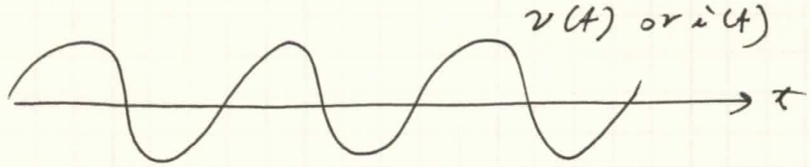
Signal Basics



v(t) and i(t) could be periodic or nonperiodic.

* Time average ([V] or [A])

$$\overline{v} = \langle v(t) \rangle \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)\,dt$$

This is general definition of time average applicable to ~~both~~ periodic, nonperiodic, deterministic or random signal.

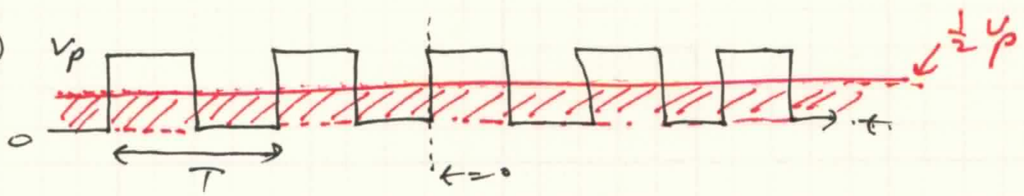
For a periodic signal,

$$\overline{v} = \langle v(t) \rangle \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)\,dt$$

$$= \frac{1}{T} \int_{0}^{T} v(t)\,dt$$

⇒ In electronic design, time-average value means DC-value.

(ex)



$$\overline{v} = \langle v(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} V_p\,dt = \lim_{T\to\infty} \frac{1}{T} \times V_p \times \frac{T}{2}$$

$$= \frac{1}{2} V_p$$

$$= \frac{1}{T} \int_{0}^{T} v(t)\,dt$$

same result.

$$= \frac{1}{T} \int_{0}^{T/2} V_p\,dt = \frac{1}{2} V_p$$

⇒ In electronic circuit design, many time we deal with DC-free AC-signals.

⇒ Time-average process is not much helpful to describe a signal quantity.

⊛ Signal power ([W])

- Instantaneous power : $p(t) = v(t)\, i(t)$

$$= \frac{v(t)^2}{R} = i(t)^2 R$$

- Time-average power

$$\overline{p} = \langle p(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} p(t)\,dt$$

$$= \frac{1}{T}\int_0^T p(t)\,dt \leftarrow \text{if } p(t) \text{ is a periodic power.}$$

- Normalized power
  - ⇒ R is normalized to 1Ω
  - ⇒ represents power of signal itself.

$$\overline{v^2} = \langle v(t)^2 \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} v(t)^2\,dt \leftarrow \text{if } v(t) \text{ is a periodic signal.}$$

- rms value of a signal

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\langle v(t)^2 \rangle} = \sqrt{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} v(t)^2\,dt}$$

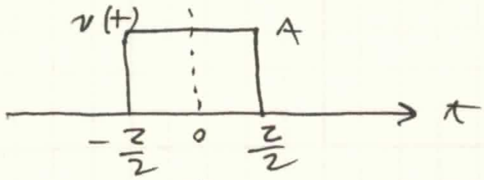
$$= \sqrt{\frac{1}{T}\int_{-T/2}^{T/2} v(t)^2\,dt} \leftarrow \text{if } v(t) \text{ is a periodic signal.}$$

⇒ rms value is very useful to describe a quantity of a DC-free periodic signal and noise.

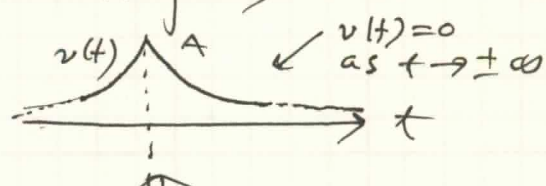
⇒ NOTE: $\boxed{v_{rms}^2 = \overline{v^2} = \text{signal power} = \text{dissipated power to } 1\Omega.}$

⊛ Signal Energy ($[J] = [W \cdot s]$)

Q) why do we need to define a signal energy?
Think about these time-limited signals



strictly time-limited signal



asymptotically time-limited signal

⇒ For both cases, $\overline{v^2} = \langle v(t)^2 \rangle$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} v^2(t)\,dt$$

$$= 0$$

⇒ For a non-periodic signal, signal power can not give us information.
(NOTE: Most physical non-periodic signals ~~will be decaded as t → ±∞~~ which have deterministic behavior (Not random signal) will be decaded as $t \to \pm\infty$.)

⇒ Energy is more meaningful quantity to describe a non-periodic signal

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} v(t)^2\,dt \leftarrow \text{signal energy normalized to } 1\Omega.$$

⇒ But, for a periodic signal this definition gives energy diverging.

⇒ therefore, we define a cycle-energy which means an energy per one cycle.

(*) Energy per cycle for a periodic signal

$$E = \text{Power} \times \text{time}$$

$$= \overline{v^2} \times T = \langle v(t)^2 \rangle \times T$$

$$= \frac{1}{T}\int_0^T v(t)^2 dt \times T$$

$$= \int_0^T v(t)^2 dt$$ ← represents Energy of a periodic signal, normalized to 1Ω.

In many text books this is defined as energy for a periodic signal. But you should keep in mind that this really means energy per cycle.

(*) Energy per unit radian for a periodic signal

$$E = \left(\overline{v^2} \times T\right) \times \frac{1}{2\pi}$$

$$= \overline{v^2} \times \frac{1}{2\pi f_0} \leftarrow f_0 = \frac{1}{T}$$

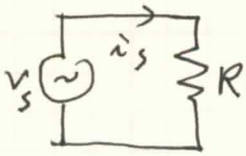
$$= \frac{\overline{v^2}}{\omega_0}$$ ← represents energy per radian of a periodic signal, normalized to 1Ω.

⇒ In general Energy in L and C means energy per unit radian.

⇒ When we compare dissipated energy in R and stored energy in L (or C), we use energy per unit radian.

⇒ This will be clear when discussing a quality factor, Q, of a R-C, R-L and R-L-C Networks.

examples



$v_s = V_p \cos\omega_0 t \rightarrow V_{s,rms} = \frac{V_p}{\sqrt{2}}$

$i_s = \frac{V_s}{R} = \frac{V_p}{R}\cos\omega_0 t \rightarrow i_{s,rms} = \frac{V_p}{\sqrt{2}R}$

$\Rightarrow P_{diss} = V_{s,rms} \cdot i_{s,rms} = \frac{1}{2}\frac{V_p^2}{R}$

This dissipated power, also called "active" power.

$\Rightarrow$ Energy per cycle (dissipated energy) = consumed energy

generally we use this as energy in R.

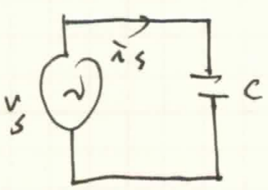
$E/\text{cycle} = P_{diss} \times T \leftarrow T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

$= \frac{1}{2}\frac{V_p^2}{R} \times T = \frac{1}{2}\frac{V_p^2}{R} \times \frac{2\pi}{\omega_0}$

But more exact meaning of energy is energy per unit radian which can be seen in C or L.

$\Rightarrow$ Energy per radian

$E/\text{cycle} \times \frac{1}{2\pi} = \frac{1}{2}\frac{V_p^2}{R} \times \frac{1}{\omega_0}$



$v_s = V_p \cos\omega_0 t \rightarrow V_{s,rms} = \frac{V_p}{\sqrt{2}}$

$i_s = j\omega_0 C V_s = j\omega C V_p \cos\omega_0 t$

$= \omega_0 C V_p \cos(\omega_0 t + \frac{\pi}{2})$

$= -\omega_0 C V_p \sin\omega_0 t \rightarrow i_{s,rms} = \frac{\omega_0 C V_p}{\sqrt{2}}$

$\Rightarrow P_{diss} = \frac{1}{T}\int_0^T v_s \cdot i_s \, dt = 0$

$\Rightarrow$ No power dissipation. But we can define a reactive power which means an imaginary power saved in C

$\Rightarrow P_{reactive} = V_{s,rms} \cdot i_{s,rms} = \frac{1}{2}\omega_0 C V_p^2$

$\Rightarrow$ Stored energy per cycle in C

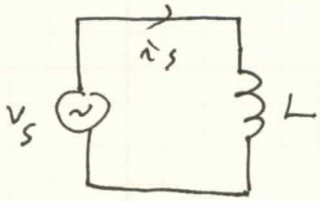
$E/\text{cycle} = \frac{1}{2}\omega_0 C V_p^2 \times T = \frac{1}{2}\omega_0 C V_p^2 \times \left(\frac{2\pi}{\omega_0}\right) = \frac{1}{2} C V_p^2 \cdot 2\pi$

This is general meaning of energy in C

$\Rightarrow$ Stored energy per radian in C

$E/\text{cycle} \times \frac{1}{2\pi} = \frac{1}{2} C V_p^2 = C V_{rms}^2$

examples



$$v_s = V_p \cos\omega_0 t \longrightarrow V_{s,rms} = \frac{V_p}{\sqrt{2}}$$

$$i_s = \frac{V_p}{j\omega_0 L}\cos\omega_0 t$$

$$= \frac{V_p}{\omega_0 L}\cos\left(\omega_0 t - \frac{\pi}{2}\right)$$

$$= \frac{V_p}{\omega_0 L}\sin\omega_0 t \longrightarrow i_{s,rms} = \frac{V_p}{\sqrt{2}\cdot\omega_0 L}$$

$$\Rightarrow P_{diss} = \frac{1}{T}\int_0^T v_s \cdot i_s \, dt = 0$$

$\Rightarrow$ No power dissipation.
But we can define a reactive power which means an imaginary power saved in L

$$\Rightarrow P_{reactive} = V_{s,rms}\cdot i_{s,rms} = \frac{1}{2}\frac{V_p^2}{\omega_0 L}$$

$\Rightarrow$ Stored energy per cycle in L

$$E/cycle = \frac{1}{2}\frac{V_p^2}{\omega_0 L}\times T = \frac{1}{2}\frac{V_p^2}{\omega_0 L}\left(\frac{2\pi}{\omega_0}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\omega_0^2}\cdot\frac{V_p^2}{L}\cdot 2\pi$$

$\Rightarrow$ Stored energy per radian in L

$$E/cycle \times \frac{1}{2\pi} = \frac{1}{2}\frac{1}{\omega_0^2}\frac{V_p^2}{L}$$

This is general meaning of ~~an~~ energy in L.

$$= \frac{1}{2}L\left(\frac{V_p^2}{\omega_0^2 L^2}\right)$$

$$= \frac{1}{2}L\, i_p^2$$

$$= L\, i_{rms}^2$$

⊛ Correlation function between signals.

⇒ Assume two different DC-free signals, $v_1(t)$ and $v_2(t)$.

⇒ We want to measure similarity between the signals (or degree of correlation between the signal).

⇒ correlation function

$$R_{v_1.v_2}(\tau) = \langle v_1(t) \times v_2(t+\tau) \rangle$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v_1(t) . v_2(t+\tau)\, dt$$

$$= \cancel{\lim} \frac{1}{T} \int_0^{T} v_1(t) . v_2(t+\tau)\, dt \leftarrow$$

(if) $v_1(t)$ and $v_2(t)$ are periodic with T-periodicity

NOTE: This correlation function is function of time offset τ.

⇒ when ~~[illegible]~~, $v_2(t) = v_1(t)$ → auto correlation function

⇒ auto correlation function

$$R_{v_1.v_1}(\tau) = \langle v_1(t) \times v_1(t+\tau) \rangle$$

NOTE: $R_{v_1.v_1}(0)$ = Normalized power of $v_1(t)$.

$$= \overline{v^2}$$

⇒ Normalized correlation function

$$R_{N,v_1.v_2}(\tau) = \frac{R_{v_1.v_2}(\tau)}{v_{1,rms} \times V_{2,rms}}$$

⇒ Correlation Coefficient ⇒ (if) $\tau = 0$

$$C = R_{N,v_1.v_2}(0) \quad , \quad -1 \le C \le 1$$

⊛ In general, if there is no correlation between V and I, then No power dissipation.

example

$v_1 = V_p \cos\omega t$

$v_2 = V_s \cos(\omega t + \phi)$

$$\Rightarrow R_{v1,v2}(0) = \frac{1}{T}\int_0^T v_1(t)\cdot v_2(t)\,dt$$

$$= \frac{1}{T}\int_0^T V_p \cos\omega t \cdot V_s \cos(\omega t + \phi)\,dt$$

$$= \frac{1}{2} V_p \cdot V_s \cos\phi$$

$$\Rightarrow R_{N,v1,v2}(0) = \frac{R_{v1,v2}(0)}{V_{1,rms} \times V_{2,rms}} = \frac{\frac{1}{2}V_p \cdot V_s \cos\phi}{\frac{1}{\sqrt{2}}V_p \cdot \frac{1}{\sqrt{2}}V_s} = \cos\phi$$

$$\Rightarrow C = R_{N,v1,v2}(0) = \cos\phi$$

if $\phi = 0$ (or π) $\rightarrow$ $C = \pm 1$ $\rightarrow$ 100% correlation

if $\phi = \frac{\pi}{2}$ $\rightarrow$ $C = 0$ $\rightarrow$ No correlation

$\Rightarrow$ ① perfect correlation

$$\begin{array}{r} v_1 = V_p \cos\omega t \\ + \quad v_2 = V_s \cos\omega t \\ \hline = (V_p + V_s)\cos\omega t \end{array}$$

Signals can be added directly in voltage or current domain. (linear-domain)

② No correlation ("independent" between signal)

$$\begin{array}{r} v_1 = V_p \cos\omega t \\ + \quad v_2 = V_s \sin\omega t \\ \hline = \sqrt{V_p^2 + V_s^2}\,\cos(\omega t + \theta) \end{array}$$

$\left(\theta = \tan^{-1}\frac{V_s}{V_p}\right)$

Signals must be added in power-domain (square-domain), then take a ~~square~~ root.

Q) How about subtraction of signals?

example

① Perfect correlation $\Rightarrow$ $\left.\begin{array}{l} v_1 = V_p \cos\omega t \\ v_2 = V_s \cos\omega t \end{array}\right\} \Rightarrow \langle v_1(t)\cdot v_2(t)\rangle = \overline{v_1 \cdot v_2} \neq 0$

② No correlation $\Rightarrow$ $\left.\begin{array}{l} v_1 = V_p \cos\omega t \\ v_2 = V_s \sin\omega t \end{array}\right\} \Rightarrow \langle v_1(t)\cdot v_2(t)\rangle = \overline{v_1 \cdot v_2} = 0$

$\Rightarrow$ Time average of product of two uncorrelated signal $= 0$.

$\Rightarrow$ this is apparent result due to no correlation (think about definition of correlation in page-9)

example

$$v_1 = V_p \cos\omega t$$

$$v_2 = V_s \cos(\omega t + \phi)$$

$\leftarrow$ we can decompose $v_2(t)$ into two parts: correlated part with v_1 and uncorrelated part.

$$= \underbrace{V_s \cos\phi \cos\omega t}_{= V_{2c}(t)} - \underbrace{V_s \sin\phi \sin\omega t}_{V_{2u}(t)}$$

$$= V_{2c}(t) + V_{2u}(t)$$

correlated part $\quad$ uncorrelated part.

$$\Rightarrow \langle V_1 \cdot V_2 \rangle = \langle V_1 (V_{2c} + V_{2u}) \rangle$$

$$= \langle V_1 \cdot V_{2c} \rangle + \cancel{\langle V_1 \cdot V_{2u} \rangle}^{\,0}$$

$$= \frac{1}{2} V_p \cdot V_s \cos\phi$$

if $|v_1| = |v_2|$ and v_1 & v_2 are correlated.

⇒ v_1 or v_2 can be decomposed into correlated and non-correlated parts.

⇒ correlated part of v_1 (or v_2) can be expressed in terms of v_2 (or v_1) and correlation coefficient C.

⇒ $v_1(t)$, $v_2(t)$ $\quad |v_1| = |v_2|$

⇒ $v_2(t) = v_{2c}(t) + v_{2u}(t)$

$= C v_1(t) + (v_2(t) - C v_1(t))$

correlated part to $v_1(t)$ $\qquad$ uncorrelated part to $v_1(t)$

ex $\quad v_1 = \cos\omega t$

$v_2 = \cos(\omega t + \phi)$

$= v_{2c}(t) + v_{2u}(t)$

$= C v_1(t) + (v_2(t) - C v_1(t))$

$= \cos\phi\cos\omega t + (\cos(\omega t + \phi) - \cos\phi\cos\omega t)$

$= \cos\phi\cos\omega t - \sin\phi\sin\omega t$

ex $\quad \overline{(v_1 + v_2)^2} = \overline{v_1^2 + v_2^2 + 2 v_1 v_2}$

$v_1 = \cos\omega t$

$v_2 = \cos(\omega t + \phi)$

$= \overline{v_1^2} + \overline{v_2^2} + \overline{2 v_1 (v_{2c} + v_{2u})}$

$= \overline{v_1^2} + \overline{v_2^2} + \overline{2 v_1 \cdot v_{2c}} + \overline{\cancel{2 v_1 \cdot v_{2u}}} \rightarrow 0$

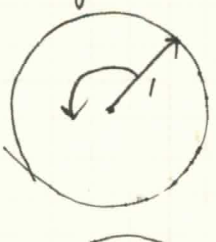
This technique is frequently used when calculating noise amounts.

$= v_{1,rms}^2 + v_{2,rms}^2 + 2 v_{1,rms} \cdot v_{2c,rms}$

$= \frac{1}{2} + \frac{1}{2} + 2 \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cos\phi = 1 + \cos\phi$

(*) positive & negative frequency

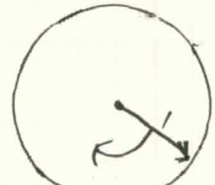
$e^{j\omega t} \rightarrow$



→ rotating counter-clock wise direction

→ $\omega = \frac{2\pi}{T}$ = rotational speed = angular frequency

$e^{-j\omega t} \rightarrow$

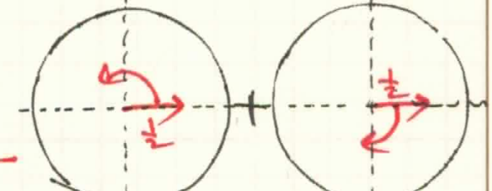


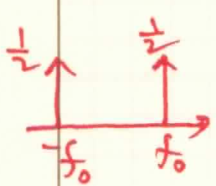
→ rotating clock-wise direction.

(*) cosine & sine

positive freq   negative freq

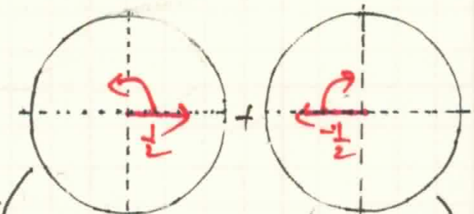
$$\cos \omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \Rightarrow$$



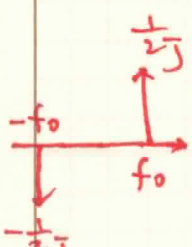


← superposition of two rotating vectors generates cosine function.

$$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Rightarrow$$

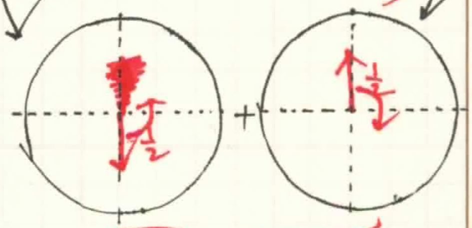


$\frac{1}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$



← superposition of these two rotating vectors generates sine function

-90° phase shift for positive freq.



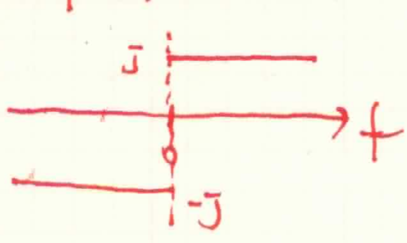
$\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

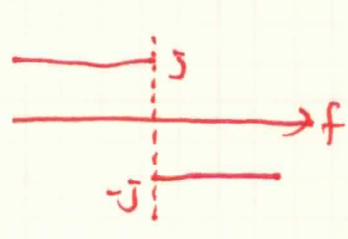
-90° phase shift for negative freq

(*) NOTE:

operator j (90°-phase shift) works ~~[illegible]~~ differently for positive and negative frequency

$+90^\circ$-phase shift -90°-phase shift





⇒ Hilbert Transform.

(*) phasor notation

$v(t) = V\cos(\omega t + \phi) = Re(V \cdot e^{j\phi} \cdot e^{j\omega t}) \Rightarrow \boxed{\mathbf{V} = Ve^{j\phi}}$

$i(t) = I\cos(\omega t + \theta) = Re(I e^{j\theta} e^{j\omega t}) \Rightarrow \boxed{\mathbf{I} = Ie^{j\theta}}$

phasor ⇒ express signal in terms of magnitude & phase only.

$\Rightarrow v(t) = \frac{1}{2}(\mathbf{V} e^{j\omega t} + \mathbf{V}^* e^{-j\omega t})$

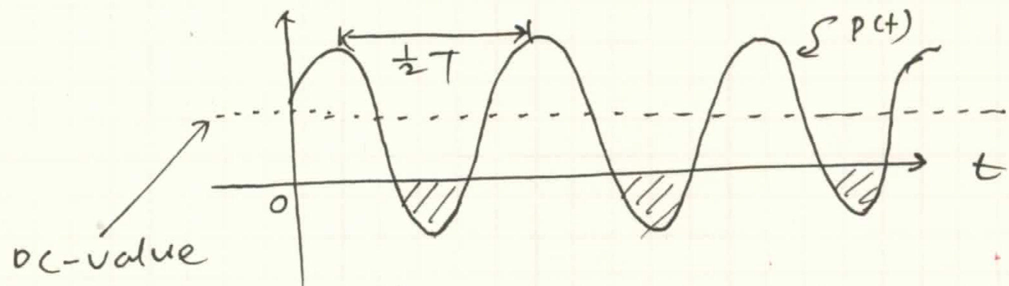
$i(t) = \frac{1}{2}(\mathbf{I} e^{j\omega t} + \mathbf{I}^* e^{-j\omega t})$

$\Rightarrow p(t) = v(t) \cdot i(t) = \frac{1}{4}(\mathbf{V}\mathbf{I}^* + \mathbf{V}^*\mathbf{I} + \mathbf{V}\mathbf{I} e^{j2\omega t} + \mathbf{V}^*\mathbf{I}^* e^{-j2\omega t})$

$= \frac{1}{2}Re(\mathbf{V}\mathbf{I}^*) + \frac{1}{2}Re(\mathbf{V}\mathbf{I} e^{j2\omega t})$

$= \frac{1}{2}Re(\mathbf{V}\mathbf{I}^*) + \frac{1}{2}Re(VI e^{j(2\omega t + \phi + \theta)})$

$= \frac{1}{2}Re(\mathbf{V}\mathbf{I}^*) + \frac{1}{2}VI\cos(2\omega t + \phi + \theta)$



$= \bar{p} = \frac{1}{2}Re(\mathbf{V}\mathbf{I}^*) = \frac{1}{2}VI\cos(\phi - \theta)$

$$\boxed{\bar{P} = P_{average} = \frac{1}{2}Re(\mathbf{V}\mathbf{I}^*)}$$

Another average power expression in terms of phasors.

In general

$$v(t) = V_1 \cos(\omega t + \phi_1) + V_2 \cos(2\omega t + \phi_2) + \cdots + V_M \cos(M\omega t + \phi_M)$$
$$i(t) = I_1 \cos(\omega t + \theta_1) + I_2 \cos(2\omega t + \theta_2) + \cdots + I_M \cos(M\omega t + \theta_M)$$

$$\Rightarrow v(t) = \mathrm{Re}\left(\sum_{n=1}^{M} \mathbb{V}_n e^{jn\omega t}\right)$$

$$i(t) = \mathrm{Re}\left(\sum_{n=1}^{M} \mathbb{I}_n e^{jn\omega t}\right)$$

$$\Rightarrow \boxed{\bar{P} = \frac{1}{2} \mathrm{Re}\left(\sum_{n=1}^{M} \mathbb{V}_n \cdot \mathbb{I}_n^*\right)}$$

average power involves $\mathbb{V}$ and $\mathbb{I}$ for the same harmonic.

when rms value used,

$$\boxed{\bar{P} = \mathrm{Re}\left(\sum_{n=1}^{M} \mathbb{V}_{n,rms} \mathbb{I}_{n,rms}^*\right)}$$

NOTE: $\frac{1}{2}$-factor disappeared.

ex $v(t) = V_1 \cos(\omega t + \phi_1) + V_2 \cos(2\omega t + \phi_2)$

$i(t) = I_1 \cos(\omega t + \theta_1) + I_2 \cos(2\omega t + \theta_2)$

$$\Rightarrow \bar{P} = \frac{1}{2} \mathrm{Re}\left(\sum_{n=1}^{2} \mathbb{V}_n \cdot \mathbb{I}_n^*\right)$$

$$= \frac{1}{2} V_1 I_1 \cos(\phi_1 - \theta_1) + \frac{1}{2} V_2 I_2 \cos(\phi_2 - \theta_2)$$