Appendix A: Maxwell's Equations and Decibel Notation

A.1 MAXWELL'S EQUATIONS

The behavior of electromagnetic (EM) waves is governed by the four laws, or equations, of electromagnetism known collectively as *Maxwell's equations*. Before stating Maxwell's equations mathematically, it is useful to describe them more heuristically. Maxwell's equations, simply stated, in words are given in Table A-1.

Maxwell's equations state that charged particles (e.g., electrons) produce electric fields (Gauss's law for electricity) and moving charged particles (currents) produce magnetic fields (Ampere's law). Electric and magnetic fields can be visualized by the forces they exert on charged particles. A charged particle in the presence of an electric field will experience an *electric force*, while a moving charged particle in the presence of a magnetic field will experience a *magnetic force*. The sum of these two forces are given by the succinct equation [1]

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{A.1}$$

where $\boldsymbol{F}$ is the EM force vector, q is the charge of the charged particle, $\boldsymbol{E}$ and $\boldsymbol{B}$ are, respectively, the vector electric and magnetic fields in which the charged particle is moving, $\boldsymbol{v}$ is the velocity vector of the charged particle, and the symbol denotes the vector cross-product.

Maxwell's equations also state that electric fields are produced by time-varying magnetic fields (Faraday's law) and vice versa, while magnetic fields are produced by time-varying electric fields (Maxwell's contribution of the Ampere-Maxwell law). This interaction gives rise to a electromagnetic waves that can propagate over long distances. Consider a charge that is oscillating back and forth (e.g., an electron in an alternating current [AC] circuit). Since this charge has a time-varying velocity (acceleration), it produces both electric and magnetic fields that change in time in accordance with Gauss's and Ampere's laws. It can be shown that if these two laws were the only laws of electromagnetism, then the amplitude of these electric and magnetic fields would be proportional to $1/R^2$, where R is the range from the charge, so that the fields would decay over a relatively short

TABLE A-1 ■ Maxwell's Equations in Words

Gauss's law for electricity	Electric charge produces electric fields.
Gauss's law for magnetism	There is no magnetic charge.
Faraday's law	Time-varying magnetic fields produce electric fields.
Ampere-Maxwell law	Moving charge (i.e., current, or Ampere's law) and time-varying electric fields (Maxwell's contribution) produce magnetic fields.

TABLE A-2 ■ Integral Form of Maxwell's Equations

Name of Law	Equation	Law in Words
Gauss's law for electricity	$\oint_{surface} \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0}$	The flux of $\boldsymbol{E}$ through any closed surface equals the net charge inside that surface, q, divided by ε_0.
Gauss's law for magnetism	$\oint_{surface} \boldsymbol{B} \cdot d\boldsymbol{A} = 0$	The flux of $\boldsymbol{B}$ through any closed surface is zero.
Faraday's Law	$\oint_{loop} \boldsymbol{E} \cdot d\boldsymbol{s} = -\int_{area} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{A}$	The circulation of $\boldsymbol{E}$ around any closed loop equals the negative time derivative of the flux of $\boldsymbol{B}$ through any area bounded by that loop.
Ampere-Maxwell law	$\oint_{loop} \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 i + \mu_0 \varepsilon_0 \int_{area} \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{A}$	The circulation of $\boldsymbol{B}$ around any closed loop equals μ_0 times the electric current, i, flowing through any area bounded by that loop plus $\mu_0\varepsilon_0$ times the time derivative of the flux of $\boldsymbol{E}$ through that area.

distance. However, Faraday's law states that the changing magnetic field also produces an electric field, while the Ampere-Maxwell law states that the changing electric field produces a magnetic field. These changing electric and magnetic fields reinforce each other far out into the space from the oscillating charge. The result is that the amplitudes are proportional to $1/R$ and therefore decay more slowly.

Mathematically, Maxwell's equations can be expressed in either differential form or integral form. Table A-2 gives the integral form as well as a more precise verbal description than given in Table A-1. The constants ε_0, μ_0 are, respectively, the *permittivity* and *permeability* of free space. Their numerical values are such that

$$1/\sqrt{\mu_0\varepsilon_0} = c \tag{A.2}$$

In free space (thus in the absence of charges and currents), Maxwell's equations can be reduced to the following vector differential equations:

$$^2\boldsymbol{E} = \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \quad \text{and} \quad {}^2\boldsymbol{B} = \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{A.3}$$

where ( ) is the *gradient* of a vector. Equations (A.3) are the so-called *wave equations*, the solutions of which are, respectively, electric and magnetic field waves (i.e., electromagnetic waves) propagating at the speed $1/\sqrt{\mu_0\varepsilon_0} = c$. A solution to the first wave equation of particular interest to radar is a traveling sinusoidal electric field wave, with the amplitude of each directional component (e.g., horizontal and vertical) having the form

$$E = E_0 \cos(kz - \omega t + \phi) \tag{A.4}$$

where k is the *wavenumber* $(2\pi/\lambda)$, ω is the radian *frequency*, and ϕ is the *phase* of the wave. These parameters and their significance are described in Section 1.3.1.

In free space, the oscillating electric, $\boldsymbol{E}$, and magnetic, $\boldsymbol{B}$, fields of the propagating EM wave are orthogonal and the direction of the propagating wave is orthogonal to both $\boldsymbol{E}$ and $\boldsymbol{B}$ (i.e., the EM wave is *transverse*). The direction of propagation is given by the direction of $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0$ (the *Poynting vector*) and the intensity (power/area) of the wave is given by the magnitude of $\boldsymbol{S}$, that is, $|\boldsymbol{S}|$. The ratio $|\boldsymbol{E}|/|\boldsymbol{B}|$ is c, the speed of the propagation of the EM wave (speed of light).[1] Therefore, if the direction of propagation and the properties of either vector field ($\boldsymbol{E}$ or $\boldsymbol{B}$) is known, then the properties of the other vector field can be determined. Consequently, the EM wave is typically described in terms of the electric field.

A.2 THE UBIQUITOUS dB

The decibel (dB) is widely used in radar technology. Many radar parameters are expressed in units of dB due to the large dynamic range of these parameters. Radar cross section (RCS) is a good example. RCS values can range from 10^{-5} m^2 (insects) to over 10^6 m^2 (aircraft carriers). This represents 11 orders of magnitude, a range of a 100 billion to one (10^{11}). In dB units, these RCS values become –50 dB and 60 dB, respectively, a range of only 110. Thus, in dB, the scale becomes significantly compressed and easier to deal with mathematically.

The value of a quantity in dB is always computed relative to some reference value. The first step in converting a value to dB is therefore to divide it by the reference value. For example, consider power, P. Before taking the logarithm of P, it is divided by a reference power P_0, say, $P_0 = 1$ watt. Now the logarithm is taken:

$$\log_{10}\left(\frac{P}{P_0}\right) \tag{A.5}$$

This is the power in "bels." Multiplying this by 10 yields the power in decibels,

$$P \text{ in dB} = 10\log_{10}\left(\frac{P}{P_0}\right) \tag{A.6}$$

To convert a value x from dB units to linear units the inverse operation is performed:

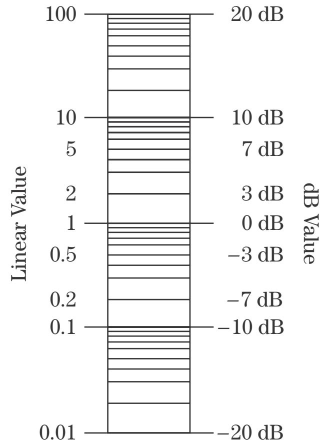
$$x_{linear} = 10^{x_{\mathrm{dB}}/10} \tag{A.7}$$

Shown in Figure A-1 is a scale comparing linear and dB values.

There is only one problem. This may be a convenient way for a transmitter engineer to express power, but a receiver engineer would have probably chosen the reference power to be $P_0 = 10^{-3}$ watt $= 1$ milliwatt. Somehow the choice of the reference parameter must be conveyed to the user. This is done by modifying the way the unit "dB" is written. In the case of power, for example, the unit is expressed as "dBW"(dB relative to 1 watt) if $P_0 = 1$ watt and "dBm" (dB relative to 1 milliwatt) if $P_0 = 1$ milliwatt.

[1] $\boldsymbol{H} = \boldsymbol{B}/\mu_0$ is also referred to as the "magnetic field" and $|\boldsymbol{E}|/|\boldsymbol{H}| \equiv Z_0 = 377$ ohms is the impedance of free space. In this case, $\boldsymbol{B}$ is sometimes referred to as the *magnetic induction* or *magnetic flux density*.

FIGURE A-1 ■ Linear and dB values.



A method to express a wide dynamic range of numbers in a convenient way

$$P\,(\mathrm{dB}) = 10\log_{10}\left(\frac{P}{P_0}\right)$$

Some features of measurements in dB to note are (1) values in dB can be determined only for positive parameters (the dB value of a negative parameter is not allowed), and (2) a negative dB value means that the linear value of the parameter is less than the reference value, that is, the ratio P/P_0 is less than 1.

Manipulating parameters expressed in dB simplifies some arithmetic operations; multiplication becomes addition and division becomes subtraction. This is due to the mathematical properties of the logarithm. For example, the liner equation $x = yz$ becomes $x = y + z$ if x, y and z are expressed in dB. Similarly, $x = y/z$ becomes $x = y - z$ if x, y and z are expressed in dB. Also, the equation $z = y^a$ becomes $x = ay$ if x and y are expressed in dB (a is not in dB). This was of great utility before the age of hand-held scientific calculators and high-speed computers, but is of little use today.

The one exception is in the determination the linear value of a parameter given in dB when a calculator is not available. Table A-3 lists several “dBs to remember.” With this table and the arithmetic properties of dBs, one can determine the approximate linear value of any parameter given in dB without resorting to a scientific calculator. For example, the linear equivalent of 7 dB can be determine by noting

$$7\ \mathrm{dB} = 3\ \mathrm{dB} + 3\ \mathrm{dB} + 1\ \mathrm{dB} \tag{A.8}$$

TABLE A-3 ■ dBs to Remember

dB	Linear
0	1
1	$1\frac{1}{4}$
3	2
10	10
$10x$	10^x
$-10x$	10^{-x}

TABLE A-4 ■ Radar Parameters Commonly Expressed in dB

Radar Parameter	dB Expression
Antenna gain	dBi (gain relative to isotropic)
Power gain	dB (power out/power in)
Power loss	dB (power in/power out)
Power	dBW (power relative to 1 watt)
	dBm (power relative to 1 milliwatt)
RCS	dBsm (RCS relative to 1 square meter)

and, from the table, that 3 dB corresponds to a linear value of 2, while 1 dB corresponds to a linear value of $1\frac{1}{4}$. From this and the arithmetic properties of dB, equation (A.8) becomes

$$7 \text{ dB} = 2 \times 2 \times 1\frac{1}{4} = 5 \tag{A.9}$$

An alternative way to reach the same conclusion is to note that

$$7 \text{ dB} = 10 \text{ dB} - 3 \text{ dB} \tag{A.10}$$

From the table, 10 dB = 10 and 3 dB = 2. Thus, equation (A.8) becomes

$$7 \text{ dB} = 10 \div 2 = 5 \tag{A.11}$$

Some radar parameters commonly expressed in dB are shown in Table A-4.

A.3 REFERENCE

[1] Lonngren, K.E., Savov, S.V., and Jost, R.J., *Fundamentals of Electromagnetics with MATLAB*, 2d ed., SciTech Publishing, Raleigh, NC, 2007.

Appendix B: Answers to Selected Problems

Chapter 1

1. $0.15x$ km.

3. 10.73 μs, 6.67 μs, 666.67 μs, 2.68 ms, 40.64 ns.

5. (a) 11.81 GHz (b) 85.714 GHz. (c) 34.88 GHz. (d) 333.33 MHz. (e) 3.33 GHz. (f) 984.25 MHz.

7. (a) 66.67 m. (b) 20 m. (c) 0.67 m. (d) 20 m. (e) 1125 m.

10. 1 kW = 30 dBW.

11. $150/x$ km.

13. (a) 2980 Hz (b) 22.0 kHz (c) 1.00 kHz (d) 707.1 Hz (e) 212.1 Hz (f) 500 Hz.

15. (a) 150 m (b) 150 m (c) 15 m (d) 1.5 m.

17. (a) 15.7 degrees (b) 4.25 degrees (c) 361.2 feet (d) 97.9 feet

Chapter 2

1. 2.06×10^{-14} W $= 2.06 \times 10^{-11}$ mW, -106.9 dBm.

3. 2.765, or 4.42 dB.

5. (a) 1.65 dB (b) 1.65 dB (c) 12.6 dB (d) 11.65 dB.

7. (a) -9.07 dB (b) -9.07 dB (c) $+1.88$ dB (d) $+0.93$ dB.

9. (a) 42.05 km (b) 28.12 km.

11. 17.5 ms.

13. 90 m^2, and -19.54 dB.

15. 62.3 dB.

17. 32.3 dB.

19. 158 kW.

Chapter 3

1. 90.

3. 1.0 seconds.

5. 1.11×10^{-2}.

7. 3 verifications.

9. 912 ms.

11. 1.75 seconds.

13. 2-of-3 detection: $P_D = 0.5$, $P_{FA} = 7.475 \times 10^{-5}$. 3-of-4 detection: $P_D = 0.6875$, $P_{FA} = 1.49 \times 10^{-4}$.

15. 9.9 dB.

Chapter 4

1. Atmospheric refraction, surface multipath.

3. Distance = 3000 m. Phase change = 62,832 radians.

5. Approximately 0.002 dB/km.

7. Radius of curvature $< \lambda/50$ for knife edge.

9. Approximately 0.126.

11. 114.6 m.

13. Specular reflection angle = 3.43°. Range extension = 0.16 m.

15. 12 dB.

17. Approximately 6 dB.

Chapter 5

1. 19.25° at $R = 10$ km, 60.2° at $R = 50$ km.

3. 100 km.

5. At L band, $\delta_C = 0.38$ rad (22°) for $\sigma_h = 10$ cm. Surface is "smooth" for any grazing angle when $\sigma_h = 1$ cm.

6. Rayleigh-resonance boundary: $a = 9.55 \times 10^{-3}$ m $= 0.955$ cm. Resonance-optics boundary: $a = 9.55$ cm.

8. 30 samples (1 kHz PRF), 6 (5 kHz), 1 (40 kHz).

Chapter 6

1. The angle of incidence from the transmitter with respect to the target surface normal is equal to the angle of reflection.

3. In a plane perpendicular to the edge.

4. For specular backscatter, surface and edge normals must point back toward the radar.

7. Edge diffraction and surface waves.

8. In free space there is equal energy in the E and H field components.

10. The permeability and permittivity, through the index of refraction.

11. Wavelength in the material is less than free space.

16. When the wave front curvature is so large that the phase change over the target dimensions is insignificant, typically less than 22.5°.

17. Zero.

20. An observer looking in the direction of propagation.

Chapter 7

1. $\sigma = \pi$ m^2. $a_t = 0.093$ m. $a_s/a_t = 10.75$.

3. $8(D/\lambda) = 40$ when $(D/\lambda) = 5$.

5. $\frac{1+2\left(1+\sqrt{2}\right)}{\left[1+\left(1+\sqrt{2}\right)\right]^2}\bar{\sigma}^2 = \frac{1}{2}\bar{\sigma}^2$

7. Exponential pdf: $P\{\sigma > 2\} = e^{-2} = 0.135$. 4-th degree chi-square: $P\{\sigma > 2\} = 5e^{-4} = 0.0916$.

9. $\Delta f = 50$ MHz.

11. $\Delta\theta = 1.14$ mrad $= 0.065°$. Choose scan-to-scan decorrelation.

Chapter 8

1. Error $\approx 2.2236\times10^{-4}$ Hz.

3. The radial velocity is initially close to 200 mph (89.4 m/s), decreases to zero when the aircraft is directly overhead, and decreases asymptotically to −200 mph (−89.4 m/s) as the aircraft flies away.

5. 1 MHz.

7. $T_d = 20$ ms. Rayleigh resolution = 50 Hz.

10. $\Delta f_d = 40$ Hz. $M = 25$.

11. Range change = 0.025 m. Phase change = $2\pi/3$ radians = 120°.

13. 329 Hz.

15. 65.1%.

17. 3 pulses. Apparent range = 20.4 km.

Chapter 9

1. $Q_t = 7.96 \times 10^{-7}$ W/m at target, same at clutter.

3. 10 GHz: $\theta_3 = 1.72°$, $D_{\max} = 10{,}970 = 40.4$ dBi. 32 GHz: $\theta_3 = 0.54°$, $D_{\max} = 112{,}460 = 50.5$ dBi.

5. $D_{\max} = 10{,}440 = 39.9$ dBi. Maximum gain = 37.9 dBi.

7. Feed diameter = 5.5 cm.

9. $\Delta x = 1.574$ cm. Number of elements = 97.

11. $\Delta\theta = 1.707°$. Pointing error = 74.5 km.

13. $\Delta\theta = 6.14° = 1.42$ beamwidths (1 m antenna), 14.2 beamwidths (10 m antenna).

15. 1600 elements. Additional cost = $487,500.

Chapter 10

1. 12.6 MHz of FM. Maximum ripple for 1 MHz of FM = 5 volts.

2. Noise density = −24 dBm/MHz. Total noise power = 3 dBm.

5. Drain voltage ripple = 570 microvolts. Gate voltage ripple = 5.7 microvolts.

7. −75 dBc.

9. Non-depressed collector mode: prime power = 3,750 W, combined efficiency = 13.3%. Depressed collector mode: prime power = 1,500 W, combined efficiency = 33.3%.

12. Answer (e).

14. Module power = 10 W. Array area = 0.56 m^2. Number of elements = 2,240.

16. Answers (a), (b), and (c).

Chapter 11

2. 50×, or 17 dB.

3. LO = 7.2 GHz. The $2f_1 - 3f_2$ intermodulation product falls within the IF band.

5. 1.75 volts.

7. $R_{\text{max}} = -158.5$ km.

9. Noise factor = 3.05.

11. $T_S = 835$ Kelvin.

13. 59 dB.

14. Spurious free dynamic range = 45 dB.

Chapter 12

1. (a) −94 dBc/Hz (b) −6.07 dB

3. Yes.

5. (a) 450×10^3 m^2 (b) 28,125 m^2 (c) yes, two others between the first and last.

7. (a) −20.5 dB (b) 33.5 dB

9. 3,333.33 Hz, −11.11 dB

11. The STALO

13. 8 sections, 6 sections.

Chapter 13

2. Filter output length = 1,039 points. Number of multiplications = 41,560.

3. Number of complex multiplications = 24,576.

5. Input rate = 120 Mbps, Output rate = 80 Mbps (16 bits), 160 Mbps (single precision), 320 Mbps (double precision)

6. $N_p \leq \dfrac{B}{2 \cdot PRF}$

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Chapter 14

1. $k_x = 20.94$ rads/m. $p = 3.33$ cycles/m.

3. $f_s = 35$ MHz.

7. 2 Hz.

9. $k_0 = 21$. $v_0 = 29.53$ m/s. Velocity error = 0.469 m/s.

12. $L_{\max} = \left\lfloor \frac{1}{2}\log_2 K \right\rfloor$. For $K = 256$, $L_{\max} = 4$.
13. Length of $y[n] = 1{,}099$ samples. FFT size = 2,048.
14. 5 samples.
16. Integrated SNR = integration gain = 20 dB. 464 samples integrated noncoherently.

Chapter 15

1. SNR = 8.
3. $T = 3.29$.
4. $P_D = 0.6922$.
6. P_{FA} does not change.
7. $P_D = 0.953$.
9. $P_{FA} = 5 \times 10^{-4}$.
11. $T = 3.035. P_D = 0.798$.
13. $\chi_1 = 8.14 = 9.1$ dB.
15. $\chi_1 = 13.14$ dB.
17. $\chi_c = -6.86$ dB.
18. $\alpha = 0.72$. Better than $\sqrt{N}$, worse than coherent integration (factor of N).

Chapter 16

1. $FAR = 2$ per second.
3. $P_{FA} = 0.0316$.
5. CFAR loss = 0.77 dB.
7. $P_D = 0.56$.
9. $P_{FA} = 0.029$.
11. $\hat{g} = 0.644$. The CFAR can reject up to 3 interfering targets.
13. $\hat{g}_{CA-CFAR} = 0.98$, $\hat{g}_{GOCA-CFAR} = 1.08$, $\hat{g}_{SOCA-CFAR} = 0.89$. $T_{CA} = 25.43$.
15. Answer (e).

Chapter 17

1. 2 kHz in all three cases.
2. Two-way range change = λ m.
4. $K = 128$. Doppler frequency sample spacing = 78.125 Hz.
6. DFT sample spacing = $\pi/32 = 0.0982$ rad/sample, 54.6875 Hz, or 1.641 m/s.
Rayleigh resolution = 116.67 Hz or 3.5 m/s.
7. (a) 12 kHz. (b) 60 kHz. (c) 5.
9. True range bin = 9.
12. $\hat{P} = 50$, $\tilde{f}_0 = -333.33$ Hz, $\hat{\sigma}_f^2 = 103{,}515$ Hz^2.

Chapter 18

1. Separate calibration: worst case uncertainty = 0.6 dB, rms = 0.26 dB. End-to-end: 0.1 dB uncertainty.

5. $J(\hat{m}) = \sigma_w^2/N$.

8. 3.3 dB.

9. −0.28 dB.

10. 32.8.

Chapter 19

1. $X_{N|N} = X_{1|N} = \frac{1}{N}\sum_{i=1}^{N} y_i$, $P_{N|N} = P_{1|N} = \frac{\sigma_w^2}{N}$.

3. $\alpha_k = \max\left\{\frac{1}{k}, \alpha_{SS}\right\}$.

5. Position bias or lag $= L_p = \left(\frac{1}{\alpha} - 1\right) V_0$.

7. Minimum MMSE: $MMSE^p = 1{,}535$, $MMSE^v = 1{,}700$. Minimum process noise covariance: $MMSE^p = 1{,}502$, $MMSE^v = 1{,}404$.

13. (a) Crossrange: $0.9 \le \kappa_1\,(0.34) \le 2.1$ at $R = 400$ km and $0.7 \le \kappa_1\,(28) \le 1.1$ at $R = 5$ km. Range: $0.7 \le \kappa_1\,(19) \le 1.1$. Choose $\kappa_1 = 1.1$ to achieve low peak errors at long ranges and minimum MMSE at short ranges in the cross range. The filter design is defined by $\sigma_v = 33\ (\text{m/s})^2$. (b) $k_1 = 0{:}9 + 0{:}7R/400$.

Chapter 20

1. Rayleigh resolution = 300 m. 0.5 amplitude corresponds to −6 dB. −3 dB range resolution = 175.7 m.

4. $X(\omega) = \tau \dfrac{\sin(\omega\tau/2)}{\omega\tau/2}$.

5. (a) Time-bandwidth product = 500. (b) 250 MHz. (c) 8.67 μs. (d) 2167 samples. (e) Gain = 27 dB, SNR = 24 dB.

8. Doppler shift = 3 kHz; radial velocity = 45 m/s.

9. Length 2, ISR = −3 dB; Length 3, ISR = −6.5 dB; Length 4, ISR = −6 dB; Length 5, ISR = −8 dB; Length 7, ISR = −9.1 dB; Length 11, ISR = −10.8 dB; Length 13, ISR = −11.5 dB.

12. Chip width = 20 ns; bandwidth = 50 MHz; Rayleigh resolution = 3 m; time-bandwidth product = 127; pulse compression gain = 21 dB; Approximate PSR = −21 dB; Doppler shift = 393.7 kHz.

14. 13 length bi-phase Barker, PSR = −22.3 dB; 69 length poly-phase Barker, PSR = −36.8 dB; 144 length P1, PSR = −31.4 dB; 169 length P1, PSR = −32.2 dB; 1023 length MLS, PSR = −30.1 dB; 102 length MPS, PSR = −26.2 dB.

Chapter 21

1. $B_r = 1$ MHz, 10 MHz, 100 MHz, and 1 GHz, respectively.

3. (a) $d = 0.143$ m (b) $PRF_{min} \approx 700$ Hz

5. (a) Change in slant range ≈ 137 m. (b) $\Delta\psi_{QT,MAX} \approx 57{,}400$ radians ≈ 9,130 cycles.

7. (a) $f_{d,min} \approx -1{,}400$ Hz, $f_{d,max} \approx +1{,}400$ Hz. (b) -1.28 rad/m $\le k_u \le +1.28$ rad/m.