# A THEORY OF FLUCTUATION NOISE*

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SUMMARY

From a consideration of the physical phenomena involved in thermionic conduction and in thermal or Johnson noise, it is deduced that the noise in a space-charge-limited valve is best expressed as a thermal noise, and it is shown that a small correction must be applied to the valve slope resistance to give the value of resistance effective as a noise source. The theoretical temperature of this resistance is then shown to be approximately half the cathode temperature. In a temperature-limited valve, both the resistance and the temperature of the conducting path are indeterminate. The " pure shot noise " formulae are applicable to this case, and represent the maximum noise which can result from the passage of a given current through the system. It is incorrect, however, to state that in space-charge-limited conditions " pure shot noise " is smoothed out, while thermal noise in the internal resistance of the valve appears as an additional factor; shot noise and thermal noise in the valve's internal resistance are essentially the same phenomenon, but are modified by the differing conditions of electron transit.

Nyquist's expression for thermal-agitation noise is derived from the atomic mechanism in the case of a metallic conductor.

## (1) INTRODUCTION

The author might perhaps be accused of temerity in attempting a theory of such weighty subjects as shot noise and thermal-agitation noise without having the backing of any experimental work; but from discussion with workers in this field there seemed to be scope for one with training as a physicist to endeavour to interpret the results of other radio workers, even though statisticians regard the problem as unpromising. Those who are familiar with the subject are asked to forgive the elementary treatment of simple shot noise with which the paper opens; it is included because it appears to be an appropriate starting point in the search for a coherent theory based on fundamental principles.

Although in the realm of mechanical engineering we do not find the atomic structure of matter forcibly brought to our notice, yet in electrical technique we have already reached the stage where the discrete nature of the electric current is clearly perceptible. As an extreme instance of the sensitivity attainable by thermionic-valve methods, an electrometer valve can be made to detect a current of the order of $10^{-18}$ ampere—say 10 electrons per second. If the indicating system could be made of sufficiently short time-constant, such a current would necessarily be found to be non-uniform, but other information is required to predict whether the variation will be a regular $0 \cdot 1$-second cycle, or an irregular distribution subject only to the constancy of the long-period mean. (Actually, these minute currents can at present be measured only by a condenser-charging method, so we are very far from the required rapid indicating system which would give the direct experimental answer to this question.) Considering only electron currents (i.e. disregarding any phenomena such as electrolysis or gas discharge, which depend in part on conduction by heavier ions) we may distinguish between electrons travelling within a conductor and those travelling through free space; in the problem of valve noise we are concerned with the latter category. Now electrons may be emitted from conductors into space by various mechanisms—radio-activity, thermal emission, photo-electric emission, and ionic or electronic bombardment (the last including secondary emission). Atomic physics shows that all these processes are essentially " random," meaning that although it is possible to predict the mean value of any quantity involved, from statistical laws based on previous observations of large numbers of events—just as actuaries will determine a statistical value for the length of life of any class of persons—yet the behaviour of the individual unit, when we come down to atomic phenomena, is as incalculable as the life of an individual human being.

It is reasonable to assume, therefore, that the electrons constituting the currents in a thermionic valve are emitted at random times. Suppose, then, that five different currents, each having a mean value of 10 electrons per second, could be observed for 2 seconds, and that the 20 electrons in each were found to be distributed over the 2 seconds in the manner represented by the curves lettered A, B, C, D, E, in Fig. 1, where the 2-second period has been divided into intervals of 1/10th second. In each current the mean number of electrons is 1 per $0 \cdot 1$-second interval, but the individual values vary between 0 and 4 electrons per interval; while adding all five currents together, giving a mean value of 5 electrons per $0 \cdot 1$ second, is seen from the curve marked " Sum " to produce variations from 2 to 12. Although this diagram has not, of course, any great quantitative significance,† it clearly illustrates that as the number of electrons with which we are dealing becomes greater (i.e. the current is increased) the rate of flow becomes *relatively* smoother, but the *absolute value* of the variations is larger. Thus, on the one hand, large currents are for ordinary purposes regarded as

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† The random distributions of Fig. 1 were obtained by drawing lots as follows. Cards numbered 1 to 20, corresponding to the 20 intervals of the period plotted, were shuffled and draws were then taken at random; after noting the number each card was replaced and shuffled, the process being repeated until 100 numbers had been drawn. The first 20 of these numbers, in order of drawing, were taken to represent the numbers of those intervals of the period during which an electron passed for current A, and successive batches of 20 for the other currents. The number of electrons attributed to the *n*th interval of $0 \cdot 1$ second is thus equal to the number of occasions on which a card numbered *n* was drawn during the 20 draws representing the particular current. The result should be genuinely " random."

uniform, while on the other hand the shot noise in a temperature-limited valve increases in proportion to the current. If in a wireless receiver the steady current of the valve in each stage of amplification were in proportion to the signal strength, so that the anode current in each valve was approximately 100 % modulated by a strong signal, this type of phenomenon would still cause a deterioration of signal/noise ratio as the sensitivity was raised, since the smaller currents in the input circuit would then be subject to a large relative fluctuation; this corresponds to the case of a photocell combined with an electron-multiplier. The increase of

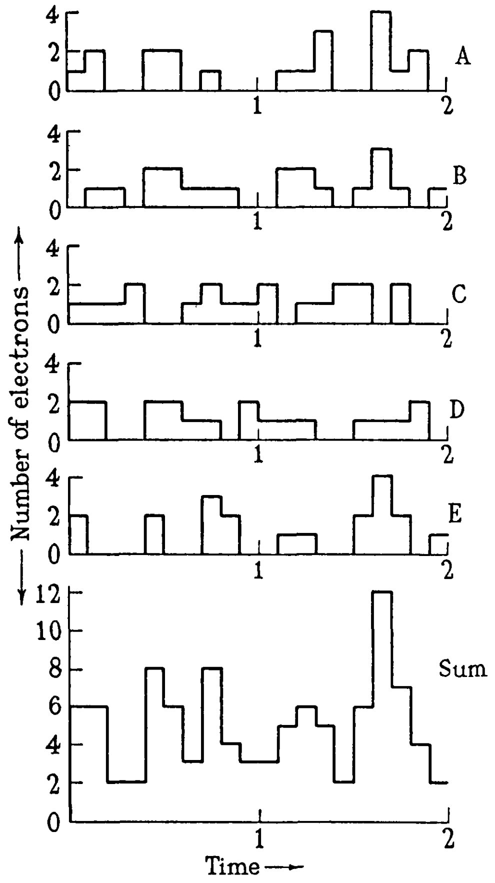


Fig. 1

noise here arises from the fact that with decreasing signal amplitude the charge conveyed across a valve per half-cycle of signal current is becoming more comparable with the charge on a single electron. But in a normal amplifier one might expect a far worse state of affairs, since the current in the first valve depends upon the type chosen, and the size of valve cannot be indefinitely reduced in proportion to the signal strength. Actually, however, the ordinary thermionic amplifying valve works under conditions of space-charge limitation (not temperature limitation), and it is known that the valve noise is then less than predicted by the simple shot-noise theories. The noise in the presence of space-charge limitation is discussed later in this paper.

## (2) TEMPERATURE-LIMITED CONDITIONS

Existing theories, which are well supported by experimental results in the temperature-limited regime, show that the mean square of fluctuation voltage at the anode of a temperature-limited diode is directly proportional to the magnitude of the current from which it arises. Moullin and Ellis,* for example, give the formula

$$\overline{V_s^2} = I_{av.}eR/(2C) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $\overline{V_s^2}$ is the mean-square shot voltage in an anode circuit, having resistance $R$ shunted by capacitance $C$, connected to a diode having temperature-limited anode current of mean value $I_{av.}$. For an oscillatory anode circuit they give

$$\overline{V_s^2} = \frac{I_{av.}eL}{2RC^2}(1 + F^2) \quad . \quad . \quad . \quad . \quad (2)$$

where $F = R/(\omega_0 L)$. Now for a good tuned circuit $R/(\omega_0 L)$ at resonance ($\omega_0 = 2\pi$ times the resonant frequency) is of the order of $10^{-2}$, so that $F^2$ is negligible compared with unity. Writing $L/(RC)$, the dynamic resistance of the circuit, as $R'$, the formula for noise in a tuned circuit becomes

$$\overline{V_s^2} = I_{av.}eR'/(2C) \quad . \quad . \quad . \quad . \quad (3)$$

which is of the same form as (1), but employing the dynamic resistance of the circuit in place of an ohmic resistance. It is interesting to note that in (3) the factor $C$ still appears in the denominator. This suggests that the use of a tuned circuit of low $L/C$ ratio with low resistance will give the best signal/noise ratio for a given amplification, i.e. for constant value of the dynamic resistance $R'$. However, this policy is limited firstly by the difficulty of lowering the resistance sufficiently to maintain the desired value of dynamic resistance, and secondly by the increasing selectivity which results from the small decrement and tends to limit the band-width. It may be said, in fact, that the reduction of noise has been obtained at the expense of decreasing the band-width.

## (3) SPACE-CHARGE-LIMITED CONDITIONS

So far we have only mentioned conditions covered by equation (1), which is based on the assumption that the time of transit of the electrons is so small as to have no effect on the noise spectrum, and that the arrival of electrons at the valve anode is " random," i.e. that the emission of an electron is not influenced by the time at which any previous electron was emitted, or arrived at the anode, but is controlled solely by the combination of the external electric field (due to potentials on the various electrodes of the valve) and its own energy of thermal agitation within the cathode. Experimental results support this equation for temperature-limited conditions; but when there is sufficient surplus emission to form a considerable space-charge it is found that the noise is substantially reduced. Indeed, the existence of a finite anode-slope resistance shows that the charge on the anode (which is dependent upon the number of electrons that have previously arrived there) must have some influence on the adventures of electrons emitted

* See Reference (1).

at a later time, and therefore casts some superficial doubt on the hypothesis of truly random emission.

It is generally agreed that with this space-charge (which is after all the practical working condition in some part of the electrode system of every amplifying valve) the envelope of noise voltage, i.e. the larger fluctuations arising from the random arrivals of numerous electrons, as illustrated in Fig. 1, must produce an opposite fluctuation in anode current, just as much as would occur with any signal voltage developed across the external circuit and appearing as a potential on the anode. (The noise envelope is of course a function of the characteristics of the external anode circuit, as well as of the fluctuation in the current arriving at the anode.) If then the valve has slope resistance $R_a$ and the external circuit is as assumed for equation (1), this effect is believed to cause a reduction in the shot-noise energy by a factor $R_a/(R_a + R)$, giving the Moullin and Ellis equations

$$V_s^2 = \frac{1}{2}I_{av.}e\left(\frac{R_a}{R_a + R}\right)\frac{R}{C} \quad . \quad . \quad . \quad . \quad (4)$$

for the total noise voltage, and

$$V_{df}^2 = 2I_{av.}e\left(\frac{R_aR}{R_a + R}\right)^2 df \quad . \quad . \quad . \quad (5)$$

for the noise voltage in a narrow band of frequencies $df$. For lack of a better name, the theory represented by equations (1) to (5) will be referred to as the " pure shot " theory, which does not explicitly consider events taking place within the space-charge, but only the arrival of electrons at the anode. According to this view, shot noise is as a first approximation expressible solely in terms of the magnitude of the current arriving at the anode [equation (1)]; after allowing for the effect of external-circuit characteristics and internal resistance of the valve on the voltage actually generated on the anode [equations (4) and (5)], any corrections required under space-charge conditions are outside the scope of this theory. Actually, however, it is agreed by all experimenters that valve noise is greatly reduced, but never completely abolished, by space-charge limitation of the anode current.

Llewellyn* and a number of American workers, on the other hand, take an entirely different view, leading to what may be briefly described as the " thermal noise plus shot noise " theory. They consider that there are two separate sources of noise within the valve. First, random emission from the cathode in the temperature-limited condition must cause an exactly corresponding random arrival at the anode, and hence give rise to " pure shot noise." But Llewellyn considers that when space-charge is present it smooths out the random emission from the cathode, so that arrival at the anode is no longer random, the smoothing being proportional to a factor $(dI/dI_c)^2$; the pure shot noise is to be multiplied by this factor to give the shot noise modified by space-charge. ($I$ is the actual anode current, and $I_c$ the total emission from the cathode.) Thus with complete space-charge limitation, $dI/dI_c = 0$, shot noise should vanish. But in addition to pure shot noise there is said to be

* See Reference (2).

" thermal noise " within the valve, for it is argued that since the valve has an effective slope resistance $R_a$ it must on general grounds of thermodynamics give rise to the amount of thermal-agitation noise (or " Johnson noise ") appropriate to such a resistance.

Experimentally, neither theory has proved satisfactory. The difficulty with the thermal-noise-plus-shot-noise theory has been that in order to correspond to the observed values of slope resistance and noise the temperature of the internal resistance must be somewhere about one-half of the cathode temperature; it has in the past seemed a natural assumption that the slope resistance should be at a temperature equal to that of the cathode, but no published paper has covered this question adequately.* The value of the total emission $I_c$ for use in the factor $dI/dI_c$ is also a matter of some difficulty, not only with dull emitters but apparently even with pure tungsten cathodes. Pearson, for example, published some experimental results in support of the thermal-noise-plus-shot-noise theory against the pure shot-noise theory.† He found it necessary to resort to an extrapolation method to eliminate the effects of external field on the temperature-limited emission from his tungsten cathode; even so, at very small currents (low filament heating) his values of $I_c$ were subject to inaccuracy, for he obtained in the worst case $dI/dI_c = 1{\cdot}03$, i.e. the current to the anode increasing 3 % faster than the reputed total emission from the cathode. The exponents of the pure-shot-noise theory, on the other hand, have not produced any alternative theory for the reduction of shot noise by space-charge.

## (4) THERMAL NOISE IN A VALVE

The present author was therefore faced with the problem of first finding the weaknesses and strengths of the two theories referred to above, and then endeavouring to construct a more adequate hypothesis. As regards thermal noise, the author is sufficient of a physicist by training to regard thermodynamics, when properly applied, as an infallible instrument; there appears to be no flaw in Nyquist's thermodynamic proof of the universality of thermal-agitation noise, whatever may be the magnitude of such noise.‡ Moullin objects that within the valve there is no collision between free electrons and fixed molecules, so that, the mechanism of ordinary resistance being absent, thermal agitation noise which arises in solid conductors should also be absent.§ But consider a resistance connected between grid and cathode of an amplifying valve; thermal noise arising from this resistance is apparent only as a varying charge on the grid of the valve, and, though we may fairly deduce therefrom that there is a random motion of electrons in the resistance, we need independent evidence to determine how that random motion is caused. Similarly, if the electrons within the anode stream of a valve have random components of velocity, they can produce a " noise " voltage which is indistinguishable from that

* A paper on this question by B. J. Thompson and D. O. North was presented at the Rochester meeting of the Institute of Radio Engineers, 16th November, 1936, but so far as the author is aware this has only been printed in abstract form.

† See Reference (3).

‡ See Reference (4). An alternative to Nyquist's derivation of the magnitude and temperature dependence of thermal-agitation noise is given at the end of this paper.

§ See Reference (1).

produced by a metallic resistance, though the mechanism of collision with fixed molecules is not present; we have to rely upon the thermodynamic proof, however, to assure us that the relation between magnitude of noise voltage and magnitude and temperature of resistance is the same in both cases. Actually some correction will be required, since a valve is not a true ohmic resistance but has a curved characteristic. Nyquist's argument is based on the power absorbed, which, when a voltage $V$ is applied to a resistance $R$, is equal to $V^2/R$; bearing this in mind, we can find the relation between slope resistance $R_a$ (i.e. tangent to the characteristic) and the value of resistance $R'$ effective for thermal noise in any particular shape of characteristic. For example, if the characteristic is

$$i = aV^{3/2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the power may be expressed as

$$P = iV = aV^{5/2} \quad . \quad . \quad . \quad . \quad (7)$$

To find the additional power absorbed when a small extra voltage, e.g. a "noise" voltage, is superimposed upon the mean applied voltage $V$, equation (7) is differentiated. Thus

$$\frac{dP}{dV} = \frac{5}{2}aV^{3/2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now for a true ohmic resistance of value $R'$ we have

$$P = V^2/R'$$

$$dP/dV = 2V/R' \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We therefore adopt as the definition of the value $R'$ of resistance effective for thermal noise in any circuit element, the expression derived from (9),

$$R' = 2V\Big/\frac{dP}{dV} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For the 3/2-power characteristic, therefore, we find from (8) and (10) that

$$R' = \frac{2V}{(5/2)aV^{3/2}} \quad . \quad . \quad . \quad . \quad (11)$$

This is related to the slope resistance $R_a$ by writing

$$\frac{1}{R_a} = \frac{di}{dV} = \frac{3}{2}aV^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (12)$$

so that combining (11) and (12) we have

$$R' = 6R_a/5 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Another important characteristic is the 5/2-power law:—

$$i = aV^{5/2} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$P = aV^{7/2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{dP}{dV} = \frac{7}{2}aV^{5/2} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$R' = 2V\Big/\frac{dP}{dV} = \frac{4}{7}\cdot\frac{1}{aV^{3/2}} \quad . \quad . \quad . \quad (17)$$

But $$1/R_a = \frac{5}{2}aV^{3/2}$$

Therefore $$R' = \frac{10}{7}R_a \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The third characteristic of practical interest is exponential:—

$$i = ae^{bV} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$1/R_a = abe^{bV}$$

$$P = aVe^{bV}$$

$$dP/dV = ae^{bV}(1 + bV) \quad . \quad . \quad . \quad . \quad (20)$$

$$R' = \frac{2bVR_a}{1 + bV} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It will be noticed that in all these cases the value of $R'$ is *greater* than $R_a$, so that the thermal noise from a valve of slope resistance $R_a$ is slightly greater than from an ohmic resistance of value $R_a$. Alternatively, if the temperature of the valve's internal resistance is calculated from the observed noise voltage and the value of $R_a$ (in place of the correct value $R'$), the temperature so deduced will be *higher* than the true temperature by a factor such as 6/5 or 10/7.

### (5) "PURE SHOT NOISE" THEORY

The next question is whether there is in addition to thermal noise a separate shot noise; the author believes there is not. Thermodynamic reasoning, as usual, indicates the overall relations between the valve, regarded as one unit, and the external circuit, without revealing the internal mechanism within the valve. The random arrival of electrons at the anode, in other words "shot noise," is the expression of the fact that the electrons have a certain random component of velocity which represents the thermal agitation of the electrons, and must therefore be related to thermal noise. It will be shown later in this paper that a proper consideration of the temperature of the valve's internal resistance makes it possible to unify these two aspects of the phenomenon. But, if so, what of the pure shot equation [equation (1) above] for which Moullin states that the sole condition for shot-noise power to be proportional to the magnitude of the mean anode current is that the arrivals of all electrons shall be random? On this view, the noise does not depend upon the magnitudes of the random components of electron velocities, i.e. the temperature of the electrons, but only on the absolute independence of the events constituted by the arrivals of the several electrons at the anode; and the general statement made by some writers that the space-charge "smooths out the irregularities of the emitted current" is unsatisfactory, since it can be shown that space-charge does not destroy the random nature of the electron emission from the virtual cathode.

Moreover, a direct antithesis to the frequency-spectrum method employed in Moullin's derivation of equation (1) is provided by the work of T. C. Fry.* Fry denies the existence of a frequency spectrum of shot noise, and states that "If electrons have been emitted in a statistically steady stream for infinite time past, the probability of the spectrum corresponding to this emission having any pre-assigned ordinate at any given frequency

* See Reference (5).

is zero, and the probability that the ordinate exceeds any finite quantity, however large, is unity." This claims to rule out any derivation in terms of a spectrum determined solely by the number of electrons emitted. But examination of the two papers (Fry's and Moullin's) reveals the critical point at which their analyses diverge to such opposite conclusions. Both authors find a Fourier integral representing the pulse produced by the passage of a single electron from cathode to anode, in which the shape of pulse does not affect those frequencies whose period is large compared with the time of transit of the electron, i.e. the frequencies which are of interest for normal radio work; they then proceed to sum the effect of a large number of electrons arriving in random phase, and the crux of the question is the length of period over which the summation is to extend. Fry sums over an *infinite* period, and hence finds that the resultant for any frequency is infinite, being the sum of an infinite number of vectors of equal magnitude but random phase. In corroboration of this result, Fry quotes Einstein's equation for the Brownian motion as a parallel case where the sum of a number of vectors in random phase tends to an infinite resultant as the number of vectors tends to infinity. Moullin, on the contrary, sums over a *finite* period such that $N$ electrons have passed, and quotes a statistical theorem that the sum of $N$ equal vectors in random phase is equal to $N^{\frac{1}{2}}$ times the magnitude of each single constituent vector. The present author believes that the period of summation should be finite and related to the time-constant of the apparatus used to measure the noise voltages; the time-constant in question should probably be that of the first integrating element in the amplifying and measuring chain, the nature of which naturally depends upon the actual apparatus in use. This rules out Fry's result, but does not thereby justify Moullin's: it remains to be determined whether the period of the summation is in fact such that the statistical theorem which he quotes is applicable.

There seems to be some doubt among statisticians as to the significance of any limiting value of a property of a collection of events which are unlimited in number. (See, for example, N. Campbell's paper on "The Statistical Theory of Errors,"* particularly pages 802 and 803 of that paper.) The problem is, that if $\epsilon$ is the deviation of the actual sum of a particular collection of $N$ unit vectors in random phase, from the statistical value of $N^{\frac{1}{2}}$, how large must $N$ be in order that the probability of exceeding a tolerable limit of error $\epsilon_0$ shall be less than a chosen small quantity $\delta$? This is a complicated question, since it involves firstly the *limit of error* $\epsilon_0$ which can be tolerated, and secondly the *probability* $\delta$ of exceeding this limit, which shall for practical purposes be regarded as zero probability. That $N$ can be sufficiently large in atomic phenomena for the difference between actual and statistical values to be imperceptible is evidenced by the successful treatment of ordinary liquids and gases in bulk as continuous fluids. A recent paper by E. N. Rowland† discusses the problem mathematically. Taking the case of random distribution of events governed solely by a constant probability (this corresponds to what was rather loosely described as "constant mean current" in connection with Fig. 1), and supposing observations to be made of the resultant effects of numbers of these events occurring in a series of limited intervals of time, the problem considered by Rowland is whether any meaning can be attached to the mean value of the resultant effect averaged over infinite time, and, if so, whether it can be related to the averages obtained over finite intervals of time. He concludes that, provided the intervals of observation are sufficiently long, we may take the mean value found in any single interval of observation as a reasonable physical estimate of the corresponding quantity averaged over all time. He does not, however, give a criterion of the time which is sufficiently long, nor the corresponding tolerance of error implied by the term "reasonable physical estimate." In the terms in which the problem was stated above, Rowland's paper gives rigorous proof of the experimental deduction from the kinetic theory of fluids, that as $N$ tends to infinity, $\epsilon_0$ and $\delta$ tend to zero; but it does not give the numerical relation between the three quantities for values of $N$ other than infinity. The "spectrum" derivation of equation (1) is therefore left as of unknown quantitative accuracy but at least qualitatively correct;* it fails to explain the reduction of noise by space-charge, but it may be that this is due to a misunderstanding of the event constituting the unit vector under conditions of space-charge.

* See Reference (6).

† *Ibid.*, (7).

## (6) PHYSICAL CONSIDERATIONS

As an introduction to an alternative method of calculation, let us return to Fry's paper, and adopt his definition that "The Schroteffekt will be measured by the difference between the energy actually dissipated in the circuit and the energy that would be dissipated if the electron stream were a uniform flow of a continuous fluid." By calculating the energy that would be dissipated in the attached circuit for a single electronic charge instantaneously transferred from cathode to anode of the valve, he deduces the expression

$$W = \nu\overline{W}_1 + W_0 \quad . \quad . \quad . \quad . \quad (22)$$

where $W$ is the total energy dissipated in the circuit, $W_0$ the steady-current energy, $\nu$ the number of electrons emitted, and $\overline{W}_1$ the mean over all electrons of the energy which would be dissipated in the anode circuit on the emission of a single electron. The noise energy is then $\nu\overline{W}_1$.

Before proceeding further, it is necessary to deal with the doubt, expressed by Fry, whether the disturbance to the external circuit caused by the arrival of a single electron can obey the same laws as the disturbance due to a large charge; for example, how can the charge on a condenser decay exponentially if that charge consists only of a single indivisible electron? This problem was also considered by Rowland, who calculated the shot noise in a circuit connected to a temperature-limited diode according to two different hypotheses: (*a*) the electrons arrive and depart suddenly from the anode, having a random distribution of length of life on the

* This is, of course, assuming the validity of Rowland's analysis, which the present author is not competent to question.

anode, subject only to the half-life constant being such as to give the correct mean value of current; or (*b*) the charge on the anode due to each electron decays exponentially, just as a charge consisting of numerous electrons would. He is surprised to find that it is hypothesis (*b*) which gives the result in accordance with the experimentally verified equation (1). But if one regards the electromagnetic field as the ultimate reality, as in Poynting's theorem for example, this is a reasonable result. The quantum of electromagnetic energy of radio frequencies is so small that the electromagnetic field may here be regarded as continuous. The unit charge remains undivided, but the effective charge on the anode gradually decreases as the electron recedes from it.

Returning to equation (22), let us consider the general conservation of energy in the system comprising the valve, anode circuit, d.c. anode supply source, and hot cathode. The d.c. anode supply is not likely to be the source of the noise energy, which is made up of alternating voltages; it might be argued that the random nature of the emission would modulate the steady anode current and thereby enable some of the energy supplied by the battery to be converted to a.c. energy, but one would then expect the noise energy to be proportional to the energy in the steady current. This is not supported by experiment, for the noise energy in a temperature-limited diode depends only on the anode current, and not on the anode voltage; it therefore seems reasonable to suppose that the energy supplied by the battery is always $W$, the same as would be expended if the current flow were uniform and continuous. The remaining energy $\nu\overline{W}_1$ involves the mean of a number of energies $W_1$, each of which is characteristic of some individual electron, not of the external circuit or applied voltages; and the most obvious form of energy which is peculiar to every particle of atomic dimensions is thermal energy. It is therefore a plausible hypothesis to suggest that $\nu\overline{W}_1$ is the thermal energy of the electron stream constituting the anode current; it is therefore a function of the temperature of the virtual conductor (or resistance) within the valve. The shot-noise energy is therefore derived from the random component of kinetic energy (i.e. the thermal energy) of the electrons in the anode-current stream, which in turn is derived from the thermal energy of the cathode. Yet despite the obvious suggestion of a " thermal " energy in $W_1$, Fry finally deduces for the noise energy in a circuit attached to a valve an expression which is a function only of current, and not of any value of energy associated with the electrons constituting the current.

The reason is that he assumed the *instantaneous* passage of each electron from cathode to anode, whereas the only way in which the thermal-agitation energy of the electrons can make itself felt is in variation of the velocity of the electrons between the valve electrodes, a possibility which is excluded by the assumption of instantaneous transit, since this implies that all electrons alike have infinite velocity.

As a purely qualitative example of the effect of transit velocity on the resultant energy expended in the circuit, consider Fig. 2, where $C$ represents the capacitance between the anode and cathode of a valve, and the resistance $R$ the external circuit. If now charge $q$ is transferred from plate " a " to plate " b " of condenser $C$ instantaneously, the energy introduced into $C$, and afterwards dissipated in $R$, is $q^2/(2C)$. Now let the charge be transferred from " a " to " b " over a finite time, according to the following rather artificial hypothesis. First a charge $\frac{1}{2}q$ is transferred instantaneously, charging $C$ to a potential $q/(2C)$, then a current flows such as to maintain this potential difference across $R$ until the remaining $\frac{1}{2}q$ has passed. The energy associated with the initial instantaneous transfer of $\frac{1}{2}q$ is $q^2/(8C)$, and will be dissipated at the end of the cycle of operations, when the condenser discharges from potential $q/(2C)$ to zero. The current required in the second stage

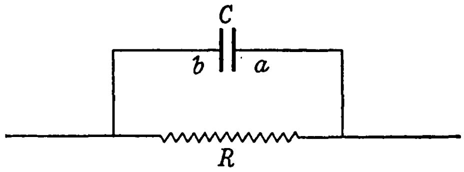


Fig. 2

is $i = q/(2CR)$ [since it is required to maintain a potential $q/(2C)$ across the resistance $R$] so that the energy dissipated by this current is

$$i^2Rt = q^2t/(4C^2R) \quad . \quad . \quad . \quad . \quad (23)$$

where $t$ is the time of current-flow and is given by

$$t = q/(2i) = (q/2)\,(2CR/q) = CR$$

Therefore
$$i^2Rt = q^2/(4C) \quad . \quad . \quad . \quad . \quad (24)$$

The total energy in the cycle of operations is $q^2/(4C) + q^2/(8C) = 3q^2/(8C)$, against $q^2/(2C)$ for instantaneous transit. Thus the electronic velocity of transit should influence the energy dissipated in the external circuit, and the noise energy as calculated by Moullin and Fry on the assumption of instantaneous transit should be a maximum value. This appears to be in general accord with experimental evidence.

It might at first be thought from the above that the reduction of noise in the presence of space charge is due to the increased time of transit of the electrons from cathode to anode; but this is not so, for the time of transit under normal working conditions is short—Moullin (*loc. cit.*) quotes $10^{-9}$ second—so that the idea of instantaneous transit would be justified on this score. But there is another way of regarding the problem. For example, consider a hypothetical valve in which the electrons are emitted *regularly*, and in such numbers that their periodicity corresponds to a frequency much higher than can be detected by radio apparatus; such a valve would give zero noise, however long or short the transit time of the electrons, for the *regular* wave-form could only produce frequencies above the range of the apparatus, whatever the shape of the constituent pulses of the wave-form. Next imagine a single gas molecule or similar body to be projected across the direction of the current flow, and stop one or two electrons by collision. The stoppage of these electrons will cause a minute pulse to be superimposed upon the steady

current-flow in the external circuit, i.e. will give rise to " noise." From this example we see that deviations of electron velocities from a mean value can cause noise, and we must therefore regard the velocities of the electrons in any space-charge-limited valve as made up of two components:—

(a) All the electrons have an equal mean velocity, corresponding to the observed steady current, and if they had no other component of velocity the noise would be zero in a practical circuit.

(b) Each electron has an individual random velocity of thermal energy, the magnitude of which controls the amount of noise arising in the circuit connected to the valve.

Temperature-limited valves, on the other hand, and any valves whose electrode systems are such that they behave in a similar way, cannot conveniently be regarded from this point of view; for it will be realized from the investigation that follows that it is not possible to assign a temperature to the anode stream of a temperature-limited valve, and in the absence of space-charge there is no significance to be attached to the " mean velocity " shared by all electrons.

## (7) TEMPERATURE OF THE ANODE CURRENT STREAM

Earlier in this paper it was stated that a valve which can behave as a resistance must for thermodynamic reasons be a source of noise energy of magnitude appropriate to its resistance and temperature. But it was pointed out that corrections are necessary when the conducting path does not obey Ohm's law, a caveat which applies very strongly, for example, to the temperature-limited diode whose " resistance " is infinite but noise energy finite. It has now been suggested that the source of the noise energy is the random motion of the electrons during their flow from cathode to anode; this is practically identical with the mechanism of thermal noise in metallic conductors.* Since it is thus fair to regard fluctuation noise in a valve having a finite resistance as a thermal effect, it is necessary to find the effective temperature of the electrons constituting the valve's anode current, i.e. the temperature of the valve's resistance.

" Temperature " must first be defined. It is a statistical property of a collection of particles, proportional to the average kinetic energy of random motion possessed by the constituent particles. The term " random motion " is intended to exclude the kinetic energy due to any velocity common to all the constituent particles. Thus a quantity of gas flowing through a pipe at high speed gains additional kinetic energy by its flow, but this does not constitute an increase of temperature, neither does the mean velocity of an electron stream from cathode to anode under the influence of an external source of e.m.f. add to the temperature of the electrons.

O. W. Richardson† showed by thermodynamic reasoning that electrons emitted from and in equilibrium with a hot conductor have a Maxwellian distribution of velocities corresponding to the same temperature as the conductor; he also found this to be in agreement with experimental evidence. It appears that this hypothesis still stands, for Hume Rothery in his book (" The Metallic State," pp. 142–3) states that " within the limits of the experimental methods the electrons emitted from pure metals in a high vacuum have velocities in accordance with the Maxwell law. . . ." It seems certain that if any two systems are capable of exchanging thermal energy, they will in equilibrium be at the same temperature. Such exchange of energy is possible between a hot conductor and electrons surrounding it, by means of both emission and absorption of electrons and reflection at the surface of the conductor.

In a thermionic valve working with a fair amount of space-charge limitation there will be a potential minimum at some point close to the cathode; all space-charge between this and the cathode has a free exchange with the cathode and is therefore at the same temperature. But the passage across the boundary line formed by the potential minimum is an irreversible process: any electron which passes this dividing line is inevitably swept across to the anode. There is therefore no thermal exchange between the cathode and the electrons between potential minimum and anode, and the temperature of these electrons might therefore differ from that of the cathode, and must now be calculated.

From the point of view of noise voltage generated in the external circuit, we are concerned only with electron velocities in the direction of the cathode-anode current stream, and therefore take the components of electron velocity normal to the cathode. This component of the energy of an electron in the outer space-charge* is that with which it left the cathode, less the energy required to pass through the potential minimum, and electrons which are received at the anode are those whose initial velocity component normal to the cathode was greater than was necessary to pass the potential minimum. Richardson (*loc. cit.*) gives the law of distribution of velocities normal to the cathode as

$$N_u du = N \,.\, 2hmue^{-hmu^2}du \quad . \quad . \quad . \quad (25)$$

where $N_u$ is the number of electrons out of a total $N$ which have velocities between $u$ and $(u + du)$, and $1/h = 2kT$. The mean energy of normal components averaged over all electrons leaving the cathode is thus $1/(2h) = kT$. The kinetic energy associated with any group $N_u du$ of electrons is, from (25),

$$W_u = \tfrac{1}{2}mu^2N_u du = Nhm^2u^3e^{-hmu^2}du \quad . \quad (26)$$

Let $u_0$ be the velocity which is just sufficient to bring an electron to the potential minimum. Then those electrons which pass over to the anode had initially an aggregate energy $W_0$ given by

$$W_0 = Nhm^2\int_{u=u_0}^{u=\infty} u^3e^{-hmu^2}du \quad . \quad . \quad . \quad (27)$$

* The difference is that in a metallic conductor the electrons are in thermal equilibrium with the molecules of the conductor, so that their temperature can be measured by any normal thermometer; in parts of the space-charge, on the other hand, there is no thermal equilibrium between electrons and molecular matter, so their temperature must be calculated from their history since leaving a body having a measurable temperature (i.e. the cathode).

† See Reference (8).

* " Outer space-charge " is a convenient name for space-charge outside the potential minimum.

The number $N_0$ of electrons passing the barrier is obtained by integrating (25). Thus

$$N_0 = 2hNm\int_{u=u_0}^{u=\infty} ue^{-hmu^2}du \quad . \quad . \quad . \quad (28)$$

$$= Ne^{-hmu_0^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Evaluation of the integral in (27) gives

$$W_0 = \tfrac{1}{2}Nme^{-hmu_0^2}\left[u_0^2 + 1/(mh)\right] \quad . \quad . \quad (30)$$

and on dividing (30) by (29) the mean initial energy of forward velocity of those electrons which ultimately pass the barrier is

$$\overline{W}_0 = \frac{W_0}{N_0} = \tfrac{1}{2}m\left[u_0^2 + 1/(mh)\right] \quad . \quad . \quad (31)$$

But each electron loses energy $\frac{1}{2}mu_0^2$ in passing the potential minimum, since $u_0$ was defined as the critical velocity. Therefore the average energy $\overline{W}$ of the electrons when they reach the outer space charge is related to their average initial energy $\overline{W}_0$ by the equation

$$\overline{W} = \overline{W}_0 - \tfrac{1}{2}mu_0^2 \quad . \quad . \quad . \quad . \quad (32)$$

Substituting (31) in (32) now gives

$$\overline{W} = 1/(2h) \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The mean kinetic energy of forward velocity of the electrons is therefore unchanged by passing through the potential minimum, remaining equal to $kT$. But, considering a metallic conductor as source of thermal noise, it is clear that treatment of an electron stream as a source of comparable thermal-agitation noise requires that the random velocities of the electrons shall be equally distributed both *forward and backward* along the direction of the current-flow. Again, referring to our definition of temperature, we are required to find a mean kinetic energy of random velocity, excluding any drift velocity common to all constituents of the system. We therefore regard the equivalent in volts of the *mean* of the emission velocities of the electrons as being added to the steady anode voltage, and the *deviations from the mean* as the source of thermal-agitation noise. If as an approximation we assume the mean electron velocity to be equal to the mean-square velocity, this means that we simply halve the value of $u$ in all the energy calculations, and divide by four the kinetic energies which depend upon $u^2$.

The new mean value of kinetic energy of random velocity along the chosen direction is then $kT/4$. But in a perfect gas the mean energy of a particle along any one of a set of three Cartesian axes is equal to the equipartition value $kT/2$; the electrons with mean random energy $kT/4$ have therefore an effective temperature $T/2$, one-half the cathode temperature.

Actually the mean velocity in a Maxwellian distribution is $\sqrt{[8/(3\pi)]}$ times the root-mean-square velocity, so the mean velocity assumed in our approximation is in error by the difference between unity and $0{\cdot}921$, i.e. by $7{\cdot}9$ %. It might at first be thought that this would cause an er or of nearly 16 % in the calculated temperature, since this depends upon the square of velocity; but actually it is a question of deducting velocity from one group of electrons and adding to another, so that the error depends upon a difference of squares only, and is small, provided the two squares are nearly equal, which is true in this case.

We have thus shown that in a valve having a resistance which is finite and a function of space charge (i.e. a space-charge-limited valve) the total fluctuation noise is equal to the thermal-agitation noise in a corresponding resistance at a temperature of approximately half the cathode temperature, subject to the following corrections:—

(1) In practical valves, end-effects cause a fraction of the current to be temperature-limited, and therefore probably more " noisy." It might be worth while to try experiments with guard rings, so as to use only the central part of the cathode where the current can be fully space-charge-limited.

(2) The relation between slope resistance and effective resistance for thermal noise must be taken into account. [See, for example, equations (13), (18), (21), which indicate that the effective resistance is normally greater than the slope resistance.]

(3) Although the simple theory of thermionic emission has been well tested for pure metals, there is a slight possibility (though no actual evidence) that with complex emitting surfaces the space-charge adjacent to the cathode might have a temperature different from that of the cathode, though probably retaining a Maxwellian distribution of velocities. This can only occur if quantum effects prevent the free exchange of energy between the space-charge and the interior of the cathode.

## (8) MECHANISM OF THERMAL-AGITATION NOISE IN SOLID CONDUCTORS

We have found that the fluctuation noise in a space-charge-limited valve may be represented as a thermal-agitation noise. Conversely, thermal-agitation noise in a metallic conductor may be represented as " shot noise " within the crystal lattice of the material, and without space-charge limitation.

Let us consider a conductor of length $\Delta l$ and sectional area $A$, having $n_0$ free electrons per unit volume, and subjected to a uniform longitudinal electric field of strength $E$. Within a solid conductor, collisions between electrons and molecules are extremely frequent, and the drift velocity acquired under an applied electric field is small compared with thermal-agitation velocity; we may therefore assume a fixed time of flight $\tau$ between collisions, depending solely upon the distance between molecules and the temperature. Denoting the electronic charge and mass by $e$ and $m$ respectively, the mean drift velocity $\bar{u}$ is given by

$$\bar{u} = eE\tau/(2m) \quad . \quad . \quad . \quad . \quad . \quad (34)$$

From the definition of electric current as $i = dQ/dt$, it follows that if there is a small finite variation of charge $\Delta Q$ in a time $\Delta t$, the average current flowing is $\bar{\imath} = \Delta Q/\Delta t$. If, further, the charge is transferred from one end to the other of a length $\Delta x$ (i.e. the current flows through a circuit of length $\Delta x$) during the time $\Delta t$, we may write

$$\bar{\imath}\Delta x = \frac{\Delta Q}{\Delta t}\Delta x = \frac{\Delta x}{\Delta t}\Delta Q = \bar{u}\Delta Q \quad . \quad . \quad (35)$$

since the ratio $\Delta x/\Delta t$ is the mean velocity. Now in the metallic conductor, $\Delta Q$ is equal to the product of the charge on an electron and the total number of electrons taking part, $\Delta x$ may be replaced by $\Delta l$ (provided the latter is taken small enough), and $\bar{u}$ is given by equation (34), so that

$$\bar{\imath}\Delta l = (A\Delta l n_0 e)\left[eE\tau/(2m)\right] \quad . \quad . \quad . \quad (36)$$

But the potential difference $V$ between the ends of the conductor is the product $E\Delta l$ of field-strength and length, and the resistance $R$ is the ratio of potential difference to current. Thus

$$R = \frac{V}{\bar{\imath}} = \frac{E\Delta l}{(An_0 e)\left[eE\tau/(2m)\right]} = \frac{2m\Delta l}{An_0 e^2\tau} \quad . \quad (37)$$

Comparing the solid-conductor problem with the shot-noise problem in a thermionic valve, we may say that equation (37) has related the mean current to the atomic mechanism, but it remains to determine the fluctuation noise due to the discrete nature of the mechanism. Provided that a current pulse is sufficiently short in duration compared with the period of frequencies capable of being observed, it is possible to evaluate the observable components of the Fourier analysis of the pulse without knowing its shape. For example, in dealing with shot noise in a temperature-limited diode, Moullin and Ellis (*loc. cit.*) showed that, knowing only the integral characteristic of the pulse due to a single electron, $\int idt = e$, the components of radio frequency can be calculated. In our present problem of the solid conductor the unit pulse is not a current, but is of the form $i\Delta l$, which will be termed a "current-element." Combining expressions (34) and (35) for a single electron, and integrating over its time of flight, we have

$$\int_0^\tau (i\Delta l)dt = e\bar{u}\tau \quad . \quad . \quad . \quad . \quad (38)$$

If, therefore, we replace $e$ in the analysis of Moullin and Ellis by $e\bar{u}\tau$, the components of the Fourier analysis will represent components of $i\Delta l$ in place of $i$. The time of flight of an electron within a conductor is even shorter than the transit time through a valve, so that the assumption that the pulse is so short that its shape has negligible effect on the radio-frequency components is a fortiori applicable in this case. If $2\pi/p$ is the period of the Fourier series, assumed to be long compared with the duration of the pulse but short compared with radio frequencies, the pulse $e\bar{u}\tau$ is found to be equivalent to a series of current-elements

$$\Sigma i\Delta l = (e\bar{u}\tau p/\pi)(\tfrac{1}{2} + \Sigma \cos npt)$$

In the present problem, unlike the shot effect, $\bar{u}$ may be either positive or negative, so that the constant term (steady current) will vanish on summing over a large number of pulses; the noise current-elements are left as

$$\Sigma i\Delta l = (e\,|\,\bar{u}\,|\,\tau p/\pi)(\Sigma \pm \cos npt) \quad . \quad . \quad (39)$$

Since the time $2\pi/p$ was taken to be very long, the order of harmonic $n$ is high, and we may replace the summation by an integral. The resultant of the noise current-elements in a group of frequencies between $n_1$ and $(n_1 + dn)$ is then given by the equation

$$(i\Delta l)_{dn} = (e\,|\,\bar{u}\,|\,\tau p/\pi)\int_{n=n_1}^{n=n_1+dn} \pm \cos npt\,.\,dn$$

Expressing this in terms of frequency, we have $np = 2\pi f$, $dn = (2\pi/p)df$,

$$\therefore \quad (i\Delta l)_{df} = 2e\,|\,\bar{u}\,|\,\tau\int_{f=f_1}^{f=f_1+df} \pm \cos 2\pi ft\,.\,df \quad . \quad . \quad (40)$$

$$(i\Delta l)^2_{df} = 4e^2\bar{u}^2\tau^2\int_{f=f_1}^{f=f_1+df} \cos^2 2\pi ft\,.\,df$$

$$= 2e^2u^2\tau^2\int_{f=f_1}^{f=f_1+df} (1 + \cos 4\pi ft)df$$

But on averaging over a considerable period we have $\int \cos 4\pi ft\,.\,df = 0$.

The mean-square value of the noise current-elements from a single pulse is therefore

$$(\bar{\imath}\Delta l)^2_{df} = 2e^2\bar{u}^2\tau^2.df \quad . \quad . \quad . \quad (41)$$

But if there are $N$ electrons taking part, and each electron collides $1/\tau$ times per second, i.e. makes $1/\tau$ separate journeys per second, there are altogether $N/\tau$ randomly-phased pulses per second. Since they are in random phase we may add the squares of the random current-elements (this is assuming that the resultant of $N$ random-phased vectors is $N^{\frac{1}{2}}$ times the unit vector), so that the resultant of the noise current-elements in frequency band $df$ has a mean-square value

$$(\bar{I}\Delta l)^2_{df} = 2Ne^2u^2\tau.df$$

and $$\bar{I}^2 = (2Ne^2u^2\tau\,.\,df)/(\Delta l)^2 \quad . \quad . \quad . \quad (42)$$

But if current $\bar{I}$ flows through resistance $R$ the corresponding potential difference is, by definition of $R$, equal to $\bar{I}R$; the mean-square noise voltage corresponding to (42) is therefore

$$\overline{V}^2_{df} = R^2\bar{I}^2_{df} = (2R^2Ne^2u^2\tau.df)/(\Delta l)^2 \quad . \quad . \quad (43)$$

Using equation (37) to eliminate $N$, $e$, and $\tau$,

$$\overline{V}^2_{df} = 2R\,.\,2m\bar{u}^2\,.\,df \quad . \quad . \quad . \quad . \quad (44)$$

It is clear that the only velocity which will contribute to the current between two points is the velocity along the direction of flow; in other words, $\bar{u}^2$ is to be taken as the mean-square of the component of thermal-agitation velocity in one specified direction. But the equipartition value of the thermal energy contributed by velocity components in one given direction is $\frac{1}{2}m\bar{u}^2 = \frac{1}{2}kT$ for each particle. Equation (44) therefore becomes

$$\overline{V}^2_{df} = 4RkT.df \quad . \quad . \quad . \quad . \quad (45)$$

which is the well-known expression deduced thermodynamically by Nyquist.

Whereas in Nyquist's derivation the internal mechanism was not involved, owing to the use of overall

energy-exchanges, in the treatment given above the mechanism is exposed, but all factors peculiar to the material, namely the number of electrons taking part in conduction, time of flight, and mass and charge of the electron, have been eliminated in terms of the resistance and the function $kT$. Equally with the thermodynamic proof, therefore, the derivation presented above indicates that the thermal-agitation noise is a function only of ohmic resistance and temperature, and not of the structure of the conductor involved. The thermionic valve is therefore included, so long as it has a determinable ohmic resistance and temperature.

It is interesting to note that in Nyquist's derivation there was no upper limit to the frequency to which the expression for noise energy was applicable; something in the nature of quantum restrictions was necessary to prevent the total energy from becoming infinite if the frequency range was extended to infinity instead of being confined to the radio band. In the derivation used above it is obvious that the expression ceases to be valid when the frequency has a period comparable with the time of flight of the electron within the conductor, for the Fourier analysis must then be modified and will depend upon the shape of the pulse. There is therefore seen to be a limit to the validity of the expression, though the limiting frequency will be somewhere in the region of heat radiation.

## (9) APPLICATION TO THE THERMIONIC VALVE

The remaining difficulty is the transition from a temperature-limited to a space-charge-limited regime in a thermionic valve: we have yet to decide what constitutes the essential distinction between the two states. It would seem that the presence of a potential minimum, however small the barrier which it imposes, is one criterion; for, provided there is such a barrier, some electrons emitted will return to the cathode, and the space-charge adjacent to the cathode will be in thermal equilibrium with it. The initial temperature of the space-charge will then be fixed at the cathode temperature, and, as we found above, the effective temperature of the outer space-charge is about half that of the inner space-charge. The transition in any real valve will be gradual, owing to lack of uniformity both of the anode-to-cathode field along the length of the cathode and of the cathode temperature.

Another criterion of the state in which the thermal-agitation treatment is applicable is that the field from the anode should terminate on space-charge, not on an actual metallic electrode. For if the field from the anode ends on a metallic electrode, the emergence of any electron from that electrode constitutes a disturbance; but if the field terminates on electrons, constituting space-charge, which are moving towards the anode with approximately uniform velocity, the presence of electrons travelling at the exact mean velocity at every point does not create a disturbance; it is only the deviations from the mean which are effective.

In a diode, but not in more complex valves, the two criteria are identical. As an example of the more complex valves, consider a screen-grid tetrode. In this the screen-anode space corresponds nearly to a temperature-limited diode with the screen as virtual cathode. Most of the anode field terminates on the screen wires, so that the injection of electrons into this space through the screen may be expected to produce a shot noise at the anode, of the magnitude predicted by equation (1). The same effect is to be expected to a less extent in triodes, depending upon the closeness of the grid·winding; this may explain the experimental results of F. C. Williams with an L.S.5 triode.*

## (10) TWO RESISTANCES IN PARALLEL

Another point which is of importance for the comparison of any theory of thermal noise with experimental results is the magnitude of the resultant fluctuation voltage from two resistances at different temperatures connected in parallel. Transferring equation (45) from voltage back to fluctuation current, which we saw was the fundamental phenomenon, we find that

$$\bar{I}^2_{df} = (4kT/R) \, . \, df$$

where $\bar{I}^2$ is the mean-square fluctuation current. But $\bar{I}^2$ is made up of a very large number of random vectors, so that to combine two such currents, arising in resistances $R_1$ and $R_2$, we merely add their squares. Thus

$$\bar{I}_0^2 = \bar{I}_1^2 + \bar{I}_2^2 = (4kT_1/R_1 + 4kT_2/R_2) \, . \, df \quad . \quad (46)$$

Equation (46) for the total fluctuation current flowing when two resistances $R_1$ and $R_2$ at temperatures $T_1$ and $T_2$ are connected in parallel reduces to

$$\bar{I}_0^2 = 4k \cdot \frac{R_2T_1 + R_1T_2}{R_1R_2} \cdot df \quad . \quad . \quad . \quad (47)$$

But the resultant resistance of the two in parallel is $R_0 = R_1R_2/(R_1 + R_2)$. The noise voltage corresponding to the current given in equation (47) is therefore

$$\bar{V}_0^2 = R_0^2\bar{I}_0^2 = 4k \cdot \frac{R_1R_2(R_2T_1 + R_1T_2)}{(R_1 + R_2)^2} \cdot df \quad . \quad (48)$$

This equation has been previously derived by Llewellyn.†

## (11) ACKNOWLEDGMENTS

Any scientific worker is indebted to his predecessors, but the present author, owing to his own lack of experimental work on the problem, is particularly indebted to the writers of previous publications, even though he has disagreed with them. In particular, his thanks are due to Mr. E. B. Moullin, who first interested him in this subject and with whom he has had a number of valuable discussions.

The author is also indebted to Marconi's Wireless Telegraph Co., Ltd., for the preparation of lantern slides used in the presentation of this paper.

## REFERENCES

(1) E. B. Moullin and H. D. M. Ellis: *Journal I.E.E.*, 1934, vol. 74, p. 323.
(2) F. B. Llewellyn: *Proceedings of the Institute of Radio Engineers*, 1930, vol. 18, p. 243.
(3) G. L. Pearson: *Physics*, 1935, vol. 6, p. 6.
(4) H. Nyquist: *Physical Review*, 1928, vol. 32, p. 110.

* See Reference (9).

† *Ibid.*, (10).

(5) T. C. FRY: *Journal of the Franklin Institute,* 1925, vol. 199, p. 203.
(6) N. CAMPBELL: *Proceedings of the Physical Society,* 1935, vol. 47, p. 800.
(7) E. N. ROWLAND: *Proceedings of the Cambridge Philosophical Society,* 1936, vol. 32, p. 580.
(8) O. W. RICHARDSON: " Emission of Electricity from Hot Bodies," 2nd edition, London, 1921.
(9) F. C. WILLIAMS: *Journal I.E.E.,* 1936, vol. 79, p. 349.
(10) F. B. LLEWELLYN: *Proceedings of the Institute of Radio Engineers,* 1930, vol. 18, p. 256.

## DISCUSSION BEFORE THE WIRELESS SECTION, 5TH JANUARY, 1938

**Mr. E. B. Moullin:** In his discussion of Fig. 1 the author points out that the deviations from the mean increase with the current, even though the flow of electrons becomes relatively smoother. It is a basic theorem of statistics that, in circumstances such as those depicted in Fig. 1, the mean-square departure from the mean increases in direct proportion to the mean value. It is both well known and remarkable that this theorem, true in the limit of very large numbers, is substantially true when the number of events is quite small. This can be illustrated well by means of the diagrams in Fig. 1. Thus we find that the mean-square departures from the mean in Cases A and D of Fig. 1 are 1·3 and 0·6 respectively, if the baseline is taken as 20 units of time. There is no reason why we should take A, B, C, or D, in preference; and therefore the natural course is to take these possible chances of succession one after the other, and then the long-time mean of the mean-square fluctuation is 0·92. According to the statistical theorem I have mentioned, if we have 5 times the current the mean-square departure should be 5 times this amount, i.e. 4·6. We have a particular case of this in the sum curve of Fig. 1; and the mean-square departure is 6·3. Thus we should expect the value 4·6, whereas we get 6·3; but this is too rough an approach to be a fair test. There are 10 possible summation curves of Cases A to E taken in pairs, 10 more possible combinations taken three at a time, and 5 combinations taken four at a time. I have made the 10 possible additions two at a time, and find the mean-square deviation of these comes to 2·27. For the 5 possible combinations four at a time I find the mean-square deviation is 4·53. As an example, let us suppose that the mean-square deviation is proportional to the mean, and equal to 4·53 when the mean is 4. Then, when the mean current is 1, 2, 4, or 5 electrons (I have not worked out the figures for 3), the deviations should be 1·14, 2·27, 4·53, and 5·67 respectively. The measured deviations for the sample curves given in Fig. 1 are in fact found to be 0·92, 2·27, 4·53, and 6·3. The agreement between these sets of values is fairly good, and the example serves to illustrate this important theorem.

I feel that Nyquist's theorem still requires to be stated more rigidly. In its present form it is certainly correct for all linear networks which do not include thermionic valves, but when such are included we still do not know how to apply it to give the correct result, though I consider the author's method of obtaining the effective value of the internal resistance (page 100) is a notable step forward to this end.

Section (5) begins by stating the author's belief that shot and thermal effects are different aspects of the same general principle. I have always thought, and I think often stated, that until we had a general theorem which would include shot and thermal effects as special cases of one general principle, this very intricate noise problem would remain in an unsatisfactory state. In my opinion Section (8) of this paper is a real step towards this general treatment, and is perhaps the most illuminating theoretical contribution to the whole subject which has been made in the last decade. I have often tried to obtain the thermal-agitation formula from the shot-effect principle, and have always failed dismally.

Lately I was talking to Prof. Schottky in Berlin, and he told me he had derived a formula for the thermal-agitation principle from a shot-voltage mechanism. The proof has not yet been published, and I shall be interested to see whether it is the same as the present author's derivation.

From one point of view it has always been difficult to understand why the passage of a steady current through a resistance did not increase the fluctuation voltage produced by it. For if most methods of calculating the shot voltage produced by a temperature-limited thermionic current (for example, that which Mr. Ellis and I produced some years ago),* are examined critically, it is difficult to find any step in the argument which could not be applied to effects inside a conductor. I think the paradox has been resolved by the analysis of Section (8) of this paper: for this brings out clearly that the number of random events in the conductor are governed by the number of free electrons in it, and not by the average current passing through it. Such a current will not produce a first-order effect on the number of random events, and will possibly produce no effect at all.

In spite of all the work that has been done, however, the background noise in an amplifier valve still cannot be calculated. In the last 6 months Prof. Schottky has produced several long papers on the shot effect in space-charge-limited conditions. I have so far been unable to disentangle fully his basic idea from the analysis in which it is clothed. Schottky's formula is appreciably better than the old classic expression of Llewellyn, but it is still in bad agreement with facts.

On page 104 the author observes: " In practical valves, end effects cause a fraction of the current to be temperature-limited, and therefore probably more ' noisy.' It might be worth while to try experiments with guard rings, so as to use only the central part of the cathode where the current can be fully space-charge-limited." That work has just been completed in Germany,† but the results do not appear to approach the theoretical value of Schottky more closely than they would have done if no precautions had been taken.

**Dr. F. C. Williams:** The author deals with this subject from the point of view of the physicist, whereas I have examined it experimentally.

* *Journal I.E.E.*, 1934, vol. 74, p. 323.
† H. JACOBY and L. KIRCHGESSNER: *Wissenschaftliche Veroffentlichungen aus den Siemens-Werken*, 1937, vol. 16, p. 42.