{in}).

Maximum current through C_{gs} leads to maximum power delivered to input port.

Conclusion: We might be able to build more voltage across C_{gs} by maximizing the power delivered to the input port.

Q) How can we do this ?



Improved Power Transfer Technique

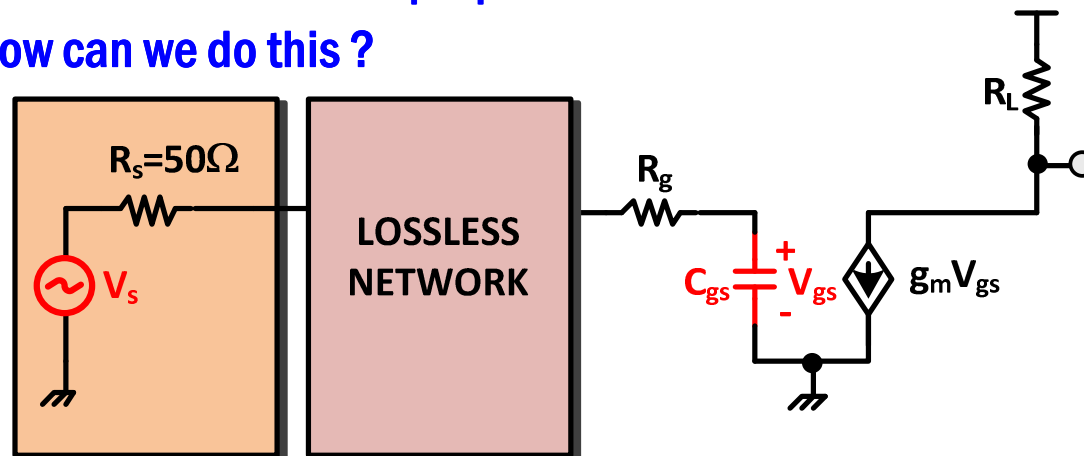
Think like this !

Voltage across will be maximized by maximizing current through C_{gs} (i_{in}).

Maximum current through C_{gs} leads to maximum power delivered to input port.

Conclusion: We might be able to build more voltage across C_{gs} by maximizing the power delivered to the input port.

Q) How can we do this ?



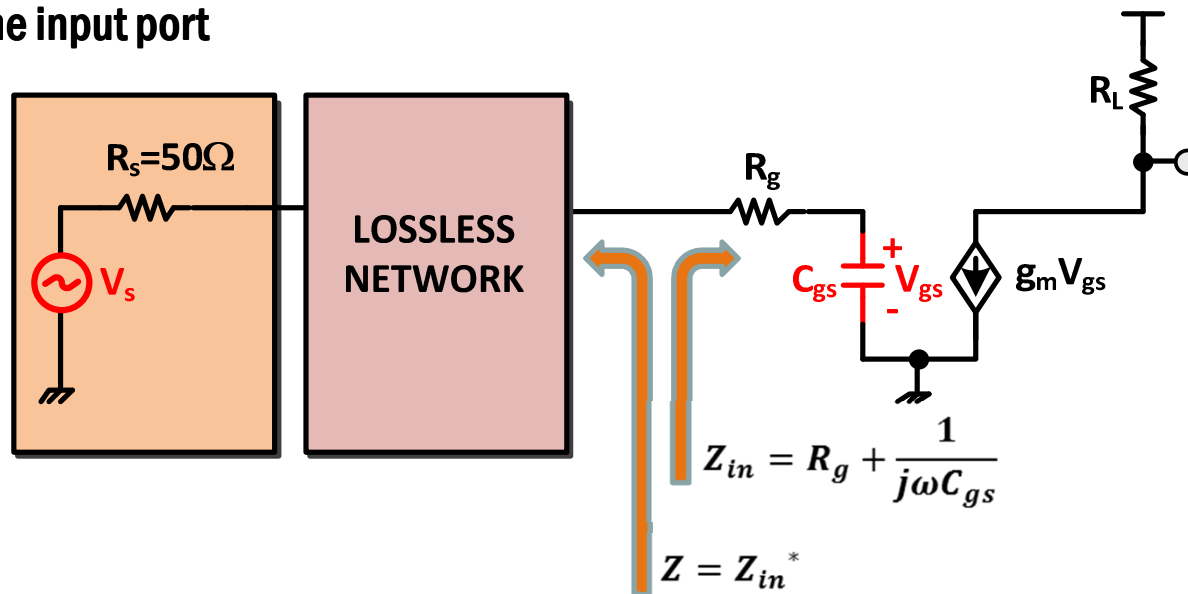
Eventually, we will find that by inserting some lossless network, we can maximize power delivered to the input port, and thereby, V_{gs} can be even larger than V_s .

Let's examine this !

Impedance Matching

Recall “**Maximum Power Transfer Theory**”

If we design the lossless network such that impedance looking back from NMOS to source is Z_{in}^* , we will get maximum power transfer from the source to the input port

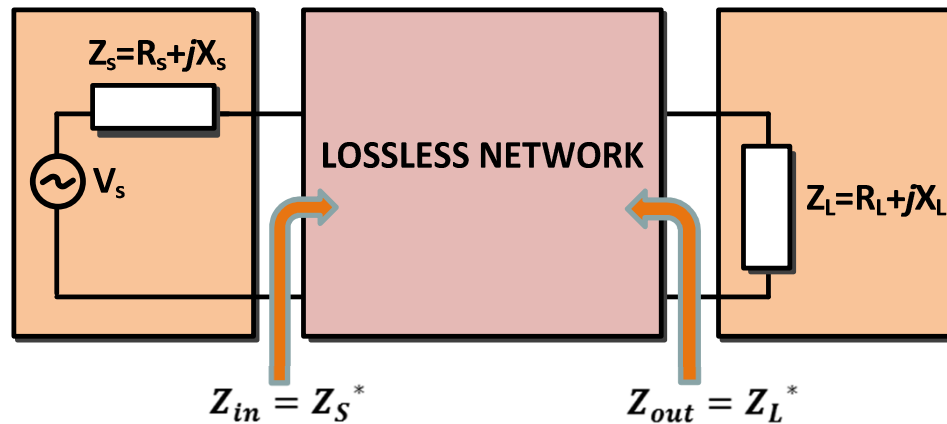


We call this as “impedance matching” and the lossless network as “impedance matching network”.

Let’s examine how to design the matching network !

Impedance Matching

Let's consider how to perform impedance matching for a general load and source impedance

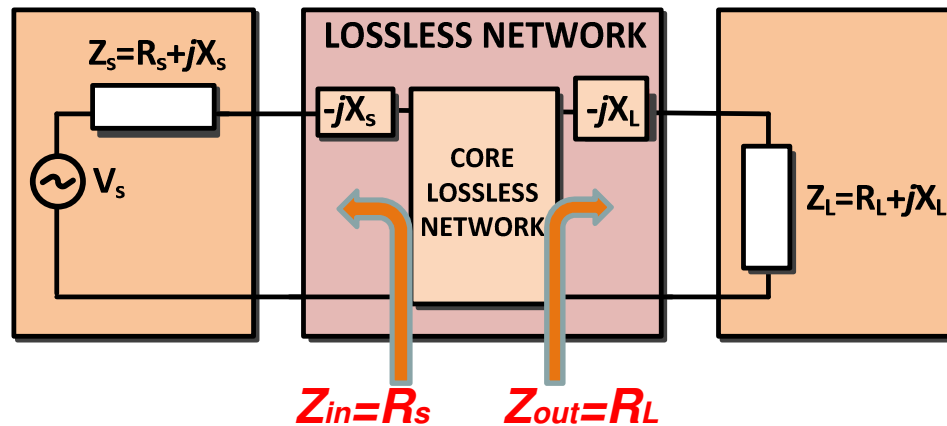


If we design a lossless network such that $Z_{out} = Z_L^*$, then $Z_{in} = Z_s^*$.
 The converse is also true, i.e., if we design the lossless network such that $Z_{in} = Z_s^*$, then $Z_{out} = Z_L^*$.

Can you come up with a simple intuitive explanation (without using equation) as to why the above statements are true ?

Impedance Matching: The First Step

First, let's cancel the imaginary parts of Z_s and Z_L . This can be done by an appropriate choice of capacitor or inductor sizes.



The problem is now reduced to how to design the core lossless network to match load resistance R_L to source resistance R_s .

We have two cases; 1) $R_s > R_L$ & 2) $R_s < R_L$.

Let's think about the case-1) first.

Impedance Up-Conversion

Case-1) $R_s > R_L$

We need to convert the smaller load impedance to a larger value of R_s (this is referred as upward impedance conversion).

Think !!!

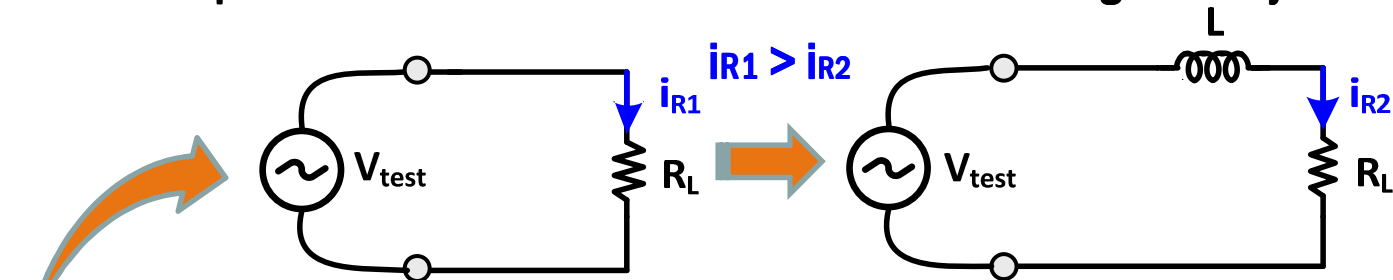
If we generate smaller voltage across R_L using the source voltage V_s , this would results in smaller current being drawn through R_L . This makes R_L look like a larger R .

How can we do this ? Note we are only allowed to use lossless elements, i.e., inductors and capacitors.

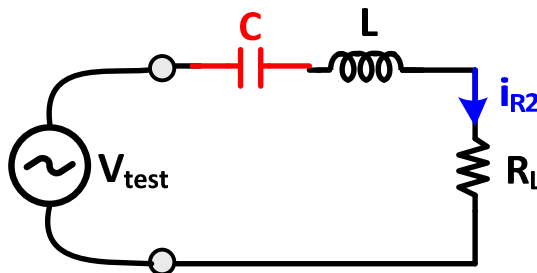
Impedance Up-Conversion

Case-1) $R_s > R_L$

Let's put an inductor in series with the resistor. This would increase input impedance which would result in smaller current being drawn by the resistor.



Now the impedance is no longer purely real value, and we need to turn the impedance into a purely real value. **How about adding C in series with the L ?**



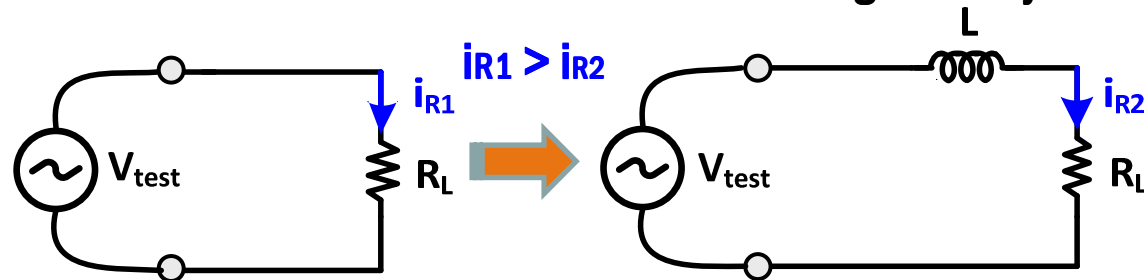
NO !!!

The C will be resonated with L in series, and undo what the L was intended to do; i.e., reduce current through the R_L .
 $i_{R1} = i_{R2}$, same as the original.

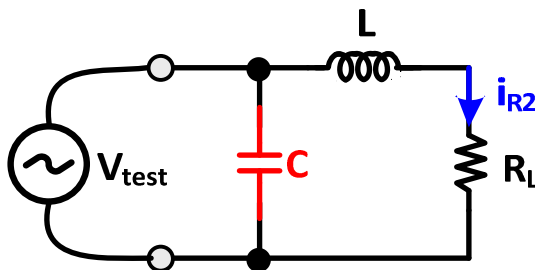
Impedance Up-Conversion

Case-1) $R_s > R_L$

Let's put an inductor in series with the resistor. This would increase input impedance which would result in smaller current being drawn by the resistor.



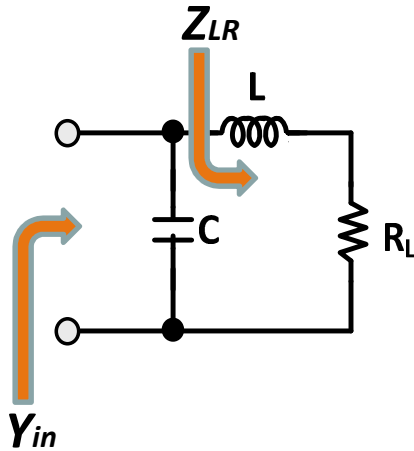
Now the impedance is no longer purely real value, and we need to turn the impedance into a purely real value. **How about adding C in parallel with the L & R_L ?**



YES !!!

The C will be resonated with L in parallel, which gives pure resistance. i_{R2} is still smaller than original, and therefore, effective R_L will be increased. Let's verify this !

Impedance Up-Conversion



Let's calculate input admittance !

$$Z_{LR} = R_L + j\omega L$$

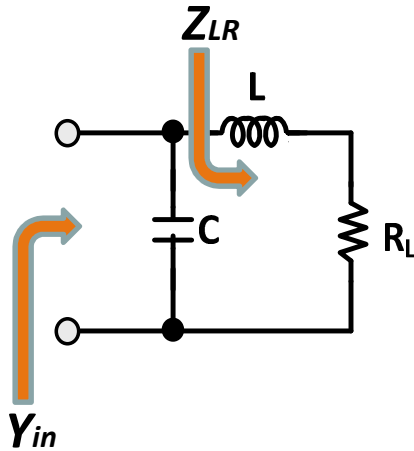
$$Y_{LR} = \frac{1}{Z_{LR}} = \frac{1}{R_L + j\omega L} = \frac{R_L}{R_L^2 + \omega^2 L^2} - \frac{j\omega L}{R_L^2 + \omega^2 L^2}$$

$$Y_{in} = Y_{LR} + j\omega C$$

$$= \frac{R_L}{R_L^2 + \omega^2 L^2} - \frac{j\omega L}{R_L^2 + \omega^2 L^2} + j\omega C$$

$$= \frac{1}{R_L} \left(\frac{1}{1 + \left(\frac{\omega L}{R_L}\right)^2} \right) + j\omega \left(C - \frac{L}{R_L^2} \left(\frac{1}{1 + \left(\frac{\omega L}{R_L}\right)^2} \right) \right)$$

Impedance Up-Conversion



From the final form of the admittance,

$$Y_{in} = \frac{1}{R_L} \left(\frac{1}{1 + \left(\frac{\omega L}{R_L} \right)^2} \right) + j\omega \left(C - \frac{L}{R_L^2} \left(\frac{1}{1 + \left(\frac{\omega L}{R_L} \right)^2} \right) \right)$$

$$R_{in} = R_L \left(1 + \left(\frac{\omega L}{R_L} \right)^2 \right)$$

Roughly, R_L is increased by a factor of square of the **impedance ratio of L & R_L** (impedance is up-converted) .

$$L = C \times R_L^2 \left(1 + \left(\frac{\omega L}{R_L} \right)^2 \right)$$

$$= C \times R_L \times R_{in}$$

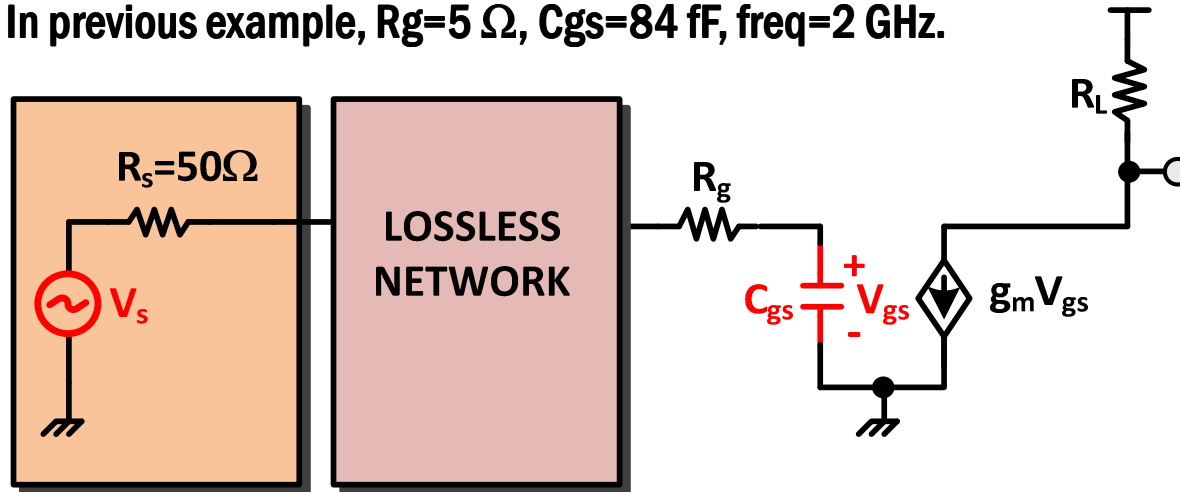
$$\left(\sqrt{\frac{L}{C}} \right)^2 = R_L \times R_{in}.$$

Imaginary part will be cancelled at this condition. **Note** $\sqrt{L/C}$ is the **characteristic impedance of L & C**.

Impedance Up-Conversion (ex)

Let's take an example.

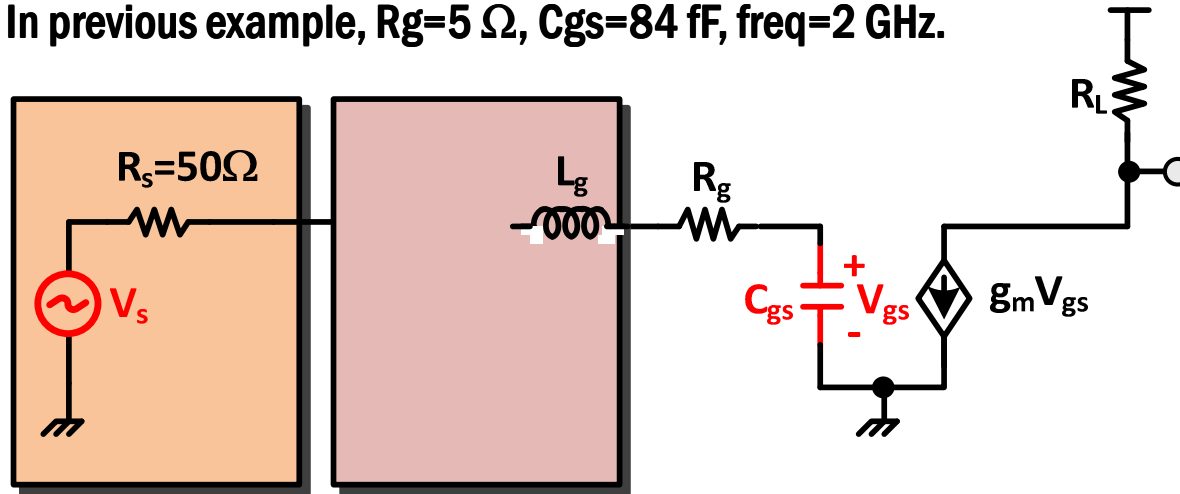
In previous example, $R_g = 5\ \Omega$, $C_{gs} = 84\ \text{fF}$, $\text{freq} = 2\ \text{GHz}$.



Impedance Up-Conversion (ex)

Let's take an example.

In previous example, $R_g=5\ \Omega$, $C_{gs}=84\ \text{fF}$, $\text{freq}=2\ \text{GHz}$.



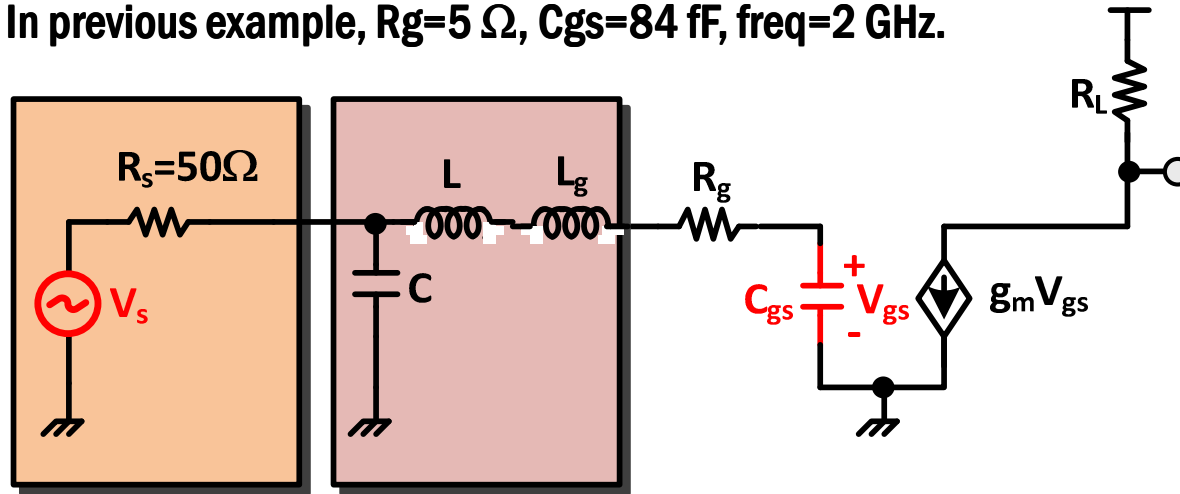
First, let's apply L_g to nullify reactance of C_{gs} at 2GHz through series resonance.

$$\omega L_g = \frac{1}{\omega C_{gs}}, @ \omega = 2\text{GHz} \quad \Rightarrow \quad L_g = 75.39\ \text{nH}$$

Impedance Up-Conversion (ex)

Let's take an example.

In previous example, $R_g = 5\ \Omega$, $C_{gs} = 84\ \text{fF}$, $\text{freq} = 2\ \text{GHz}$.



Next, apply L & C to convert R_g ($5\ \Omega$) to R_s ($50\ \Omega$).

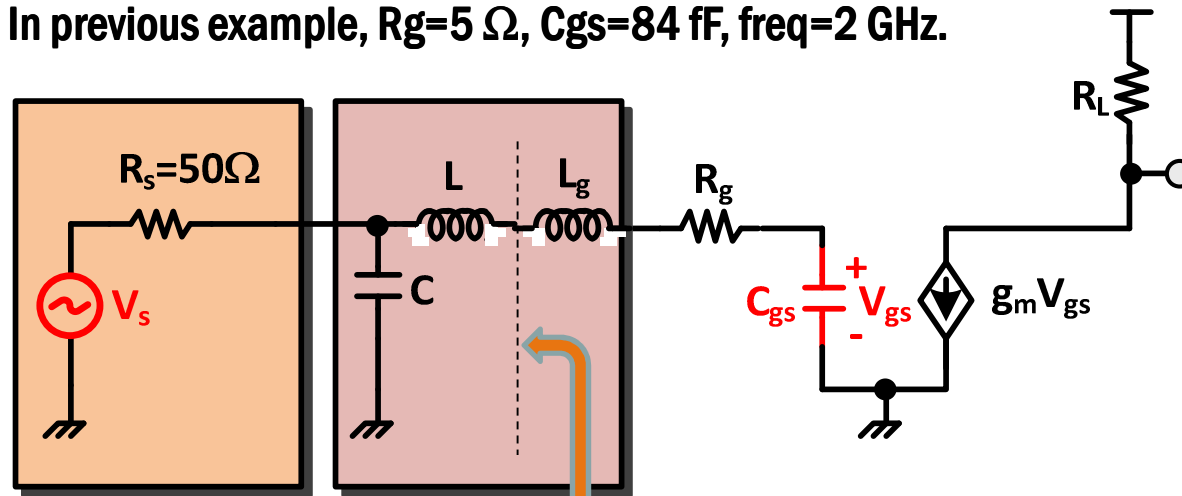
$$R_s = R_g \left(1 + \left(\frac{\omega L}{R_g} \right)^2 \right), \text{ @ } \omega = 2\text{GHz} \quad \Rightarrow \quad L = 2.39\ \text{nH}$$

$$\left(\sqrt{L/C} \right)^2 = R_g \times R_s \quad \Rightarrow \quad C = 9.56\ \text{pF}$$

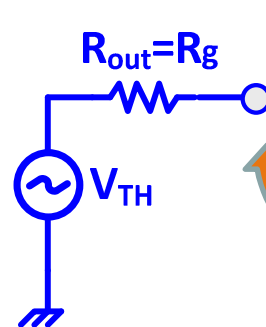
Impedance Up-Conversion (ex)

Now, let's calculate voltage across C_{gs} , V_{gs} .

In previous example, $R_g=5\ \Omega$, $C_{gs}=84\ \text{fF}$, $\text{freq}=2\ \text{GHz}$.



Now after matching,
 $R_{out} = R_g$

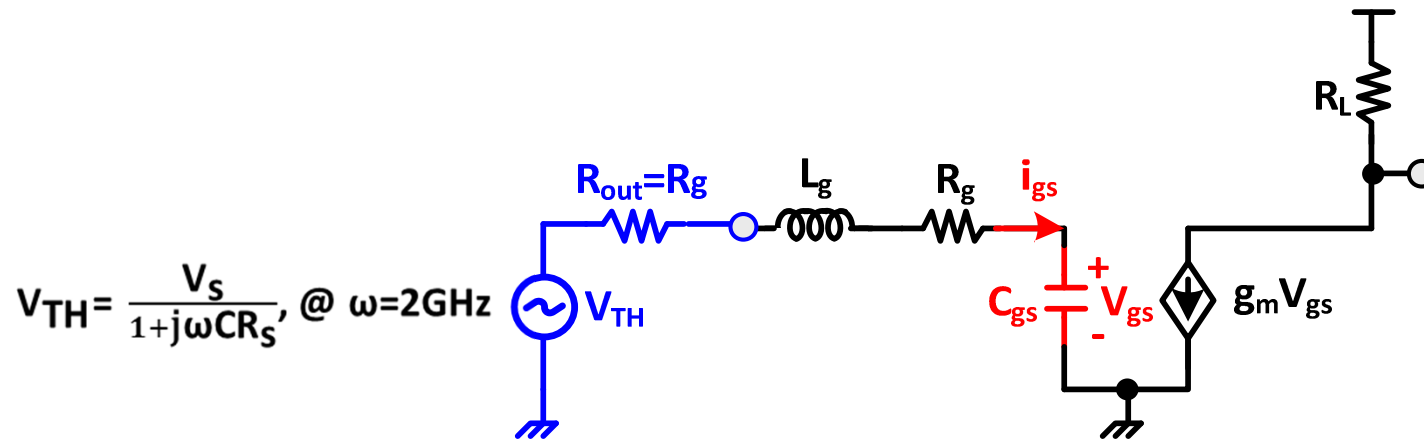


$$V_{TH} = \frac{1/j\omega C}{R_s + 1/j\omega C} V_s = \frac{V_s}{1 + j\omega C R_s}, \text{ @ } \omega = 2\text{GHz}$$

Change to Thevenin Equivalence

Impedance Up-Conversion (ex)

Now, let's calculate voltage across C_{gs} , V_{gs} .



$$i_{gs} = \frac{1}{2R_g} \frac{V_s}{1 + j\omega CR_s}$$

In previous example,
 $C=9.56\text{ pF}$, $R_s=50\ \Omega$, $R_g=5\ \Omega$, $C_{gs}=84\text{ fF}$, $\text{freq}=2\text{ GHz}$.

$$V_{gs} = \left| \frac{1}{j\omega C_{gs}} \times i_{gs} \right| = \left| \frac{1}{j\omega C_{gs}} \times \frac{1}{2R_g} \frac{V_s}{1+j\omega CR_s} \right| = 15.56 V_s$$

More than $15 \times V_s$ using
impedance matching !!

Impedance Matching for Analog Design

Q) Why is impedance matching not used in analog circuits?

In this example, just change **freq=2 MHz (from 2 GHz)**

$R_s=50\ \Omega$, $R_g=5\ \Omega$, $C_{gs}=84\ \text{fF}$.

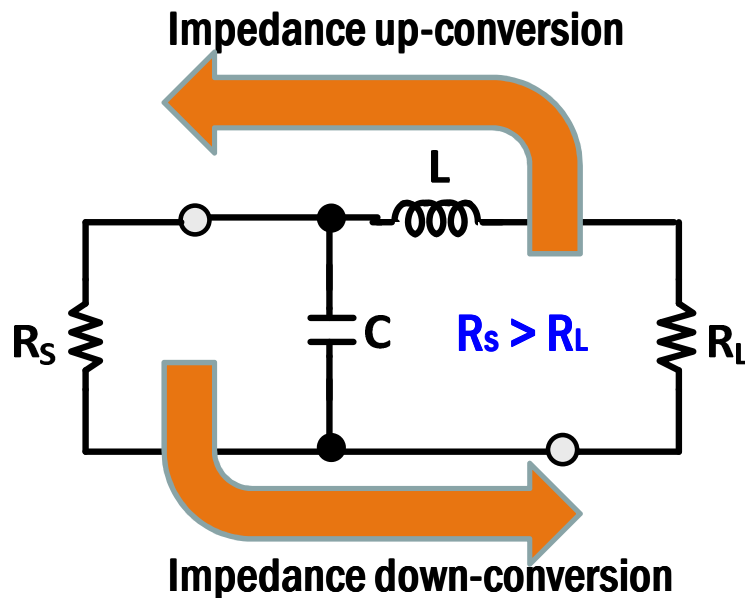
$$R_s = R_g \left(1 + \left(\frac{\omega L}{R_g} \right)^2 \right), @ \ \omega=2\text{MHz} \quad \Rightarrow \quad L=2.39\ \mu\text{H}$$

$$\left(\sqrt{L/C} \right)^2 = R_g \times R_s \quad \Rightarrow \quad C=9.56\ \text{nF}$$

These values are almost impossible to realize in IC.

Also, note that large values of L imply large values of parasitic resistors to the inductor. This large parasitic resistor will take up most of power delivered by the source; hence, only a small fraction of the power will be delivered to the transistor input port (impractical for most cases of analog IC designs).

Impedance Down-Conversion



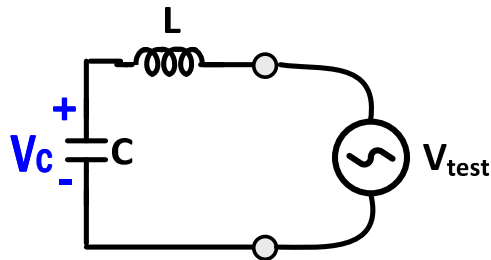
Impedance down-conversion is just reciprocal of the up-conversion.

Q) Can you explain intuitively how the impedance down-conversion work ?

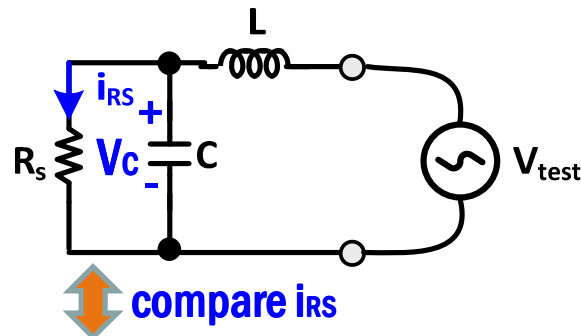
We call this type of L-C matching circuit as L-matching circuit.

Impedance Down-Conversion

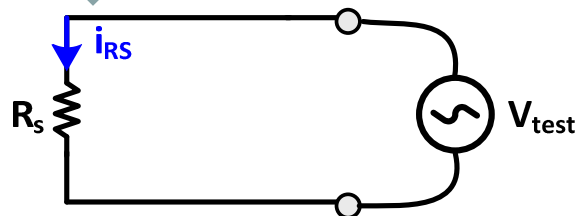
Let's think about this !



When we apply very small test signal across the L & C series circuit, a large voltage will be developed across C due to a series resonance.



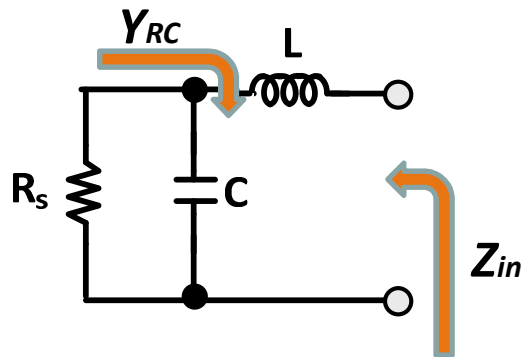
When we place resistor, R_s , across C, the capacitor will provide a large current to the resistor. This current can be much larger than that driven directly by V_{test} . This makes the R_s looks smaller effectively, when looking at the source side.



This is the behind logic of the downward impedance conversion.

Impedance Down-Conversion

Let's drive input impedance Z_{in} .



$$Y_{RC} = \frac{1}{R_s} + j\omega C$$

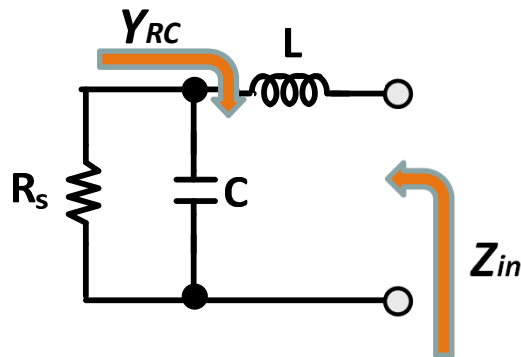
$$Z_{RC} = \frac{R_s}{1 + (\omega C R_s)^2} - \frac{j\omega C R_s^2}{1 + (\omega C R_s)^2}$$

$$Z_{in} = \frac{R_s}{1 + (\omega C R_s)^2} - \frac{j\omega C R_s^2}{1 + (\omega C R_s)^2} + j\omega L$$

$$= \frac{R_s}{1 + (\omega C R_s)^2} - \frac{j\omega C R_s^2}{1 + (\omega C R_s)^2} + j\omega L$$

$$= \frac{R_s}{1 + (\omega C R_s)^2} + j\omega \left(L - \frac{C R_s^2}{1 + (\omega C R_s)^2} \right)$$

Impedance Down-Conversion



From the final form of Z_{in} ,

$$Z_{in} = \frac{R_s}{1 + (\omega C R_s)^2} + j\omega \left(L - \frac{C R_s^2}{1 + (\omega C R_s)^2} \right)$$

$$R_{in} = \frac{R_s}{1 + (\omega C R_s)^2}$$

Roughly, R_s is decreased by a factor of square of the **admittance ratio of C & R_s** (impedance is down-converted) .

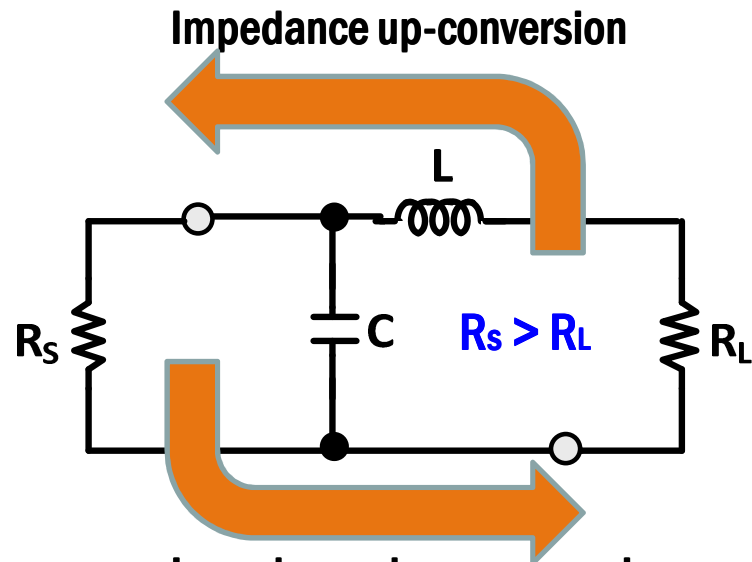
$$L = C \times \frac{R_s^2}{1 + (\omega C R_s)^2}$$

$$= C \times R_s \times R_{in}$$

$$\left(\sqrt{\frac{L}{C}} \right)^2 = R_s \times R_{in}.$$

Imaginary part will be cancelled at this condition. **Note** $\sqrt{L/C}$ is the characteristic impedance of L & C.

Impedance Conversion Summary



Impedance down-conversion

$$R_L = \frac{R_s}{1 + (\omega C R_s)^2}$$

Change form to admittance

$$Y_L = Y_s (1 + (\omega C R_s)^2)$$

Up-conversion:

$$R_s = R_L \left(1 + \left(\frac{\omega L}{R_L} \right)^2 \right) = R_L (1 + Q_{RL}^2)$$

Down-conversion:

$$Y_L = Y_s (1 + (\omega C R_s)^2) = Y_s (1 + Q_{RC}^2)$$

 Note for similarity !

Q_{RL} : Quality factor of series of L- R_L

Q_{RC} : Quality factor of parallel of C- R_s

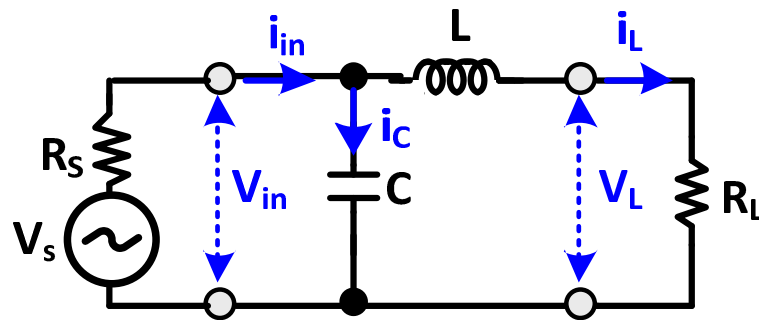
Impedance (admittance) up & down conversion is always related with a factor of $(1 + Q^2)$.

Q) What is Q ?

To answer this, we need to consider frequency response of the matching network , first.

Frequency Response of Matching Network

Let's drive in-out transfer function, V_L/V_s .



$$i_{in} = i_C + i_L$$

$$\frac{V_s - V_{in}}{R_s} = j\omega C V_{in} + \frac{V_{in}}{R_L + j\omega L}$$

$$V_{in} = \left(\frac{1}{1 + j\omega C R_s + \frac{R_s}{R_L + j\omega L}} \right) \times V_s$$

$$V_L = \frac{R_L}{R_L + j\omega L} V_{in}$$

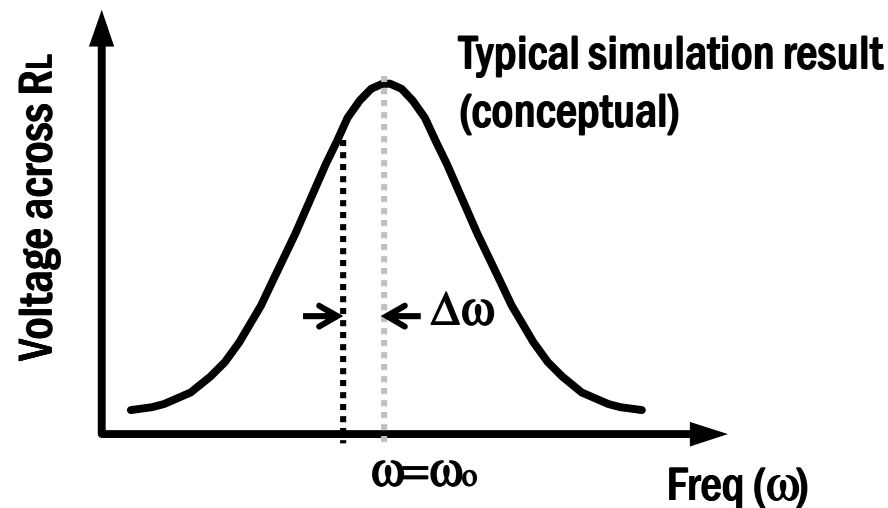
$$\left| \frac{V_L(\omega)}{V_s(\omega)} \right| =$$

$$\frac{R_L}{\sqrt{\left((R_L + R_s) + (R_L - R_s) \left(\frac{\omega}{\omega_o} \right)^2 \right)^2 - 4R_L(R_L - R_s) \left(\frac{\omega}{\omega_o} \right)^2}}$$

$$, \text{ where } \omega_o = \frac{1}{\sqrt{LC}}.$$

$$= \frac{R_L}{R_L + j\omega L} \times \left(\frac{1}{1 + j\omega C R_s + \frac{R_s}{R_L + j\omega L}} \right) \times V_s$$

Bandwidth of Matching Network

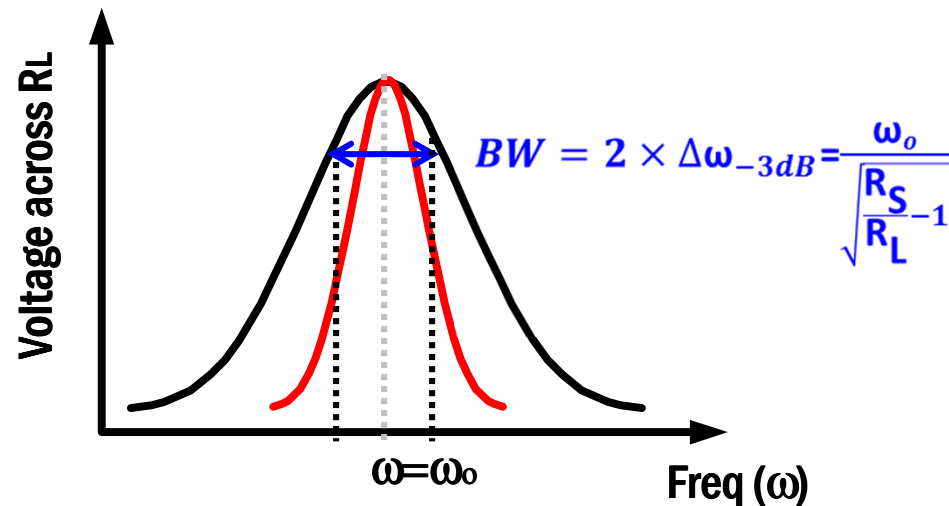


Observation: Circuit delivers power to the load at other frequencies around $\omega = \omega_0$ which is resonance frequency of the circuit. This means there is a power loss around at $\omega = \omega_0 + \Delta\omega$.

-3dB frequency can be calculated by equating this, $\left| \frac{V_L(\omega)}{V_S(\omega)} \right| = \frac{1}{\sqrt{2}}$

$$\Delta\omega_{-3dB} = \frac{1/2 \omega_0}{\sqrt{\frac{R_S}{R_L} - 1}} \quad \Rightarrow \quad BW = 2 \times \Delta\omega_{-3dB} = \frac{\omega_0}{\sqrt{\frac{R_S}{R_L} - 1}}$$

Bandwidth of Matching Network



Observation: For a fixed impedance transform ratio (R_S/R_L), the bandwidth (BW) is fixed.

In many RF systems, narrowband operation is preferable, since we don't want to lose any input power around signal bandwidth. Also, narrower band means more filtering of unwanted signals.

Q) Can we design matching network to have more narrower band operation (red-curve) for a given fixed impedance transform ratio?

Quality Factor (Q)

To answer the question, we need to introduce a very important concept in RF systems, referred to as “Quality Factor (Q)”.

In its simplest form, Q is used to measure the “quality” of capacitors and inductors

- An inductor/capacitor with large resistor has poor Q
- An inductor/capacitor with zero series resistance has infinite Q

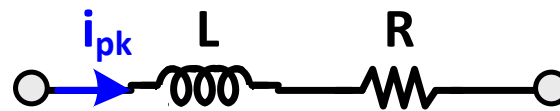
Let us now define the quality factor for a general network :

$$Q = 2\pi \times \frac{\text{Maximum energy stored in network per cycle}}{\text{Energy dissipated in network per cycle}}$$
$$= 2\pi f \times \frac{\text{Maximum energy stored in network per cycle}}{\text{Average power dissipated in network}}$$

Recall, Energy (J) = Average power (W) x Cycle period (T)

Q of Lossy Inductor

Let's consider simple example of a lossy inductor.



Maximum energy stored in L per cycle $= \frac{1}{2} \times L \times i_{pk}^2$

Average power dissipated in the network $= \frac{1}{2} \times i_{pk}^2 \times R$

$$Q = 2\pi f \times \frac{\text{Maximum energy stored in network per cycle}}{\text{Average power dissipated in network}}$$

$$= 2\pi f \times \frac{\frac{1}{2} \times L \times i_{pk}^2}{\frac{1}{2} \times i_{pk}^2 \times R} = \frac{\omega L}{R}$$

Q of Lossy Capacitor

Let's consider simple example of a lossy capacitor.



$$\text{Maximum energy stored in } C \text{ per cycle} = \frac{1}{2} \times C \times v_{pk}^2 = \frac{1}{2} \times C \times \left(\frac{i_{pk}}{\omega C} \right)^2$$

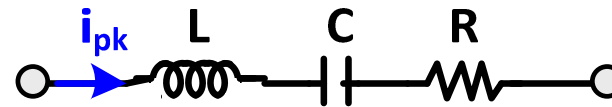
$$\text{Average power dissipated in the network} = \frac{1}{2} \times i_{pk}^2 \times R$$

$$Q = 2\pi f \times \frac{\text{Maximum energy stored in network per cycle}}{\text{Average power dissipated in network}}$$

$$= 2\pi f \times \frac{\frac{1}{2} \times C \times \left(\frac{i_{pk}}{\omega C} \right)^2}{\frac{1}{2} \times i_{pk}^2 \times R} = \frac{1}{\omega CR}$$

Q of Series LCR Network

Let's consider another simple example of a series L-C-R network.



Maximum energy stored in L & C per cycle $= \frac{1}{2} \times C \times \left(\frac{i_{pk}}{\omega C}\right)^2 + \frac{1}{2} \times L \times i_{pk}^2$ (???)

Average power dissipated in the network $= \frac{1}{2} \times i_{pk}^2 \times R$

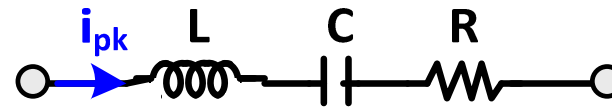
$$Q = 2\pi f \times \frac{\text{Maximum energy stored in network per cycle}}{\text{Average power dissipated in network}}$$

$$= 2\pi f \times \frac{\frac{1}{2} \times C \times \left(\frac{i_{pk}}{\omega C}\right)^2 + \frac{1}{2} \times L \times i_{pk}^2}{\frac{1}{2} \times i_{pk}^2 \times R} = \frac{\frac{1}{\omega C} + \omega L}{R} = \frac{2 \times \sqrt{\frac{L}{C}}}{R} \text{ @resonance (???)}$$

This is not correct ! What was wrong ? Don't apply the formula in a blind manner !

Q of Series LCR Network

Note: in inductor and capacitor, voltage and current have 90° (quadrature, or orthogonal) phase relationship.



inductor current, $i_L = i_{pk} \cos \omega t$
 capacitor voltage, $v_C = \frac{1}{C} \int i_L dt = \frac{i_{pk}}{C\omega} \sin \omega t$

90° phase difference

Maximum energy stored in L & C per cycle = $\frac{1}{2} \times L \times i_L^2 + \frac{1}{2} \times C \times v_C^2$
 $= \frac{1}{2} L (i_{pk} \cos \omega t)^2 + \frac{1}{2} C \left(\frac{i_{pk}}{C\omega} \sin \omega t \right)^2$

$= \frac{1}{2} L i_{pk}^2 = \frac{1}{2} \sqrt{\frac{L}{C}} \times i_{pk}^2, @ \text{resonance}$

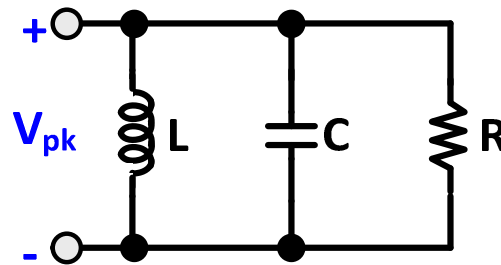
Average power dissipated in the network = $\frac{1}{2} \times i_{pk}^2 \times R$

$Q = \omega \frac{\frac{1}{2} L i_{pk}^2}{\frac{1}{2} R i_{pk}^2} = \frac{\omega L}{R} = \frac{\sqrt{L/C}}{R}$

$Q = \frac{\sqrt{L/C}}{R},$
 @resonance

Q of Parallel LCR Network

Another example of parallel LCR network.



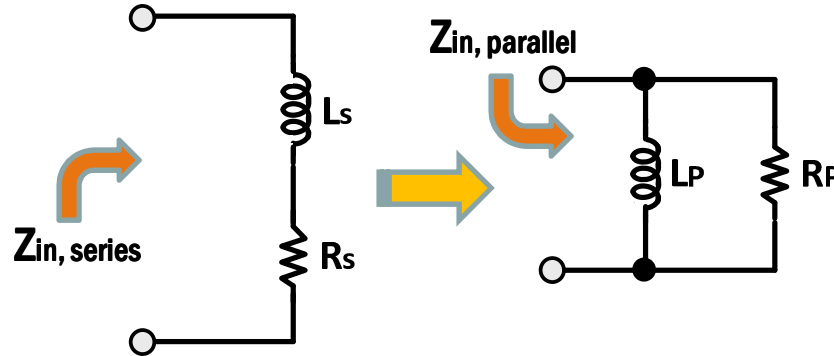
Following the same manner with applying AC-voltage, V_{pk} , you can verify that the Q of the parallel of LCR network at resonance frequency will be given as,

$$Q = \frac{R}{\sqrt{\frac{L}{C}}}, @\text{resonance}$$

Note, $\sqrt{L/C}$ is the characteristic impedance of L-C network.

Series L-R to Parallel L-R

It is quite useful to change series network to parallel or vice versa, which allow great simplification of circuit analysis.



If $Z_{in, series} = Z_{in, parallel}$, then two circuits are identical.

$$Z_{in, series} = R_S + j\omega L_S$$

$$Z_{in, parallel} = \frac{R_P \times j\omega L_P}{R_P + j\omega L_P} = \frac{R_P (\omega L_P)^2 + j\omega L_P R_P^2}{R_P^2 + (\omega L_P)^2}$$

$$R_P = R_S \times \left(1 + \left(\frac{\omega L_S}{R_S} \right)^2 \right) \\ = R_S \times (1 + Q_S^2)$$

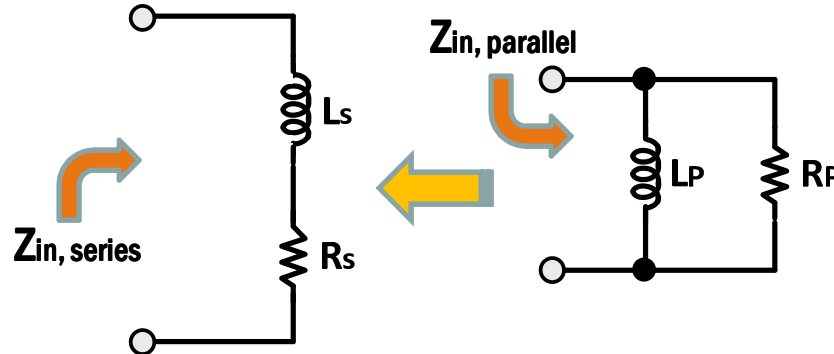
Resistance is increased in parallel from by a factor of Q_S^2 (if $Q_S \gg 1$)

$$L_P = L_S \times \left(1 + \left(\frac{R_S}{\omega L_S} \right)^2 \right) \\ = L_S \times \left(1 + \frac{1}{Q_S^2} \right)$$

Inductance will be about the same in parallel form, if $Q_S \gg 1$

Parallel L-R to Series L-R

It is quite useful to change series network to parallel or vice versa, which allow great simplification of circuit analysis.



If $Z_{in, series} = Z_{in, parallel}$, then two circuits are identical.

$$Z_{in, series} = R_S + j\omega L_S$$

$$Z_{in, parallel} = \frac{R_p \times j\omega L_p}{R_p + j\omega L_p} = \frac{R_p (\omega L_p)^2 + j\omega L_p R_p^2}{R_p^2 + (\omega L_p)^2}$$

$$R_S = \frac{R_p}{\left(1 + \left(\frac{\omega L_S}{R_p}\right)^2\right)} = \frac{R_p}{(1 + Q_p^2)}$$

Resistance is decreased in series from by a factor of Q_p^2 (if $Q_p \gg 1$)

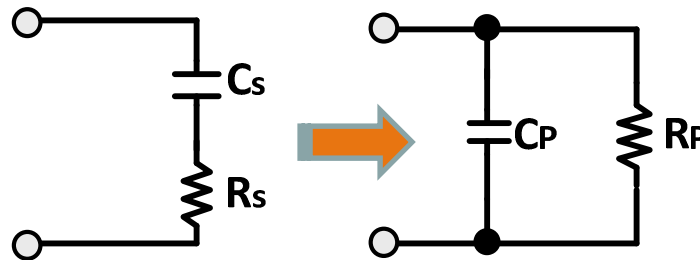
$$L_S = \frac{L_p}{\left(1 + \left(\frac{R_p}{\omega L_p}\right)^2\right)} = \frac{L_p}{\left(1 + \frac{1}{Q_p^2}\right)}$$

Inductance will be about the same in series form, if $Q_p \gg 1$

You can also verify that, $Q_s = Q_p$

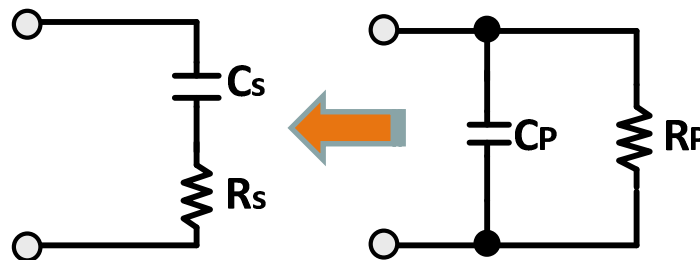
Series C-R to/from Parallel C-R

You are encouraged to verify this.



$$\begin{aligned}
 R_p &= R_s \times \left(1 + \left(\frac{1}{\omega C_s R_s} \right)^2 \right) \\
 &= R_s \times (1 + Q_s^2)
 \end{aligned}$$

$$\begin{aligned}
 C_p &= C_s \times (1 + (\omega C_s R_s)^2) \\
 &= C_s \times \left(1 + \frac{1}{Q_s^2} \right)
 \end{aligned}$$

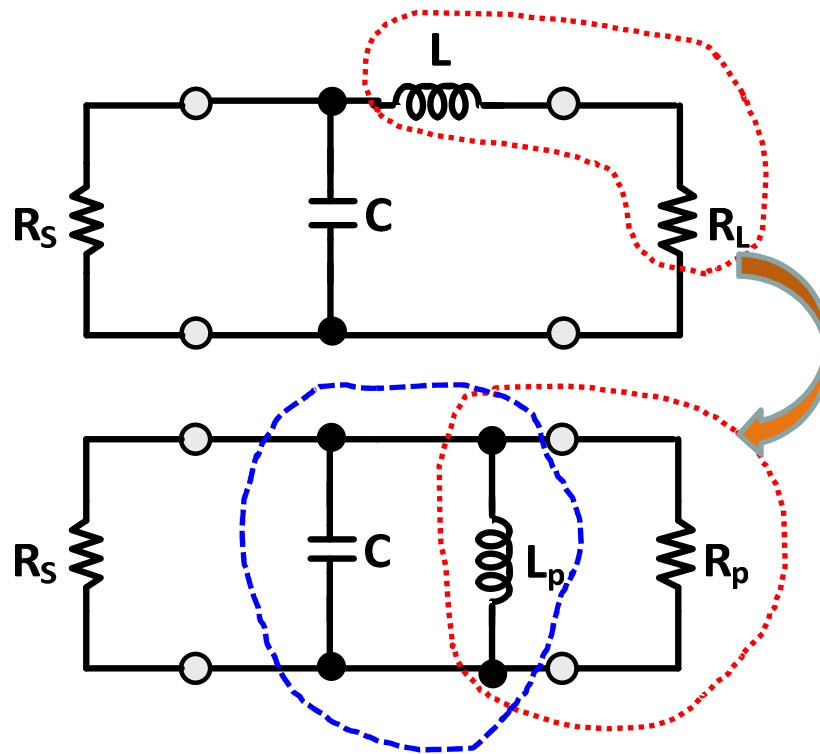


$$R_s = \frac{R_p}{(1 + (\omega C_p R_p)^2)} = \frac{R_p}{(1 + Q_p^2)}$$

$$C_s = \frac{C_p}{\left(1 + \left(\frac{1}{\omega C_p R_p} \right)^2 \right)} = \frac{C_p}{\left(1 + \frac{1}{Q_p^2} \right)}$$

Point is that if network Q is large enough ($Q_s = Q_p \gg 1$), there will be little change in the capacitance, but big change in resistance between series & parallel forms (key to impedance up/down conversion).

Q of Matching Network (Z up-conversion)



Let's revisit this matching circuit
($R_s > R_L$).

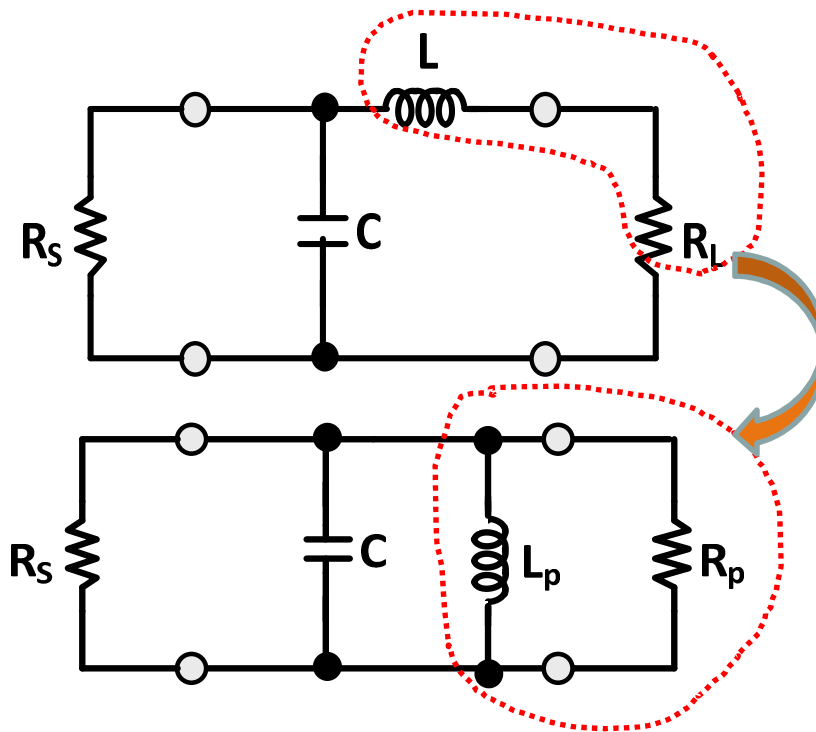
Impedance up-conversion:

$$\begin{aligned}
 R_p &= R_L \left(1 + \left(\frac{\omega L}{R_L} \right)^2 \right) = R_L (1 + Q_{RL}^2) \\
 &\approx R_L \times Q_{RL}^2, \text{ if } Q_{RL} \gg 1 \\
 &= R_s
 \end{aligned}$$

$$\begin{aligned}
 L_p &= L \left(1 + \left(\frac{R_L}{\omega L} \right)^2 \right) = L \left(1 + \frac{1}{Q_{RL}^2} \right) \\
 &\approx L, \text{ if } Q_{RL} \gg 1
 \end{aligned}$$

These L-C will be resonated at the target frequency. As a result there will be pure matched resistances, R_s and R_p .

“Loaded Q” vs. “Unloaded Q”



Let's use this matching circuit to illustrate between “loaded Q” and “unloaded Q”.

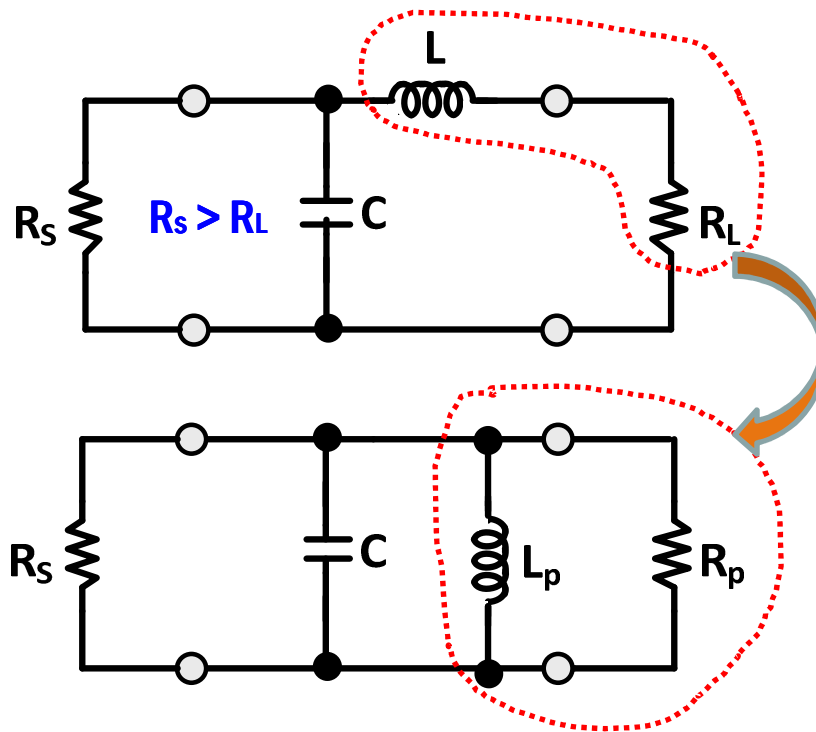
Unloaded Q: the driving source impedance is not included in the measurement of Q.

$$Q_{unloaded} = \frac{R_p}{\sqrt{\frac{L_p}{C}}}, @resonance$$

Loaded Q: the driving source impedance is included in the measurement of Q.

$$Q_{loaded} = \frac{R_p // R_s}{\sqrt{\frac{L_p}{C}}}, @resonance$$

Various Forms of Q (unloaded)



When we call Q (or network Q), it usually means unloaded Q.

$$Q = \frac{\omega_o L}{R_L} = \frac{\sqrt{\frac{L}{C}}}{R_S} \Rightarrow \text{Expressed in series R-L}$$

$$= \frac{R_p}{\omega_o L_p} = \frac{R_p}{\sqrt{\frac{L_p}{C}}} \Rightarrow \text{Expressed in parallel R-L}$$

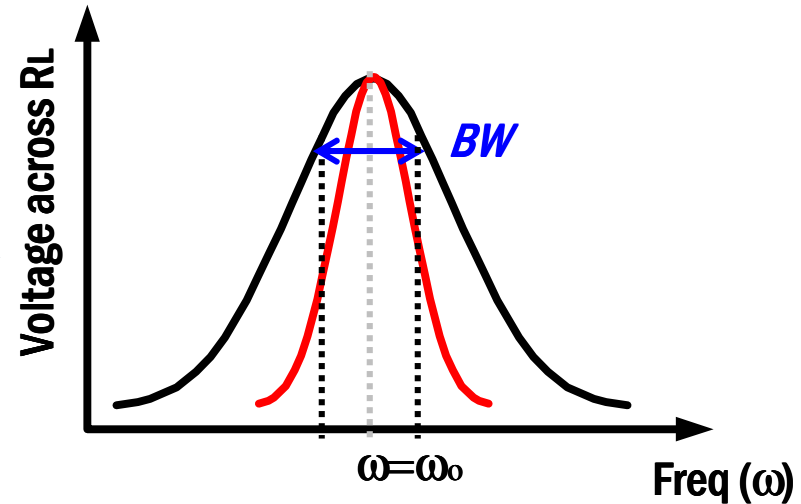
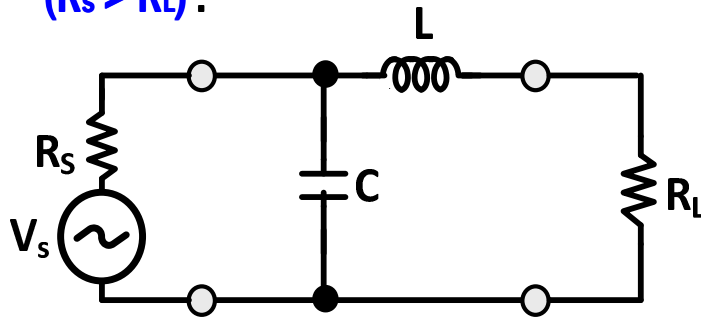
$$= \sqrt{\frac{R_S}{R_L}} - 1 \Rightarrow \text{Expressed in terms of impedance transform ratio}$$

$$, \text{ where } \omega_o = \frac{1}{\sqrt{LC}}$$

Impedance Matching With Higher Q

Revisiting bandwidth consideration

($R_S > R_L$):



$$Q = \sqrt{\frac{R_S}{R_L} - 1}$$

$$BW = 2 \times \Delta\omega_{-3dB} = \frac{\omega_o}{\sqrt{\frac{R_S}{R_L} - 1}} = \frac{\omega_o}{Q}$$

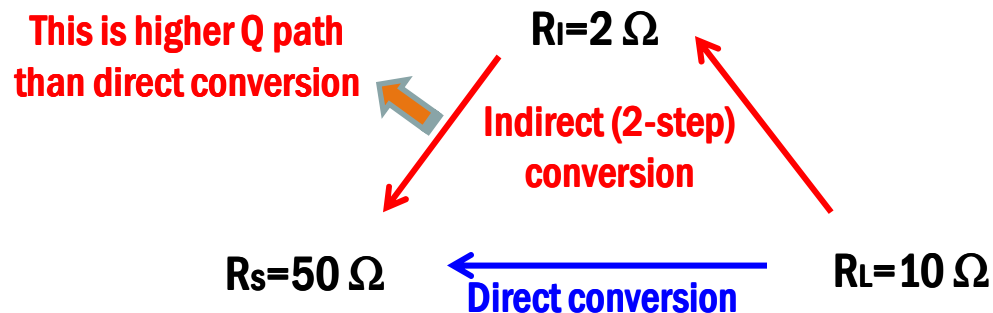
The higher the Q of the network, the narrower the BW of the matching network.
For a given impedance ratio, the BW is fixed.

Q) How can we design a network with higher Q for a given R_S and R_L ?

An Idea for Impedance Matching With Higher Q

Let's propose this !

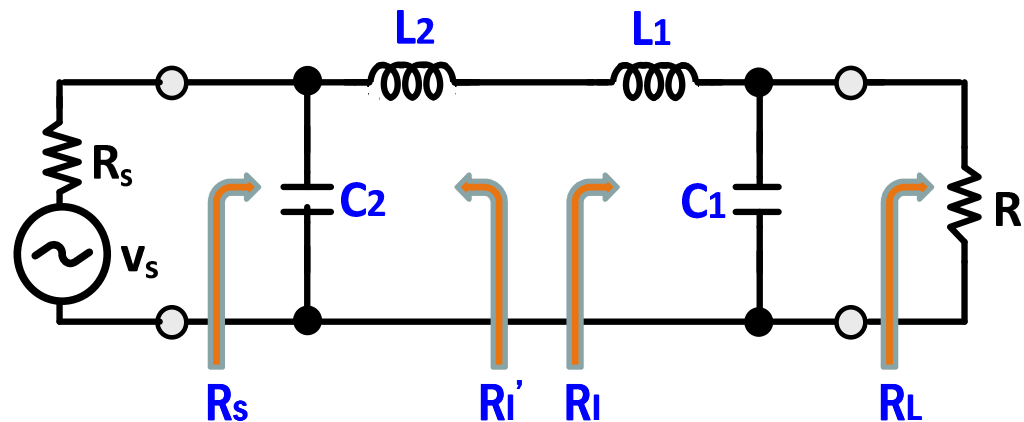
- The network Q depends on the impedance ratio.
- For single L-C matching circuit, the impedances of R_s and R_L are fixed, and so is Q.
- What if we down-convert the R_L to a smaller value of R_I (intermediate impedance), and then up-convert to the desired R_s in second stage (2-step conversion) ?



Let's examine this idea to see if this would allow us to design a network with higher Q for a give impedance transform from R_L to R_s .

Implementation of Higher Q Impedance Matching (π -Matching)

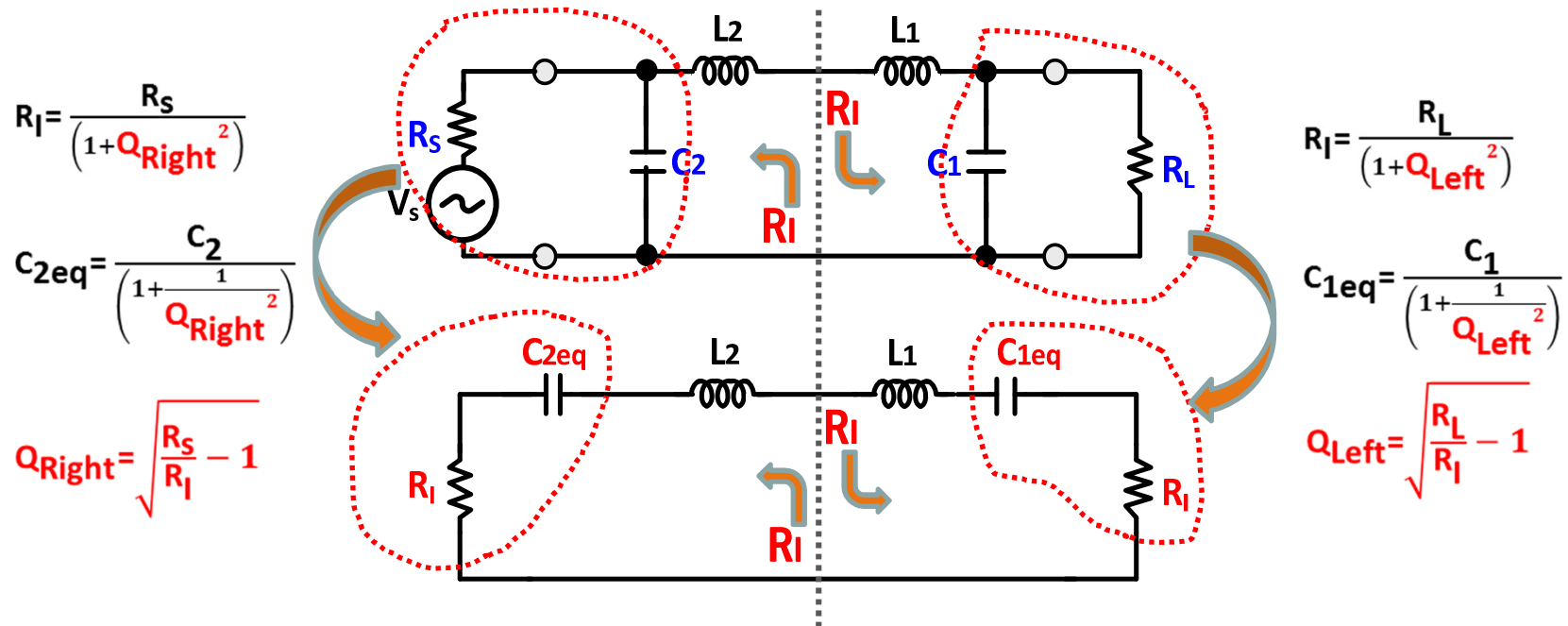
Let's cascade two L-matching networks such that first L-matching (**L1-C1**) down-converts the load impedance to R_I , and the second L-matching (**L2-C2**) up-converts the impedance to the desired value.



Apparently, $R_I' = R_I$ (why ?)

Let's continue the analysis.

Q of Π -Matching Network (Up-Conversion)

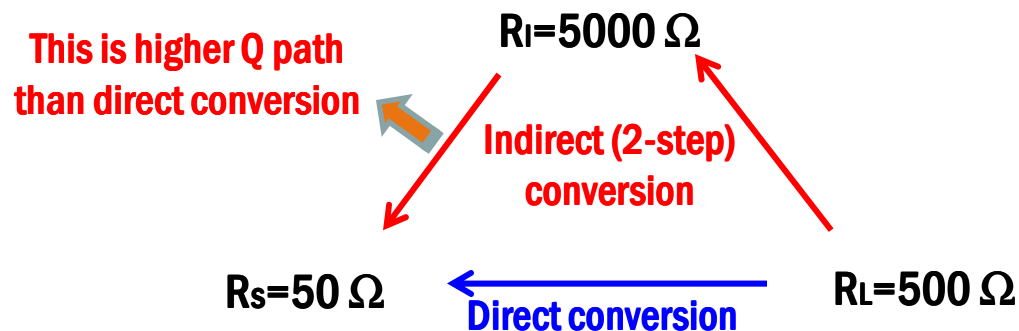


$$Q = 2\pi f \times \frac{\text{Maximum energy stored in network on the (left+right) sides}}{\text{Average power dissipated in network on the right side only}}$$

$$= Q_{Left} + Q_{Right} = \sqrt{\frac{R_L}{R_I} - 1} + \sqrt{\frac{R_s}{R_I} - 1} = \frac{\omega_o(L_1 + L_2)}{R_I} = \frac{1}{\omega_o(C_{1eq} // C_{2eq})R_I}$$

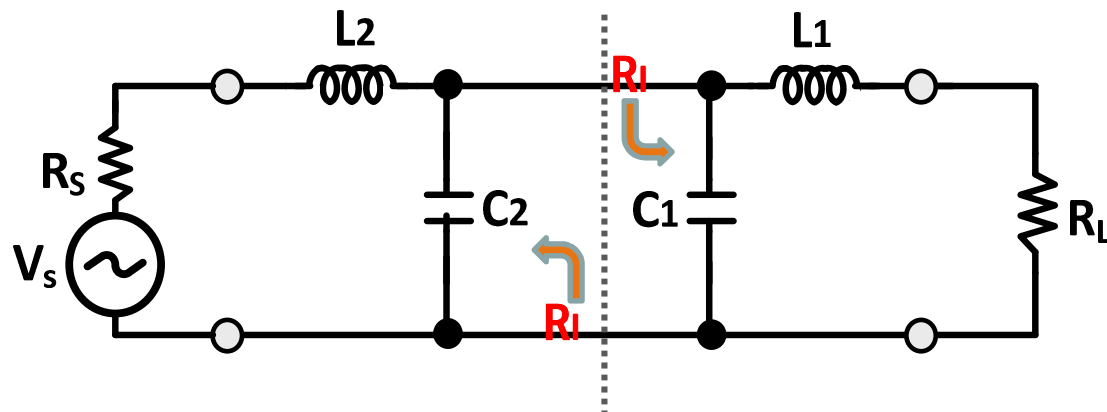
Q of T-matching Network (Down-Conversion)

Similarly, we can think 2-step high Q matching network to down-convert from high load resistance to smaller source resistance; First, up-convert to higher impedance, and then down-convert to the desired impedance.



Q of T-matching Network (Down-Conversion)

L_1 - C_1 up-converts R_L to R_I , after that, C_2 - L_2 down-converts R_I to R_S .
(Students are encouraged to verify this.)

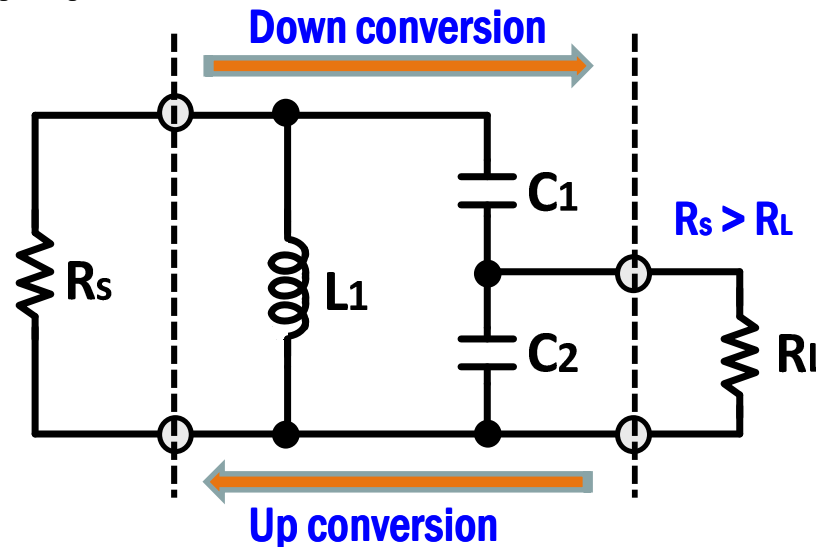


$$Q = 2\pi f \times \frac{\text{Maximum energy stored in network on the (left+right) sides}}{\text{Average power dissipated in network on the right side only}}$$

$$= Q_{Left} + Q_{Right} = \sqrt{\frac{R_I}{R_L} - 1} + \sqrt{\frac{R_I}{R_S} - 1} = \omega_o (C_1 + C_2) R_I = \frac{R_I}{\omega_o (L_{1eq} // L_{2eq})}$$

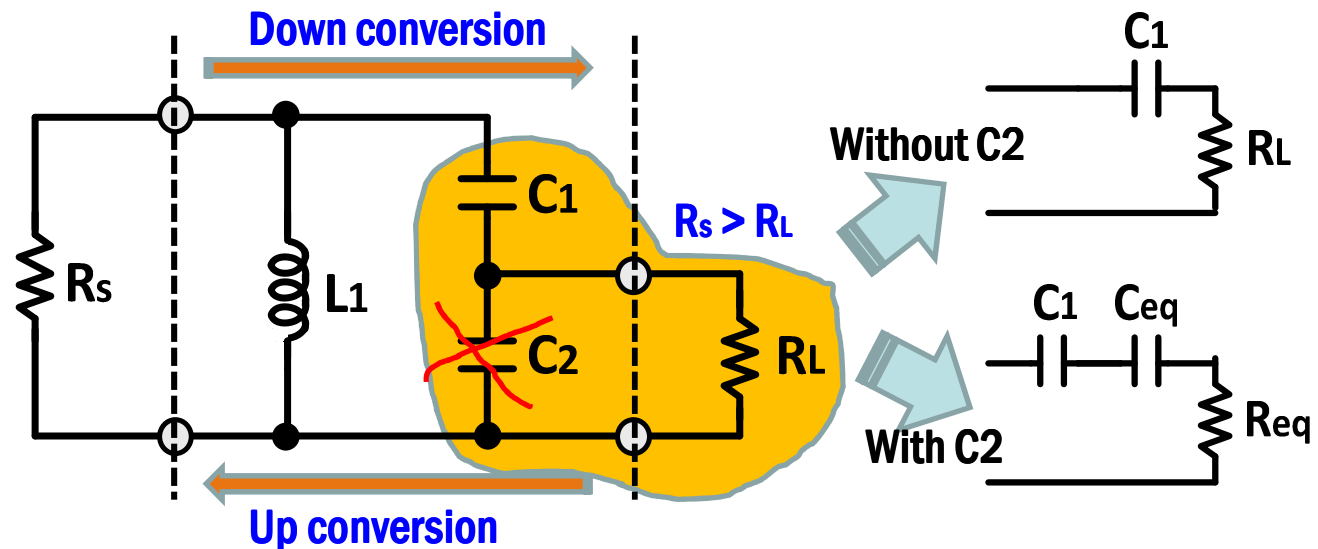
Other Matching NWs: Tapped Cap. Resonator

So far we have seen that Π - and T-matching networks provide more flexibility in choosing Q of the network. This is yet another example for higher Q matching network, and called as “capacitor tapped resonator transformer”.



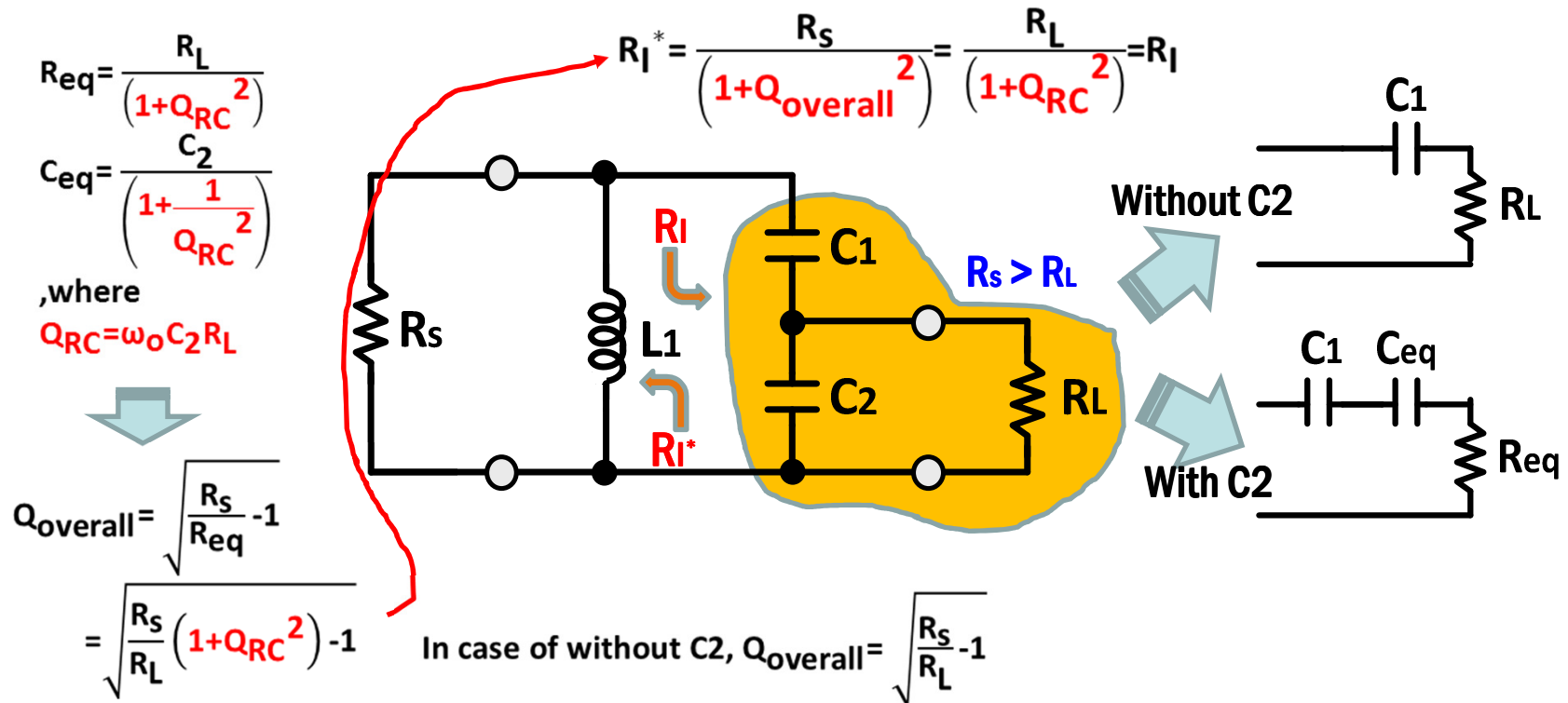
Other Matching NWs: Tapped Cap. Resonator

Note, without C_2 , it is a simple L-C matching circuit. But with C_2 , the C_2 will reduce R_L effectively by a factor of $(1+Q^2)$, where Q is unloaded- Q formed by C_2 - R_L parallel network.



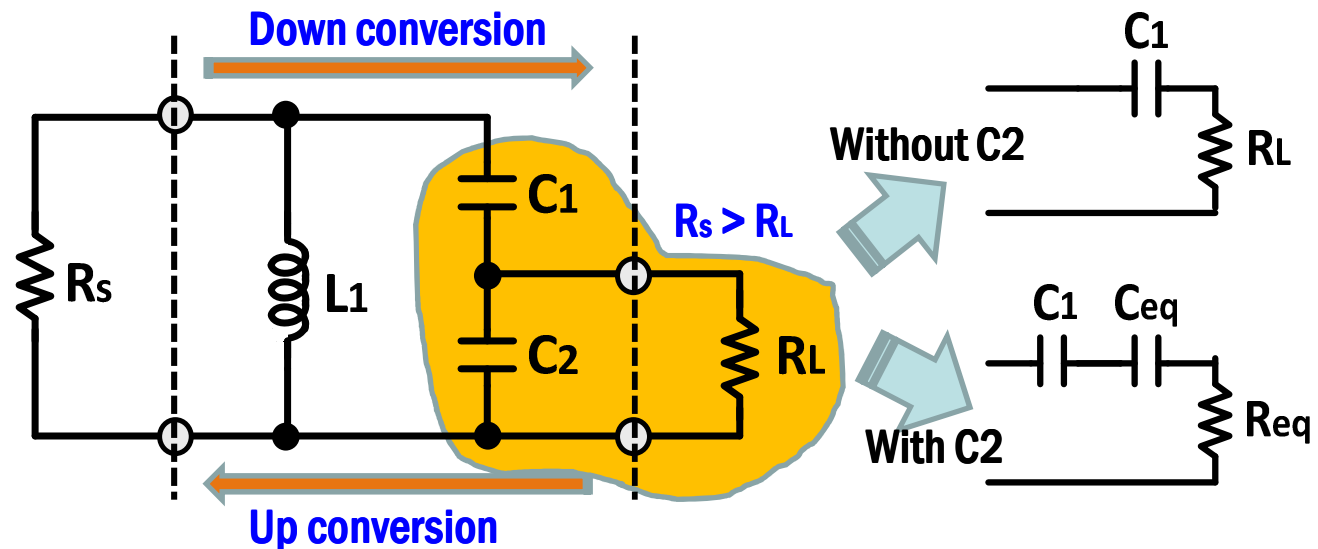
Other Matching NWs: Tapped Cap. Resonator

This is similar concept of 2-step conversion in Π - or T-matching network;
 i.e., $R_L \rightarrow R_I (= R_{eq}) \rightarrow R_S$.



Other Matching NWs: Tapped Cap. Resonator

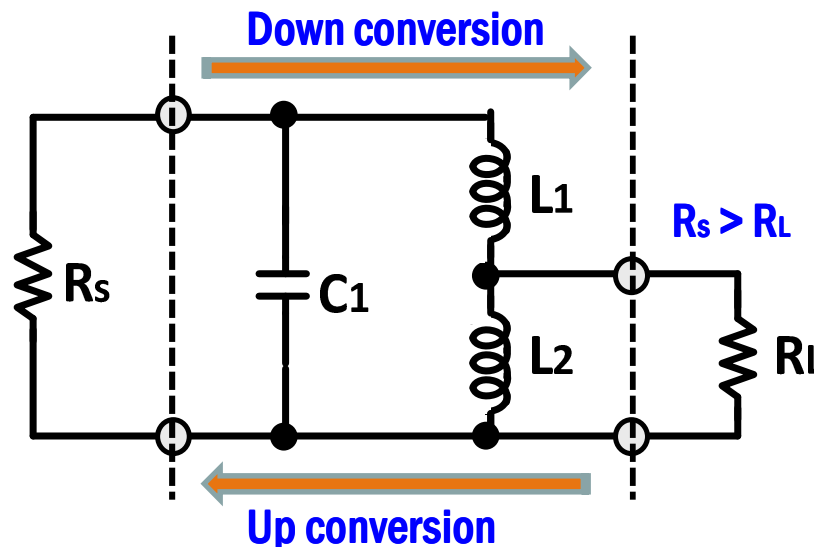
Also, it can be easily verified that the impedance transform ratio will be given by the capacitance ratio, approximately; i.e., $\frac{R_S}{R_L} \sim \left(1 + \frac{C_2}{C_1}\right)^2$, if $C_2 \neq 0$.



This tapped capacitor resonator is used popularly in “Colpitts VCO” designs, which we will discuss later.

Other Matching NWs: Tapped Ind. Resonator

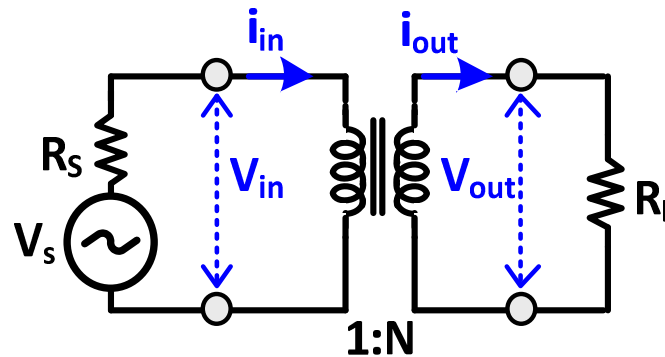
Using the same logic and mathematical flow, you can verify that this transformer network, called as “tapped inductor resonator”, also provides higher Q-matching.



This topology is not popular compared with previous topology, since it needs two inductors which will take more space than previous one in IC designs (But students are encouraged to develop math for this).

Other Matching NWs: Transformer

Transformer is also widely used for impedance matching at board level as an external component, mainly at low-frequencies.

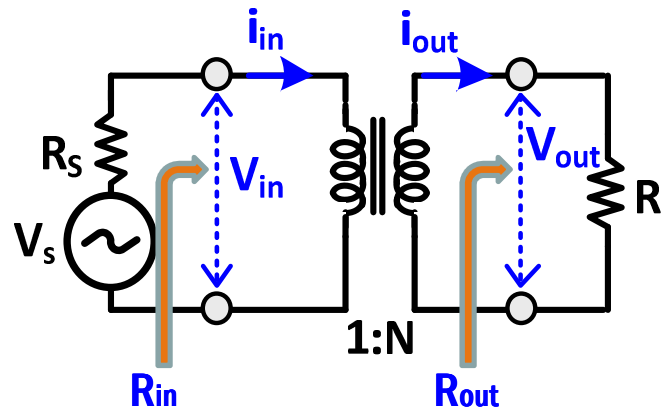


Transformer basics:

- Voltage will be amplified by a factor of turn ratio (N) at the output; i.e., $V_{out} = N \times V_{in}$.
- ideal transformer is pure passive and no loss, therefore there will not be any power loss; i.e., $P_{in} = V_{in} \times i_{in} = V_{out} \times i_{out} = P_{out}$ & $i_{out} = i_{in}/N$ (why?).

Other Matching NWs: Transformer

Let's find input and output impedance relationships.



$$R_{in} = \frac{V_{in}}{i_{in}} = \frac{\frac{V_{out}}{N}}{N \times i_{out}} = \frac{1}{N^2} \times \frac{V_{out}}{i_{out}} = \frac{1}{N^2} \times R_L$$

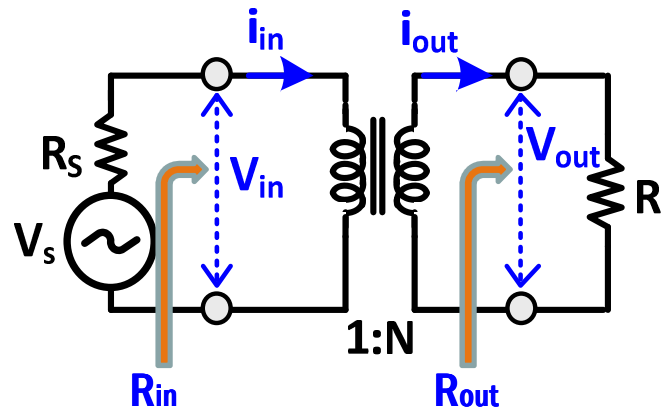
Maximum input power will be delivered to the load via transformer when input impedance is matched, i.e, $R_s = R_{in}$ (recall “maximum power transfer theory”);

$$R_s = R_{in} = \frac{1}{N^2} \times R_L \quad \Rightarrow \quad \boxed{\frac{R_L}{R_s} = N^2}$$

Once R_L and R_s is given, you can determine N for maximum power delivered to load.

Other Matching NWs: Transformer

Let's find input and output impedance relationships.

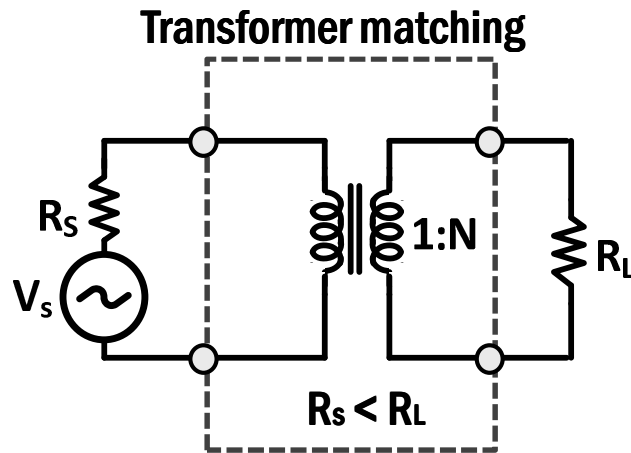


$$R_{in} = \frac{V_{in}}{i_{in}} = \frac{\frac{V_{out}}{N}}{N \times i_{out}} = \frac{1}{N^2} \times \frac{V_{out}}{i_{out}} = \frac{1}{N^2} \times R_L$$

Q) How much is voltage gain in this system?

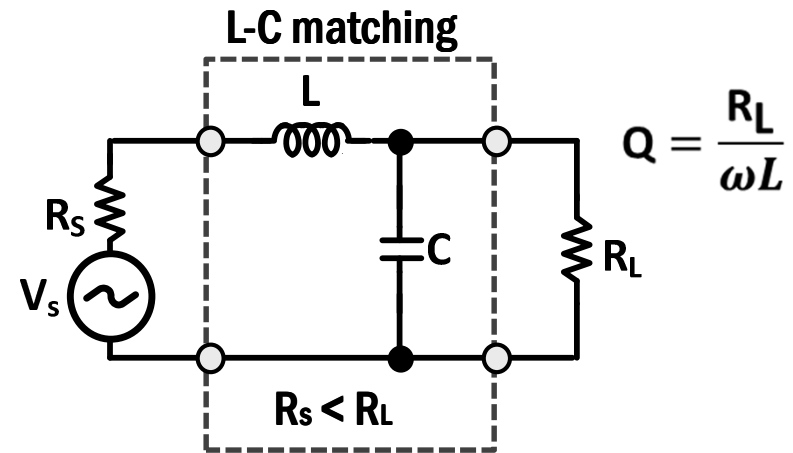
$$V_{out} = N \times V_{in} = N \times \left(\frac{1}{2} V_s \right) \Rightarrow \frac{V_{out}}{V_s} = \frac{1}{2} N = \frac{1}{2} \sqrt{\frac{R_L}{R_s}}$$

Similarity btw Transformer and LC matching



$$\frac{R_L}{R_s} = N^2$$

$$\frac{V_{out}}{V_s} = \frac{1}{2} N = \frac{1}{2} \sqrt{\frac{R_L}{R_s}}$$



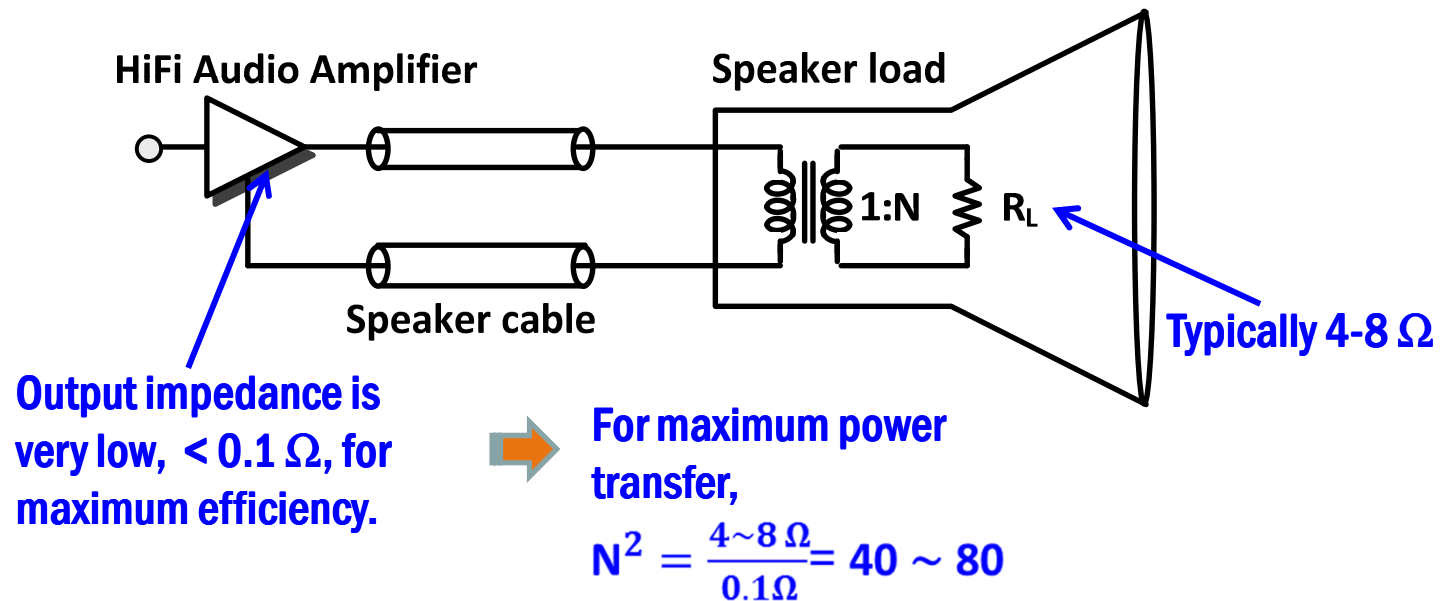
$$\frac{R_L}{R_s} = Q^2 + 1 \approx Q^2$$

$$\frac{V_{out}}{V_s} = \frac{1}{2} Q \approx \frac{1}{2} \sqrt{\frac{R_L}{R_s}}$$

Note: N in the transformer and Q in the LC matching network play similar role.

Transformer Matching Example: Audio App.

One popular use of transformer matching is in the loudspeaker driving systems.



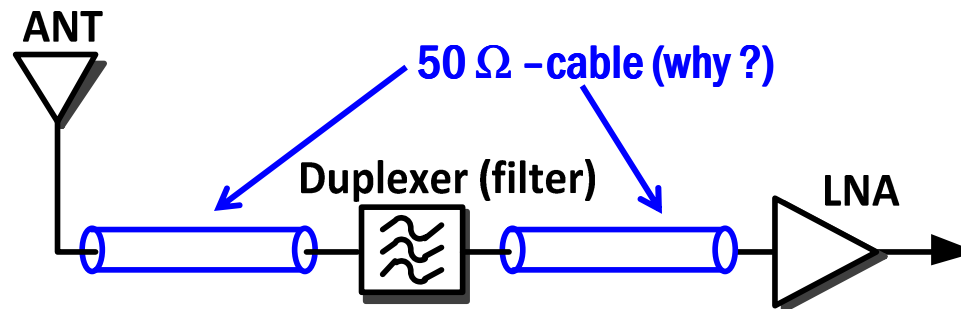
In RFIC ($< 5\text{GHz}$), impedance conversion using transformer is not popular, because of unavailability of good quality on-chip transformer.

50 Ω Question ?

Very often in RF IC designs, it is required that the input & output ports are matched with 50 Ω . One might ask

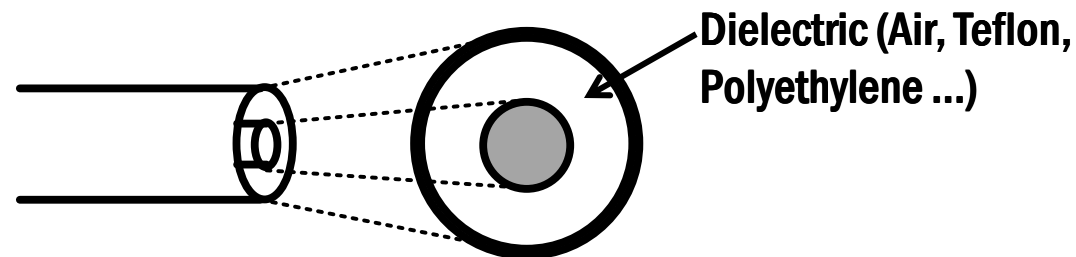
- Why 50 Ω ?
- Why not 500 Ω ?

Once we understand the reason for 50 Ω , we can judiciously choose when to follow the 50 Ω match, and when to choose another level of impedance match condition (500 Ω for example).



Maximum Power Delivered Using Coax Cable

Let's examine a coaxial cable with a characteristic impedance of Z_0 :



Maximum power (P) delivered to the load using such a coaxial cable can be expressed as:

$$P \propto \frac{V^2}{Z_0}.$$

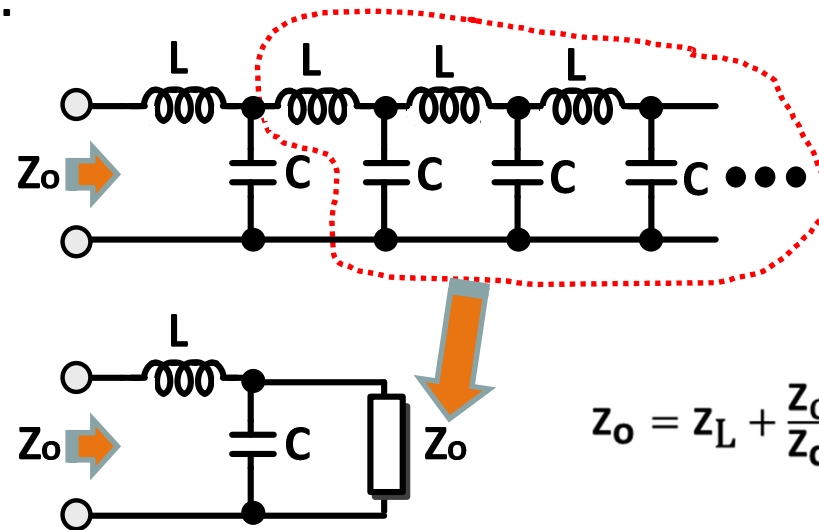
Maximum power will be delivered to the load when

- V is maximized, and
- Z_0 is minimized.

Let's calculate the characteristic impedance of the coaxial cable.

Characteristic Impedance of Coaxial T-Line

The coaxial transmission line can be represented as shown in figure below:



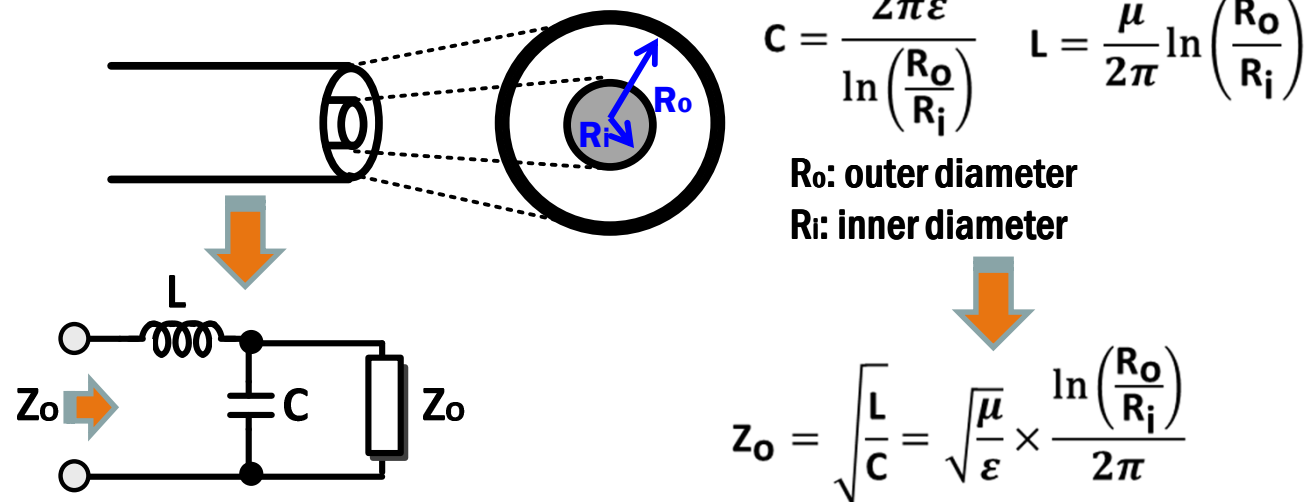
$$Z_o = Z_L + \frac{Z_C \times Z_o}{Z_C + Z_o}, \text{ where } Z_C = \frac{1}{j\omega C}, Z_L = j\omega L$$

$$\approx \sqrt{Z_C \times Z_L} = \boxed{\sqrt{\frac{L}{C}}} \text{ if } \frac{1}{\sqrt{LC}} \gg \omega$$

L & C represent line inductance and capacitance with infinitesimal length (this is common technique to extract characteristic impedance of a T-line).

Characteristic Impedance of Coaxial T-Line

The capacitance and inductance of a coaxial cable are given as
(refer to any microwave textbook):



$$C = \frac{2\pi\epsilon}{\ln\left(\frac{R_o}{R_i}\right)} \quad L = \frac{\mu}{2\pi} \ln\left(\frac{R_o}{R_i}\right)$$

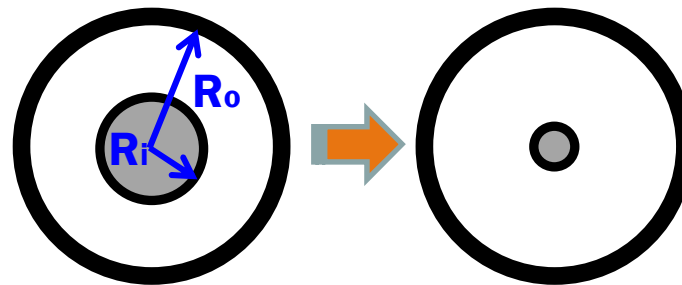
R_o : outer diameter
 R_i : inner diameter

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi}$$

To maximize power delivered to load, V needs to be maximized in the coaxial transmission line. But there is a limit of V because of breakdown of the dielectric beyond some point of V across the inner and outer diameter.

Maximum Electric Field in a Coaxial Cable

When we increase V , electric field will be increased between inner and outer conductors. And the maximum E-field depends on the radius of the inner and outer conductors as well as the voltage across the conductors.



What happens when we decrease the diameter of the inner conductor?

- **Ans)** The electric field increases significantly as the radius of the inner conductor, R_i , is made smaller and smaller.
- **Ans)** This increase in E-field strength eventually causes a breakdown of the dielectric between the two conductors (at a fixed voltage).

$$E_{\max} = \frac{V}{R_i \times \ln\left(\frac{R_o}{R_i}\right)} \quad \Rightarrow \quad V_{\max} = E_{\max} \times R_i \times \ln\left(\frac{R_o}{R_i}\right)$$

Z_o for Max. Power Handling

Now we have everything to re-evaluate the maximum power deliverable to the load.

$$P \propto \frac{V_{\max}^2}{Z_o} = \frac{\left(E_{\max} \times R_i \times \ln\left(\frac{R_o}{R_i}\right)\right)^2}{\sqrt{\frac{\mu}{\varepsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi}} = \left(\frac{E_{\max}^2}{\frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}}}\right) R_i^2 \ln\left(\frac{R_o}{R_i}\right)$$

The condition for maximum of power delivered can be obtained by differentiating the P with respect to R_i and setting it to zero:

$$\frac{dP}{dR_i} \propto \left(\frac{E_{\max}^2}{\frac{1}{2\pi} \sqrt{\frac{\mu}{\varepsilon}}}\right) \frac{d}{dR_i} \left(R_i^2 \ln\left(\frac{R_o}{R_i}\right)\right) = 0 \quad \Rightarrow \quad \frac{R_o}{R_i} = \sqrt{e}$$

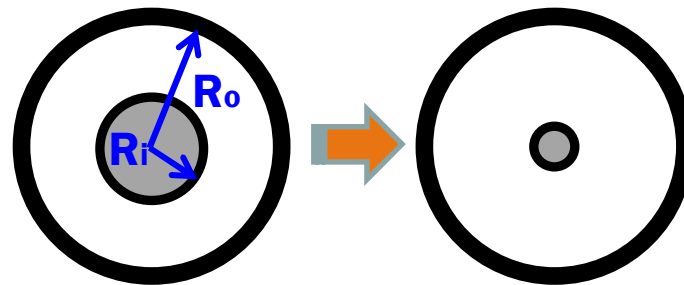
This result gives the optimum characteristic impedance for the maximum power handling capability as:

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi} = \sqrt{\frac{\mu}{\varepsilon}} \times \frac{\ln(\sqrt{e})}{2\pi} = 30 \, \Omega$$

= 60 Ω

Minimum Attenuation using Coaxial Cable

Let's re-examine the coax cable from the standpoint of signal attenuation. The metal conductors used for the coax cable have finite resistance which causes signal attenuation, even for the case of perfect matching.



The resistance of the cable keeps increasing as R_i and R_o decreased (note that at RF, current flows through only the skin of the conductors). As the resistance of the cable keeps increasing, the signal gets more attenuated.

$$\text{Resistance per unit length} = R = \frac{1}{2\pi\delta\sigma} \left(\frac{1}{R_i} + \frac{1}{R_o} \right)$$

where, δ =skin depth, σ =conductivity of metal

Z_o for Min. Attenuation

The attenuation constant is given by: $\alpha = \frac{R}{2Z_o} = \frac{\frac{1}{2\pi\delta\sigma} \left(\frac{1}{R_i} + \frac{1}{R_o} \right)}{2\sqrt{\frac{\mu}{\epsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi}}$

To find the optimum condition for the minimum attenuation, we differentiate the expression for attenuation with respect to R_i :

$$\frac{d\alpha}{dR_i} = \frac{d}{dR_i} \left(\frac{\frac{1}{2\pi\delta\sigma} \left(\frac{1}{R_i} + \frac{1}{R_o} \right)}{2\sqrt{\frac{\mu}{\epsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi}} \right) \stackrel{=0}{\Rightarrow} \ln\left(\frac{R_o}{R_i}\right) = 1 + \frac{R_o}{R_i}$$

$$\Rightarrow \frac{R_o}{R_i} = 3.6$$

This result gives the optimum characteristic impedance for the minimum signal attenuation as:

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \times \frac{\ln\left(\frac{R_o}{R_i}\right)}{2\pi} = \sqrt{\frac{\mu}{\epsilon}} \times \frac{\ln(3.6)}{2\pi} = 77 \, \Omega$$

Conclusion from Analysis of “50 Ω Question”

Some conclusions (and conventions in RF systems) can now be drawn:

- **For TV reception, signal attenuation is an issue because the signals are weak , therefore, TV equipment uses 75- Ω system to minimize attenuation.**
- **For systems involved in pure transmission, a 30- Ω systems allows maximum signal handling capability.**
- **For systems involved in both transmission and reception of signals, both signal attenuation (for received signal) and signal power handling capability (for transmission signals) are important. And therefore, 50- Ω system (approximate average of 30- Ω and 77- Ω) is chosen.**

Conclusion from Analysis of “50 Ω Question”

Typical wireless RF systems (including measure instruments) use 50- Ω systems. And 50- Ω interface needs to be followed with external environment.

However, if signals are not going out of the chip, 50- Ω specification need not be followed. Note, in IC environment typical interface impedance is 0.1 ~ 1 k Ω range. So, if you try to match every interface with 50- Ω inside chip, you might end up with wasting a lot of signal power.

One of main philosophies in IC design paradigm of SoC is to minimize 50- Ω interface and hence to save power. In old days of radios based on discrete elements on board level, each function blocks need substantial power consumption to drive 50- Ω interface to other external blocks

Filter Characteristics Vs. Impedance Matching

Typical insertion loss of duplexer will be 1~2 dB range and its performance can be significantly affected by the degree of the impedance matching between LNA.

