Thermal Noise in Linear, Lossy, Electromagnetic Media

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Citation: J. Appl. Phys. **33**, 2572 (1962); doi: 10.1063/1.1729022
View online: http://dx.doi.org/10.1063/1.1729022
View Table of Contents: http://jap.aip.org/resource/1/JAPIAU/v33/i8
Published by the American Institute of Physics.

The reverse procedure results in a decrease in the $1/f$ noise.

On this basis the n-type surface layer produced by ion bombardment[9] causes the $1/f$ noise from a freshly bombarded n-type bulk specimen to be low in magnitude, and high in magnitude from a p-type bulk specimen. This is exactly what was observed; the ion bombardment either increased or decreased the $1/f$ noise, depending on the sign of the bulk conductivity.

In the absence of surface contamination, an ion-bombarded and annealed surface is slightly p type.[9] The annealing of the ion-bombardment damaged surface then changes the surface conductivity from n to p type. This change in the surface conductivity is also accompanied by a change in the $1/f$ noise, a decrease for the p-type bulk specimen, and a slight increase for the n-type bulk specimen. The large increase in the $1/f$ noise from the n-type silicon that was observed after the formation of the boron-doped p skin was due to the strong inversion layer on the surface caused by the surface doping.

The elimination of the slow surface states by surface cleaning does not appreciably affect either the magnitude or the spectrum of the $1/f$ noise from these semiconductor specimens. The changes in the noise observed during the cleaning cycle are explainable in terms of the changes in the sign of the surface conductivity relative to that of the bulk. It therefore appears that $1/f$ noise may possibly be due to some process occurring in the space-charge region. The exact mechanism responsible for the production of this noise and its spectrum is still a matter of conjecture, however.

ACKNOWLEDGMENTS

I would like to express my appreciation to J. T. Law for several discussions on the cleaning technique and to H. C. Montgomery for a critical reading of the manuscript.

JOURNAL OF APPLIED PHYSICS VOLUME 33, NUMBER 8 AUGUST 1962

Thermal Noise in Linear, Lossy, Electromagnetic Media*

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(Received February 5, 1962)

The thermal radiation of lossy media is described by a random current-source term that is introduced into Maxwell's equations. The media treated are linear, in general anisotropic, nonuniform, and such that the constituent relations are not local relations, but are described in terms of a Green's operator. For such media, in thermodynamic equilibrium, it is shown that simple relations exist between the correlation and spectral density matrices of the random current-source field and the Green's conductivity operator. These relations are analogous to Nyquist's theorem of lumped circuits.

1. INTRODUCTION

THERMAL radiation from dissipative bodies is due to the random thermal motion of the charges in the bodies. It allows two bodies at different temperatures to exchange energy and eventually reach thermodynamic equilibrium even if there is no material contact between the bodies. Once the two bodies have reached the same temperature, no more energy is exchanged between them. This equilibrium is sustained because each body absorbs as much power from the radiation of the other body as it transfers power to the other body by its own radiation.

There are many cases of interest for which the thermal radiation from a body needs to be determined. If the body is at a uniform temperature, one approach that may be used for studying the radiation may be called the integral approach. The body as a whole is considered to be nonradiating, and the power that it absorbs from its surroundings, which is assumed to be at the temperature of the body, can be computed. This power is set equal to the power radiated by the body. In this approach we make no attempt to determine the noise current fluctuations that are the cause of the thermal radiation. In those cases in which the temperature of the body is nonuniform this approach fails. In most cases, the radiated power is given by integrals that are difficult to evaluate and the simple underlying physical principles are therefore obscured.

Another approach, which may be called the "Nyquist-source treatment," focuses attention upon the sources of the radiation, the relevant statistical properties of which are determined. Once these are known, the determination of the radiation is conceptually a simple problem, although usually mathematical difficulties arise.

A step toward the determination of the current

* This work was supported in part by the U. S. Army Signal Corps, the U. S. Air Force Office of Scientific Research, and the Office of Naval Research.

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fluctuations in a linear, dissipative medium has been taken by S. M. Rytov.[1] He considered *ordinary conducting media* and showed that by postulating some simple correlations for the current fluctuations in such media, a correct description of the thermal electromagnetic field can be obtained. Recently, one of the authors has been able to determine the correlations of the current fluctuations in all *uniform, linear, dissipative* media.[2]

It is the purpose of this paper to generalize these results to *nonuniform media*. We include anisotropic media, dispersive media, and media for which the relation between the current density and the electromagnetic field is not a local relation. Within certain limits,[2] this approach can be used to treat cases with nonuniform temperature distributions. Another attractive feature of such a treatment is the simple relation between the statistical properties of the source fluctuations and the loss characteristics of the medium.

2. LINEAR, LOSSY, ELECTROMAGNETIC MEDIA

The equations needed to solve electromagnetic problems in a material medium are Maxwell's equations to which the constituent relations are added. The constituent relations express the total (conduction and displacement) electric and magnetic current densities $\mathbf{j}(\mathbf{r},t)$ and $\mathbf{k}(\mathbf{r},t)$ in terms of the fields that induce them. Maxwell's equations are

$$\begin{aligned}\nabla\times\mathbf{e}(\mathbf{r},t)&=-\mathbf{k}(\mathbf{r},t)\\ \nabla\times\mathbf{h}(\mathbf{r},t)&=\mathbf{j}(\mathbf{r},t).\end{aligned}\tag{1}$$

We use lower case letters to denote time-dependent quantities. The complex representations of these quantities in the sinusoidal steady state will be denoted by corresponding capital letters.

We shall study the most general form assumed by the electric current density and the electromagnetic field. We shall not treat explicitly the case in which the magnetic field causes an electric current, nor the case in which electric field would cause a magnetic current. The generalization of the results obtained here to cover such cases is straightforward. For similar reasons, we shall limit ourselves to losses caused by the electric currents only.

Consider a lossy medium occupying a region M of space (Fig. 1). The region M is part of a region C that contains, except for M, a lossless dielectric medium. We suppose that the lossy medium has the following properties:

(A) The electric field $\mathbf{e}(\mathbf{r},t)$ in M determines uniquely the total induced current-density field $\mathbf{j}(\mathbf{r},t)$ in M.

(B) The lossy medium is linear. A superposition of two electric fields gives rise to a current field that is the superposition of the current fields induced by the electric fields taken separately.

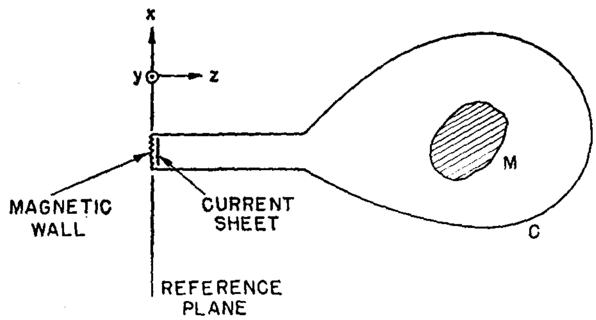

FIG. 1. Enclosure surrounding medium M.

(C) The lossy medium is time-invariant. As a consequence of the foregoing assumptions, the constituent relation for the region C is of the form

$$\boldsymbol{j}(\mathbf{r},t)=\int_{-\infty}^{+\infty}d\tau\int_C d\mathbf{s}\,\boldsymbol{y}(\mathbf{r},\mathbf{s},t)\boldsymbol{e}(\mathbf{s},\tau).\tag{2}$$

The integration over the variables $\mathbf{s}$ includes all volume elements $d\mathbf{s}$ of C. We have represented the vectors $\mathbf{j}(\mathbf{r},t)$ and $\mathbf{e}(\mathbf{r},t)$, by the (3×1) column matrices $\boldsymbol{j}(\mathbf{r},t)$ and $\boldsymbol{e}(\mathbf{r},t)$ in boldface italics. The elements of these matrices are the components of the vectors along the coordinate directions of an orthogonal Cartesian frame —the (3×3) matrix $\boldsymbol{y}(\mathbf{r},\mathbf{s},t)$ is the matrix representation of the Green's "admittivity" tensor (the tensor relating the total current, convection plus displacement, to the electric field). If the electric field is given by

$$\boldsymbol{e}(\mathbf{r},t)=\delta(\mathbf{r}-\mathbf{s})\delta(t-\tau)\boldsymbol{e}_s,\tag{3}$$

where $\boldsymbol{e}_s$ stands for any constant (3×1) column matrix, the current response is

$$\boldsymbol{j}(\mathbf{r},t)=\boldsymbol{y}(\mathbf{r},\mathbf{s},t-\tau)\boldsymbol{e}_s.\tag{4}$$

The fact that $\boldsymbol{y}(\mathbf{r},\mathbf{s},t-\tau)$ depends upon $(t-\tau)$, and not explicitly upon t, is due to the time-invariance of the medium.

In the sinusoidal steady state of frequency f, we represent time functions by their complex representations, for example,

$$\boldsymbol{e}(\mathbf{r},t)=\mathrm{Re}[\boldsymbol{E}(\mathbf{r})e^{i2\pi ft}].\tag{5}$$

The constitutent relation becomes

$$\boldsymbol{J}(\mathbf{r})=\int_C d\mathbf{s}\,\boldsymbol{Y}(\mathbf{r},\mathbf{s},f)\boldsymbol{E}(\mathbf{s}).\tag{6}$$

The new Green's "admittivity" matrix $\boldsymbol{Y}(\mathbf{r},\mathbf{s},f)$ is the Fourier transform of $\boldsymbol{y}(\mathbf{r},\mathbf{s},t)$, for example,

$$\boldsymbol{Y}(\mathbf{r},\mathbf{s},f)=\int_{-\infty}^{+\infty}dT\,\boldsymbol{y}(\mathbf{r},\mathbf{s},T)e^{-i2\pi fT}.\tag{7}$$

[1] S. M. Rytov, *Theory of Electric Fluctuations and Thermal Radiation* (Electronics Research Directorate, U. S. Air Force, Bedford, Massachusetts, 1959), translation.
[2] H. A. Haus, J. Appl. Phys. **32**, 493 (1961).

The average power loss in the medium per unit frequency is

$$P=\tfrac{1}{4}\int_C d\mathbf{r}E^+(\mathbf{r})J(\mathbf{r})+J^+(\mathbf{r})E(\mathbf{r})$$
$$=\tfrac{1}{4}\iint_C d\mathbf{r}d\mathbf{s}E^+(\mathbf{r})[Y(\mathbf{r},\mathbf{s},f)+Y^+(\mathbf{s},\mathbf{r},f)]\cdot E(\mathbf{s}). \quad (8)$$

The superscript plus indicates the Hermitian adjoint matrix.

It is clear that the same value for P is obtained if the integration in (8) is extended over M only instead of over C. The loss matrix $[Y^+(\mathbf{r},\mathbf{s},f)+Y^+(\mathbf{s},\mathbf{r},f)]$ is dependent on the lossy medium M, and is independent of the choice of C. For lossless media, the loss matrix vanishes.

Although Maxwell's equations and the constitutent relations are sufficient to solve most electromagnetic problems, they are insufficient for noise studies. The current density derived from the constitutent relations represents only the current driven by the electromagnetic field. Besides these driven currents we have to consider the current density fluctuations caused by the random motion of the charges. They can be taken into account by introducing in Maxwell's equations a random driving current-density distribution, $\mathbf{j}_n(\mathbf{r},t)$, which is independent of the electromagnetic field. It is, in general, necessary to introduce such Langevin terms for both the magnetic current density and the electric current density. Because we restrict ourselves for simplicity to the case in which all losses are due to the electric field, we do not have to introduce magnetic current-density fluctuations. The equations describing electromagnetic phenomena in the medium, including noise phenomena, are

$$\begin{aligned}\nabla\times\mathbf{e}(\mathbf{r},t)&=-\mathbf{k}(\mathbf{r},t),\\ \nabla\times\mathbf{h}(\mathbf{r},t)&=\mathbf{j}(\mathbf{r},t)+\mathbf{j}_n(\mathbf{r},t).\end{aligned} \quad (9)$$

The driven current densities $\mathbf{j}(\mathbf{r},t)$ and $\mathbf{k}(\mathbf{r},t)$ have matrix representations given by relations of the form (2). Our problem is reduced to that of finding the properties of $\mathbf{j}_n(\mathbf{r},t)$ when the medium under study is in thermodynamic equilibrium.

3. OPERATOR FORMALISM

It turns out to be convenient to introduce some short notations. Let us consider a relation of the form

$$U(\mathbf{r})=\int_C d\mathbf{s}A(\mathbf{r},\mathbf{s})V(\mathbf{s}). \quad (10)$$

Such a relation establishes a linear transformation between the vector fields $\mathbf{V}(\mathbf{r})$ and $\mathbf{U}(\mathbf{r})$ that enter the relation in their matrix representations. This relation therefore defines a linear operator denoted by A. We then write (10) in the following short notation

$$U=AV. \quad (11)$$

We also introduce a special notation for some integrals. We set

$$U^+V=\int_C d\mathbf{r}U^+(\mathbf{r})V(\mathbf{r}), \quad (12)$$

where $U^+(\mathbf{r})$ is the Hermitian conjugate of $U(\mathbf{r})$, the row matrix obtained from $U(\mathbf{r})$ by transposition and complex conjugation. It is easy to see that

$$U^+V=(V^+U)^*. \quad (13)$$

Here the star denotes complex conjugate.

If in Eq. (10) we use instead of the matrix $A(\mathbf{r},\mathbf{s})$ the matrix $A^+(\mathbf{s},\mathbf{r})$ the Hermitian conjugate with interchanged variables $\mathbf{r}$ and $\mathbf{s}$, we shall denote the resulting operation by A^+V. We have

$$U^+(AV)=(A^+U)^+V, \quad (14)$$

and

$$(AB)^+=B^+A^+. \quad (15)$$

In (15), B is, like A, a linear operator. The operator $\nabla\times$ also transforms a vector field into another vector field and the transformation is linear. We shall denote this operator by the symbol ***curl***.

The reader will have noticed that the introduced notations can have a natural interpretation when one considers the set of vector fields $\mathbf{U}(\mathbf{r})$ defined over C as constituting an abstract vector space. A *vector*, such as U, of that abstract vector space represents a *vector field* $\mathbf{U}(\mathbf{r})$ defined over the region C of physical space. The scalar product of two abstract vectors U and V is defined by relation (12). The linear operators are equivalent to linear transformations in abstract vector space.

If we use the introduced formalism, the constituent relations (2) and (6) become

$$j(t)=\int_{-\infty}^{+\infty}d\tau y(t-\tau)e(\tau), \quad (16)$$

and

$$J=YE. \quad (17)$$

The power loss per unit frequency is given by

$$P=\tfrac{1}{4}[E^+Y+Y^+E]. \quad (18)$$

We shall assume henceforth that the loss operator $G=\tfrac{1}{2}[Y+Y^+]$ is positive definite. This implies that any electric field gives rise to some loss. It might be that, because of idealizations introduced in the mathematical description of a medium, G is only positive semidefinite. The introduction into such a medium of an arbitrarily small conductivity would restore the definiteness of G. We believe that the results apply even when G is positive semidefinite.

4. OPEN-CIRCUIT VOLTAGE AND IMPEDANCE OF A LOSSY CAVITY: STEADY-STATE ANALYSIS

If we surround the region C by perfectly conducting walls, extending one side into a uniform waveguide, we obtain a cavity that is partly filled with the lossy medium and fed by a waveguide. We impress at the cross section $z=0$ in the waveguide an H field pertaining to one single-waveguide mode with the normalized transverse field pattern $\mathbf{h}_T(x,y)$. This situation can be described by an equivalent circuit with one terminal pair to the derivation of which we now turn.

The stated boundary value problem may be formulated as a problem involving a *closed* cavity with internal driving currents. For this purpose, we close the waveguide at the reference plane $z=0$ by a magnetic wall. At an infinitesimal distance to the right of the magnetic wall we introduce a current sheet. The current density belonging to that sheet is

$$\mathbf{j}_d(\mathbf{r},t)=-\delta(z-0+)\mathbf{e}_T(x,y)i_0(t), \tag{19}$$

where $\mathbf{e}_T(x,y)$ is the normalized transverse electric field pattern of the mode under consideration, $\mathbf{e}_T(x,y)=-\mathbf{i}_z\times\mathbf{h}_T(x,y)$. The transverse magnetic field at the right of the current sheet becomes

$$\mathbf{h}_T(x,y,0+,t)=\mathbf{h}_T(x,y)i_0(t). \tag{20}$$

This excitation produces an electric field $\mathbf{e}(x,y,z,t)$ throughout the cavity that leads to a field pattern at the reference plane $\mathbf{e}(x,y,0+,t)$. The amplitude $v_0(t)$ of the electric field of the mode whose magnetic field is used to excite the cavity is

$$v_0(t)=\int_{\substack{\text{reference}\\ \text{plane}}} dxdy\,\mathbf{e}_T(x,y)\cdot\mathbf{e}(x,y,0+,t). \tag{21}$$

Part of this voltage will depend linearly upon $i_0(t)$; the other part will be caused by the random current field $\mathbf{j}_n(\mathbf{r},t)$ in the lossy medium. When these relations are determined we shall have found the one terminal pair equivalent circuit.

We introduce a singular vector field, defined as

$$\mathbf{W}(\mathbf{r})=\delta(z-0+)\mathbf{e}_T(x,y), \tag{22}$$

in terms of which (19) and (22) can be written as

$$j_d=-i_0(t)\boldsymbol{W}, \tag{23}$$

and

$$v_0(t)=\boldsymbol{W}^+\boldsymbol{e}(t). \tag{24}$$

Before treating the case in which $\mathbf{j}_n(\mathbf{r},t)$ has a random time dependence, we compute the output voltage and impedance of the cavity one-port, when all quantities have a sinusoidal time dependence of some frequency f. The electromagnetic field is determined by the equations

$$\boldsymbol{curlE}=-j\boldsymbol{XH}, \tag{25}$$

$$\boldsymbol{curlH}=\boldsymbol{YE}+\boldsymbol{J}_d+\boldsymbol{J}_n=j\boldsymbol{BE}+\boldsymbol{GE}+\boldsymbol{J}_d+\boldsymbol{J}_n, \tag{26}$$

and we take into account the boundary conditions at the cavity walls. We have split up the electric current operator $\boldsymbol{Y}$ into its loss part $\boldsymbol{G}$ and its lossless part $j\boldsymbol{B}$. We have

$$\boldsymbol{G}=\tfrac{1}{2}[\boldsymbol{Y}+\boldsymbol{Y}^+], \tag{27}$$

and

$$j\boldsymbol{B}=\tfrac{1}{2}[\boldsymbol{Y}-\boldsymbol{Y}^+]. \tag{28}$$

The operators $\boldsymbol{G}$ and $\boldsymbol{B}$ are Hermitian. Because the magnetic current operator is lossless, it can be written in the form $j\boldsymbol{X}$, where $\boldsymbol{X}$ is a Hermitian operator.

The problem now becomes equivalent to solving simultaneously the equations

$$\begin{aligned}\boldsymbol{curlE}&=-j\boldsymbol{XH},\\ \boldsymbol{curlH}&=j\boldsymbol{BE}+\boldsymbol{I},\end{aligned} \tag{29}$$

$$\boldsymbol{I}=\boldsymbol{GE}+\boldsymbol{J}_d+\boldsymbol{J}_n, \tag{30}$$

and taking the boundary conditions into account.

Equations (29) are Maxwell's equations for a cavity filled by a lossless medium having $j\boldsymbol{X}$ and $j\boldsymbol{B}$ as the magnetic and electric constituent operators. The cavity has lossless walls and is driven by a current density field $\boldsymbol{I}$. The losses of the real cavity are taken into account by the fact that $\boldsymbol{I}$ depends upon $\boldsymbol{E}$ through Eq. (30). The solution of our problem will be in the form

$$\boldsymbol{E}=\boldsymbol{ZI}. \tag{31}$$

The operator $\boldsymbol{Z}$ is derived from the Green's electric field matrix $\boldsymbol{Z}(\mathbf{r},\mathbf{s})$ of the ficticious cavity that is filled by the lossless medium characterized by the constituent operators $j\boldsymbol{X}$ and $j\boldsymbol{B}$. If the driving current is

$$\boldsymbol{I}(\mathbf{r})=\delta(\mathbf{r}-\mathbf{s})\boldsymbol{I}_s, \tag{32}$$

where $\boldsymbol{I}_s$ represents any constant 3×1 column matrix, the electric field in the cavity is

$$\boldsymbol{E}(\mathbf{r})=\boldsymbol{Z}(\mathbf{r},\mathbf{s})\boldsymbol{I}_s. \tag{33}$$

Because the (fictitious) cavity is lossless, the power dissipated in the cavity is zero whatever the current distribution in the cavity may be. It follows that $\boldsymbol{Z}$ is anti-Hermitian, for example,

$$\boldsymbol{Z}+\boldsymbol{Z}^+=0. \tag{34}$$

We eliminate $\boldsymbol{I}$ between (30) and (31). We find that

$$(\boldsymbol{1}-\boldsymbol{ZG})\boldsymbol{E}=\boldsymbol{ZJ}_n+\boldsymbol{ZJ}_d. \tag{35}$$

The operator $\boldsymbol{1}$ represents the identity, that is, it transforms a vector field into itself. The positive definiteness of $\boldsymbol{G}$ ensures that $(\boldsymbol{1}-\boldsymbol{ZG})$ has an inverse operator; this we denote $\boldsymbol{Q}$. The formal solution to our problem is then given by

$$\boldsymbol{E}=\boldsymbol{QZJ}_n+\boldsymbol{QZJ}_d=\boldsymbol{QZJ}_n-I_0\boldsymbol{QZW}. \tag{36}$$

The voltage across the terminals of the cavity one-port is, then,

$$V_0=\boldsymbol{W}^+\boldsymbol{E}=\boldsymbol{W}^+\boldsymbol{QZJ}_n-I_0\boldsymbol{W}^+\boldsymbol{QZW}. \tag{37}$$

Part of V_0 arises from the driving current I_0, part of it is caused by the internal driving current field J_n. The impedance $Z_0(f)$ of the one-port is given by

$$Z_0(f) = -W^+QZW. \tag{38}$$

The open-circuit voltage is obtained by setting in (37) the driving current I_0 equal to zero. The open-circuit voltage is then given by

$$V_{0,n} = W^+QZJ_n. \tag{39}$$

Let us compute the resistive part $R_0(f)$ of the impedance:

$$\begin{aligned} 2R_0(f) &= Z_0(f) + Z_0^*(f) \\ &= -W^+(QZ + Z^+Q^+)W. \end{aligned} \tag{40}$$

From the definition of Q, it follows that

$$Q(1 - ZG) = 1 \tag{41}$$

$$(1 - G^+Z^+)Q^+ = 1. \tag{42}$$

We multiply Eq. (41) by Z^+Q^+ from the right, and Eq. (42) by QZ from the left. We add the two expressions and, noting that Z is anti-Hermitian, we obtain

$$QZ + Z^+Q^+ = -2QZGZ^+Q^+. \tag{43}$$

Therefore,

$$2R_0(f) = 2W^+QZGZ^+Q^+W. \tag{44}$$

5. NOISE OUTPUT VOLTAGE OF A LOSSY CAVITY

The fact that we have an expression, (39), for the open-circuit voltage of the one-port when $\mathbf{j}_n(\mathbf{r},t)$ has a sinusoidal time variation of any frequency, allows us to find the spectral density[3] of this voltage when $j_n(\mathbf{r},t)$ is random. According to the results of the Appendix, we have

$$\phi_0(f) = W^+QZ\phi_nZ^+Q^+W. \tag{45}$$

The symbol $\phi_0(f)$ is the spectral density of the open-circuit noise voltage $v_{0,n}(t)$ of the one-port. It is the Fourier transform of the autocorrelation function of $v_{0,n}(t)$. The spectral operator $\boldsymbol{\phi}_n$ of the random current field $j_n(\mathbf{r},t)$ is defined as follows. We form the correlation and spectral matrices, $\boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},T)$ and $\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f)$ of $j_n(\mathbf{r},t)$.

$$\boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},T) = \langle j_n(\mathbf{r}, t+T) j_n^+(\mathbf{s},t)\rangle_{\text{av}} \tag{46}$$

$$\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f) = \int_{-\infty}^{+\infty} dT\, \boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},T) e^{-i2\pi fT}. \tag{47}$$

The spectral *operator* $\boldsymbol{\phi}_n$ is then derived from the spectral matrix $\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f)$ according to Sec. 3. The operator $\boldsymbol{\phi}_n$ is Hermitian and positive (semi) definite.

[3] Y. W. Lee, *Statistical Theory of Communication* (John Wiley & Sons, Inc., New York, 1960).

6. STATISTICAL PROPERTIES OF THE NOISE CURRENTS IN LOSSY MEDIA IN THERMODYNAMIC EQUILIBRIUM

We have obtained an expression, (44), for the resistive component of the impedance of a linear one-port in which the lossy medium is imbedded. We have also obtained the spectral density, (45), of the open-circuit noise voltage in terms of the spectral operator of the noise currents in the medium. According to a generalization of Nyquist's theorem[4] that is due to H. B. Callen *et al.*,[5] the spectral density of the open-circuit noise voltage of a one-port in thermodynamic equilibrium is proportional to the resistive part of the impedance of the one-port. If we denote by θ, the temperature of the one-port, expressed in energy units by means of Boltzmann's constant, we have

$$\phi_0(f) = 2\theta R_0(f). \tag{48}$$

Quantum effects[6] can be accounted for by replacing θ in Eq. (48) with

$$hf/[\exp(hf/\theta) - 1]. \tag{49}$$

The symbol h stands for Planck's constant.

Applying Callen's formula to the case at hand, we find from (44) and (45),

$$W^+QZ\phi_nZ^+Q^+W = 2\theta W^+QZGZ^+Q^+W. \tag{50}$$

The operator G is independent of the electromagnetic structure in which the lossy medium is imbedded. It is, therefore, an operator characteristic of the lossy medium. This is not the case for W, Q, and Z which are entities that can be changed at will by changing the shape or volume of the cavity and of the waveguide. Therefore we set

$$\boldsymbol{\phi}_n = 2\theta G. \tag{51}$$

The spectral operator $\boldsymbol{\phi}_n$ becomes then, just as G, an operator characteristic of the lossy medium. One could argue that it is not obvious that the vector field Z^+Q^+W can be changed in an arbitrary way and that (51) is, therefore, not a necessary consequence of (50). It is clear, however, that postulating the existence of a random driving current-density field $j_n(\mathbf{r},t)$, having a spectral operator given by (51), results in a correct prediction of the noise output spectrum of any one-port in which the lossy material is imbedded. One may further show, by a similar analysis in which a multiple mode excitation is used that leads to a multiterminal pair network, that (51) always predicts the correct Nyquist expression for the resulting multiport. It

[4] H. Nyquist, Phys. Rev. **32**, 110 (1928).
[5] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[6] J. C. Slater, "Report on Noise and the Reception of Pulses," MIT Radiation Laboratory Report, February 3, 1941.

should be noted that the operator $2\theta\boldsymbol{G}$ has the essential properties of a spectral operator, that is, it is Hermitian and is at least positive semidefinite. We now express the operators $\boldsymbol{\phi}_n$ and $\boldsymbol{G}$ in terms of the matrices $\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f)$ and $\boldsymbol{Y}(\mathbf{r},\mathbf{s},f)$, and from Eq. (51) we obtain

$$\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f)=\theta[\boldsymbol{Y}(\mathbf{r},\mathbf{s},f)+\boldsymbol{Y}^+(\mathbf{s},\mathbf{r},f)]. \tag{52}$$

By Fourier transforming this result, we obtain

$$\boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},T)=\theta[\boldsymbol{y}(\mathbf{r},\mathbf{s},T)+\boldsymbol{y}(\mathbf{s},\mathbf{r},-T)]. \tag{53}$$

Relations (52) and (53) express the spectral matrix and the correlation matrix of the noise current density sources in a linear, lossy, electromagnetic medium in thermodynamic equilibrium. We find that these matrices are related in a simple way to the temperature and to the Green's conductivity matrices of the medium.

It has been shown by Callen that the average value of the open-circuit noise voltage is zero. This requirement is seen to be fulfilled, if we require the average value of the driving current density $\boldsymbol{j}_n(\mathbf{r},t)$ to be zero,

$$\langle \boldsymbol{j}_n(\mathbf{r},t)\rangle_{\mathrm{av}}=0. \tag{54}$$

ACKNOWLEDGMENT

This research was carried out during the tenure by one of the authors (Marc C. Vanwormhoudt) of a NATO Fellowship. He wishes to express his sincere gratitude to NATO for having enabled him to take part in this research.

APPENDIX

We consider a linear system for which a real vector field $\boldsymbol{j}_n(t)$ constitutes the input, while the output is a real scalar quantity $v_0(t)$ derived from $\boldsymbol{j}_n(t)$ by means of the following convolution integral

$$v_0(t)=\int_{-\infty}^{+\infty} d\tau\,\boldsymbol{m}^+(\tau)\boldsymbol{j}_n(t-\tau). \tag{A1}$$

The vector field $\boldsymbol{m}(\mathbf{r},t)$, denoted by $\boldsymbol{m}(\tau)$ can be considered as a generalized response function of the system. We suppose that $\boldsymbol{m}(t)$ has a Fourier transform.

The autocorrelation function of $v_0(t)$ is

$$\begin{aligned}\varphi_0(T)&=\langle v_0(t+T)v_0(t)\rangle_{\mathrm{av}}\\ &=\iint_C d\mathbf{r}d\mathbf{s}\iint_{-\infty}^{+\infty} d\tau_1 d\tau_2\,\boldsymbol{m}^+(\mathbf{r},\tau_1)\\ &\qquad\times\langle \boldsymbol{j}_n(\mathbf{r},\,t+T-\tau_1)\boldsymbol{j}_n^+(\mathbf{s},\,t-\tau_2)\rangle_{\mathrm{av}}\boldsymbol{m}(\mathbf{s},\tau)\\ &=\iint_C d\mathbf{r}d\mathbf{s}\iint_{-\infty}^{+\infty} d\tau_1 d\tau_2\,\boldsymbol{m}^+(\mathbf{r},\tau_1)\\ &\qquad\times\boldsymbol{\varphi}_n(\mathbf{r},\,\mathbf{s},\,T+\tau_1-\tau_1)\boldsymbol{m}(\mathbf{s},\tau_2).\end{aligned} \tag{A2}$$

The matrix $\boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},T)$ is defined by (46). The spectral density of $v_0(t)$, according to the Wiener-Khinchine theorem,[3] is found by taking the Fourier transform of $\varphi_0(T)$. We obtain

$$\begin{aligned}\phi_0(f)&=\int_{-\infty}^{+\infty} dT e^{-j2\pi fT}\varphi_0(T)\\ &=\iint_C d\mathbf{s}d\mathbf{r}\left[\int_{-\infty}^{+\infty} d\tau_1 e^{j2\pi f\tau_1}\boldsymbol{m}(\mathbf{r},\tau_1)\right]^+\boldsymbol{\phi}_n(\mathbf{r},\mathbf{s},f)\\ &\qquad\times\left[\int_{-\infty}^{+\infty} d\tau_2 e^{j2\pi f\tau_2}\boldsymbol{m}(\mathbf{s},\tau_2)\right].\end{aligned} \tag{A3}$$

In this relation $\boldsymbol{\varphi}_n(\mathbf{r},\mathbf{s},f)$ is defined as in (47). We introduce the field $\boldsymbol{M}(f)$ that is the Fourier transform of $\boldsymbol{m}(t)$. We then obtain

$$\phi_0(f)=\boldsymbol{M}^+(-f)\boldsymbol{\phi}_n\boldsymbol{M}(-f). \tag{A4}$$

This relation is valid, even when $\mathbf{j}_n(\mathbf{r},t)$ has a random time-dependence.

The vector field $\boldsymbol{M}(-f)$ is obtained by analyzing the system under sinusoidal, steady-state conditions. If $\boldsymbol{j}_n(t)$ and $v_0(t)$ have a sinusoidal time dependence, their complex representations $\boldsymbol{J}_n$ and V_0 are related, according to (A1), by

$$V_0=\boldsymbol{M}^+(-f)\boldsymbol{J}_n. \tag{A5}$$

The expression of V_0 in terms of $\boldsymbol{J}_n$ therefore, determines $\boldsymbol{M}(-f)$, and allows us to find $\phi_0(f)$ in the stochastic case.