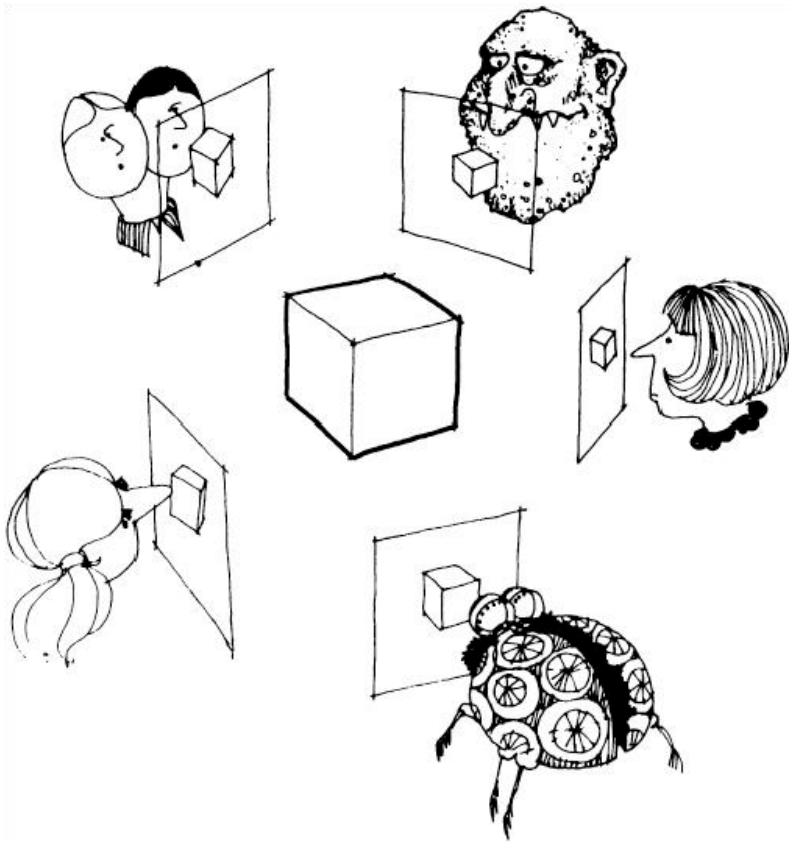
Projective Geometry and Camera Models



Computer Vision

Jia-Bin Huang, Virginia Tech

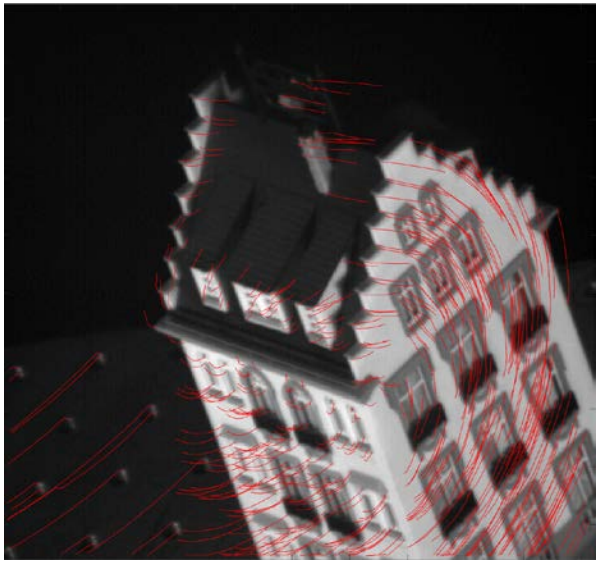
Many slides from S. Seitz and D. Hoiem

Administrative stuffs

- HWs
  - HW 1 will be back this week
  - HW 2 due 11:55 PM on Oct 3.
  - Submit your results on shape alignment: link
  - Frequently asked questions for HW 2

- Think about your final projects
  - work in groups of 2-4
  - should evince independent effort to learn about a new topic, try something new, or apply to an application of interest
  - Proposals will be due Oct 23

HW 2: Feature Tracking



```
function featureTracking
% Main function for feature tracking
folder = '.\images';
im = readImages(folder, 0:50);

tau = 0.06;                                  % Threshold for harris corner detection
[pt_x, pt_y] = getKeypoints(im{1}, tau);     % Prob 1.1: keypoint detection

ws = 7;                                      % Tracking ws x ws patches
[track_x, track_y] = ...                     % Prob 1.2 Keypoint tracking
    trackPoints(pt_x, pt_y, im, ws);

% Visualizing the feature tracks on the first and the last frame
figure(2), imagesc(im{1}), hold off, axis image, colormap gray
hold on, plot(track_x', track_y', 'r');
```

HW 2 – Feature/Keypoint detection

- Compute second moment matrix

$$\mu(\sigma_I, \sigma_D) = g(\sigma_I) * \begin{bmatrix} I_x^2(\sigma_D) & I_x I_y(\sigma_D) \\ I_x I_y(\sigma_D) & I_y^2(\sigma_D) \end{bmatrix}$$

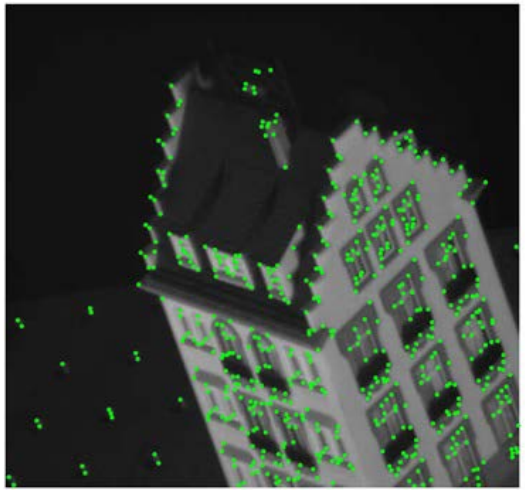
- Harris corner criterion

$$har = \det[\mu(\sigma_I, \sigma_D)] - \alpha[\mathrm{trace}(\mu(\sigma_I, \sigma_D))^2] =$$

$$g(I_x^2)g(I_y^2) - [g(I_x I_y)]^2 - \alpha[g(I_x^2) + g(I_y^2)]^2$$

$$\alpha = 0.04$$

- Threshold
- Non-maximum suppression



```
function [keyXs, keyYs] = getKeypoints(im, tau)
% im:  input image; tau: threshold
% keyXs, keyYs: detected keypoints, with dimension [N] x [1]

% 0. Smooth image (optional)

% 1. Compute image gradients. Hint: can use “gradient”

% 2. Compute Ix*Ix, Iy*Iy, Ix*Iy

% 3. Smooth Ix*Ix, Iy*Iy, Ix*Iy (locally weighted average)

% 4. Compute Harris corner score. Normalize the max to 1

% 5. Non-maximum suppression. Hint: use “ordfilt2”

% 6. Find positions whose values larger than tau. Hint: use
     “find”
end
```

```
function [track_x, track_y] = trackPoints(pt_x, pt_y, im, ws)
% Tracking initial points (pt_x, pt_y) across the image sequence
% track_x: [Number of keypoints] x [nim]
% track_y: [Number of keypoints] x [nim]

% Initialization
N     = numel(pt_x);                    % Number of keypoints
nim = numel(im);                        % Number of images
track_x = zeros(N, nim);
track_y = zeros(N, nim);
track_x(:, 1) = pt_x(:);
track_y(:, 1) = pt_y(:);

for t = 1:nim-1                         % Tracking points from t to t+1
   [track_x(:, t+1), track_y(:, t+1)] = ...
         getNextPoints(track_x(:, t), track_y(:, t), im{t}, im{t+1}, ws);
end
```

Iterative L-K algorithm

Original (x,y) position

$I_t = I(x', y', t+1) - I(x, y, t)$

1. Initialize (x',y') = (x,y)
2. Compute (u,v) by

$$\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

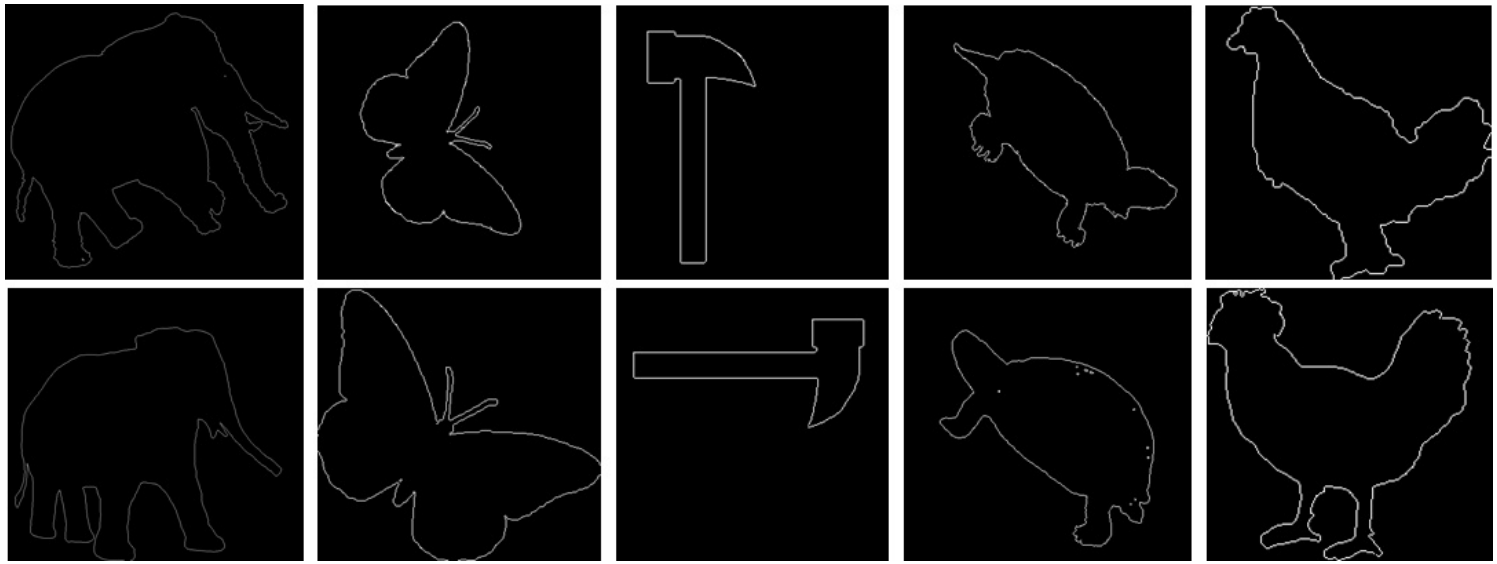
2nd moment matrix for feature patch in first image

displacement

3. Shift window by (u, v): x′=x′+u; y′=y′+v;
4. Recalculate I_t
5. Repeat steps 2-4 until small change
   - Use interpolation for subpixel values

```
function [x2, y2] = getNextPoints(x, y, im1, im2, ws) % Iterative Lucas-Kanade feature tracking
% (x,  y) : initialized keypoint position in im1; (x2, y2): tracked keypoint positions in im2
% ws: patch window size

% 1. Compute gradients from Im1 (get Ix and Iy)
% 2. Grab patches of Ix, Iy, and im1.
Hint 1: use "[X, Y] = meshgrid(-hw:hw,-hw:hw);" to get patch index, where hw = floor(ws/2);
Hint 2: use "interp2" to sample non-integer positions.

for iter = 1:numIter                    % 5 iterations should be sufficient
% Check if tracked patch are outside the image. Only track valid patches.
% For each keypoint
%  - grab patch1 (centered at x1, y1), grab patch2 (centered at x2,y2)
% - compute It = patch2 – patch1
% -  grab Ix, Iy (centered at x1, y1)
% - Set up matrix A and vector b
% -  Solve linear system d = A\b.
% -  x2(p)=x2(p)+d(1);       y2(p)=y2(p)+d(2);   -> Update the increment
end
```

HW 2 – Shape Alignment

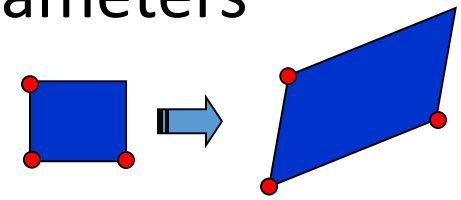


- Global transformation (similarity, affine, perspective)
- Iterative closest point algorithm

Fitting an affine transformation

$$\begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ x_i' \\ y_i' \\ \cdots \end{bmatrix}$$

- Linear system with six unknowns
- Each match gives us two linearly independent equations: need at least three to solve for the transformation parameters



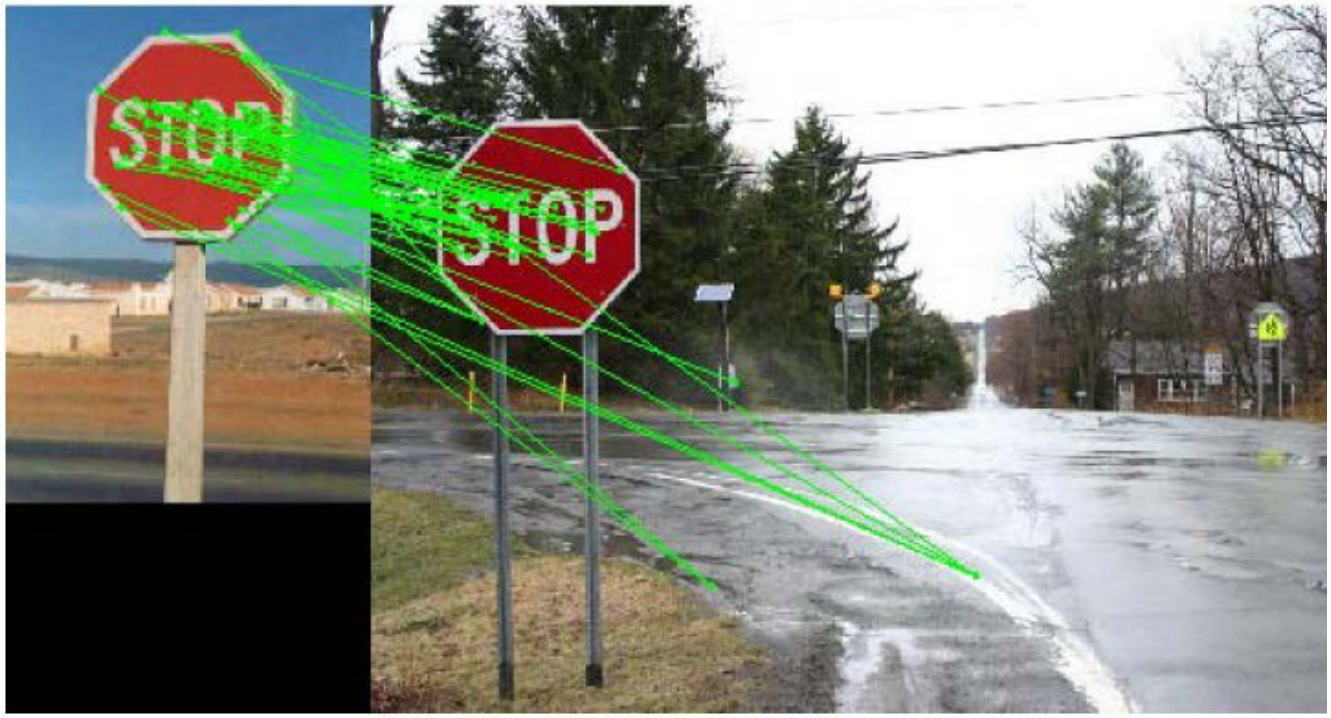
```
function Tfm = align_shape(im1, im2)
% im1: input edge image 1
% im2: input edge image 2
% Output: transformation Tfm [3] x [3]

% 1. Find edge points in im1 and im2. Hint: use "find"
% 2. Compute initial transformation (e.g., compute translation and
scaling by center of mass, variance within each image)

for i = 1: 50

% 3. Get nearest neighbors: for each point p_i find corresponding
```

$$\mathrm{match}(i) = \underset{j}{\mathrm{argmin}}\, \mathrm{dist}(pi, qj)$$

```
% 4. Compute transformation T based on matches

% 5. Warp points p according to T

end
```

HW 2 – Local Feature Matching

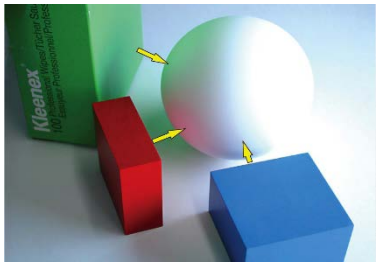
- Implement distance ratio test feature matching algorithm

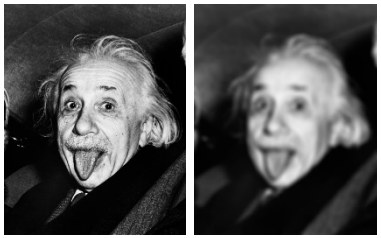


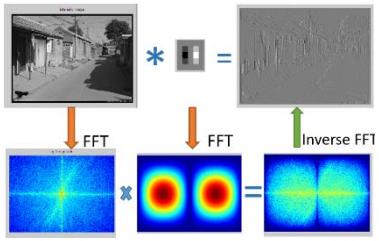
```
function featureMatching
im1 = imread('stop1.jpg');
im2 = imread('stop2.jpg');
% Load pre-computed SIFT features
load('SIFT_features.mat');
% Descriptor1, Descriptor2: SIFT features
% Frame1, Frame2: position, scale, rotation

% For every descriptor in im1, find the 1st nearest neighbor
and the 2nd nearest neighbor in im2.

% Compute distance ratio score.

% Threshold and get the matches: a 2 x N array of indices
that indicates which keypoints from image1 match which
points in image 2
figure(1), hold off, clf
plotmatches(im2double(im1),im2double(im2),Frame1,Frame2,mat
ches); % Display the matched keypoints
```

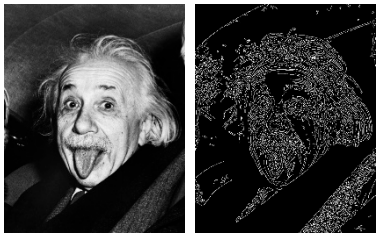
Review: Interpreting Intensity

- **Light and color**
  - What an image records
- **Filtering in spatial domain**
  - Filtering = weighted sum of neighboring pixels
  - Smoothing, sharpening, measuring texture
- **Filtering in frequency domain**
  - Filtering = change frequency of the input image
  - Denoising, sampling, image compression
- **Image pyramid and template matching**
  - Filtering = a way to find a template
  - Image pyramids for coarse-to-fine search and multi-scale detection
- **Edge detection**
  - Canny edge = smooth -> derivative -> thin -> threshold -> link
  - Finding straight lines, binary image analysis



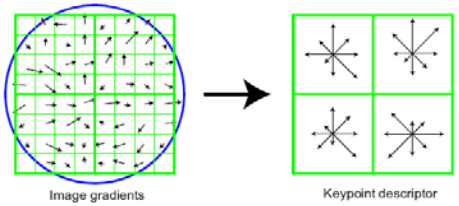


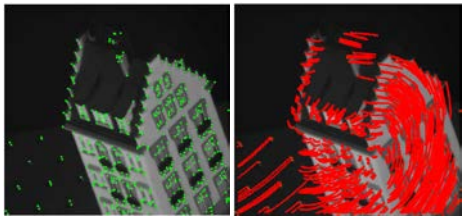


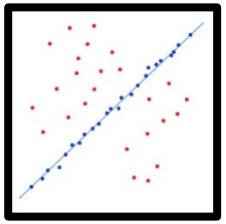


Review: Correspondence and Alignment

- **Interest points**
  - Find *distinct* and *repeatable* points in images
  - Harris-> corners, DoG -> blobs
  - SIFT -> feature descriptor
- **Feature tracking and optical flow**
  - Find motion of a keypoint/pixel over time
  - Lucas-Kanade:
    - brightness consistency, small motion, spatial coherence
  - Handle large motion:
    - iterative update + pyramid search
- **Fitting and alignment**
  - find the transformation parameters that best align matched points
- **Object instance recognition**
  - Keypoint-based object instance recognition and search



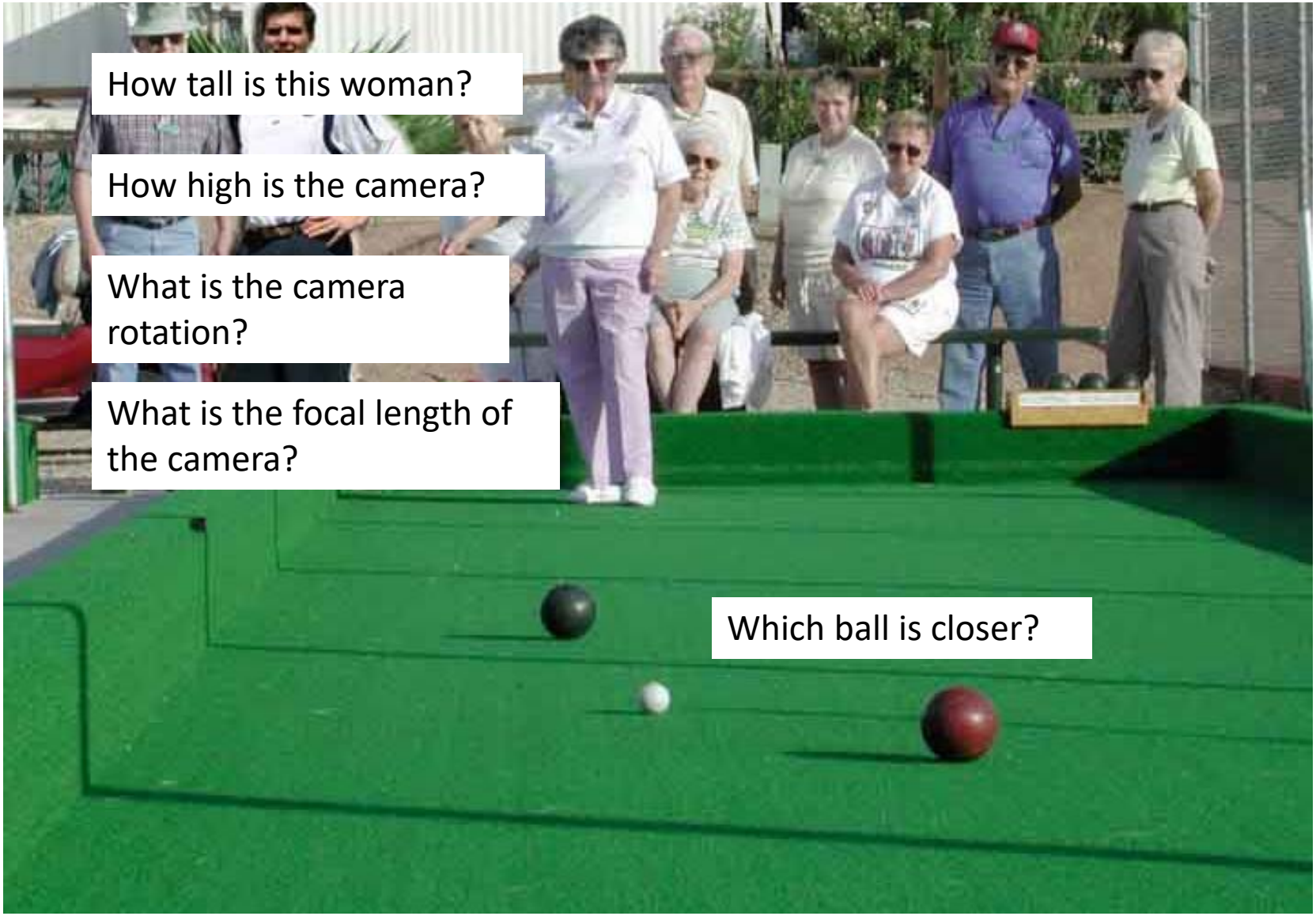




Perspective and 3D Geometry

- **Projective geometry and camera models**
  - What's the mapping between image and world coordiantes?
- **Single view metrology and camera calibration**
  - How can we measure the size of 3D objects in an image?
  - How can we estimate the camera parameters?
- **Photo stitching**
  - What's the mapping from two images taken without camera translation?
- **Epipolar Geometry and Stereo Vision**
  - What's the mapping from two images taken with camera translation?
- **Structure from motion**
  - How can we recover 3D points from multiple images?

Next two classes: Single-view Geometry

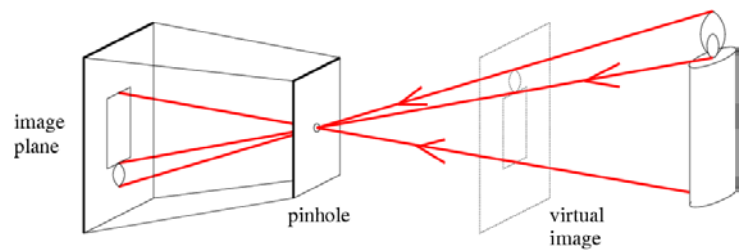


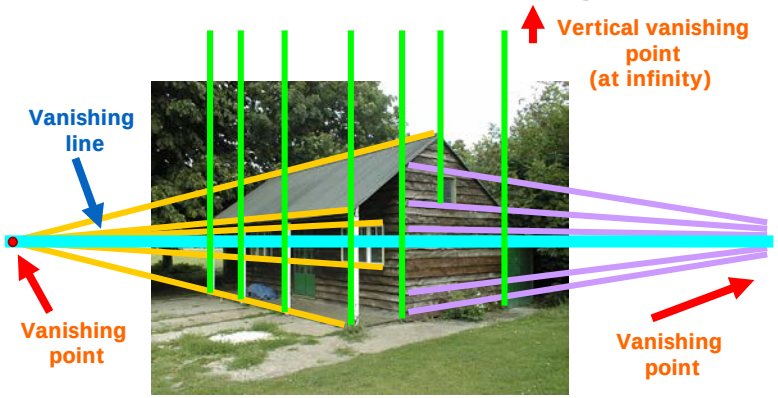
Today's class
Mapping between image and world coordinates

- Pinhole camera model

- Projective geometry
  - Vanishing points and lines

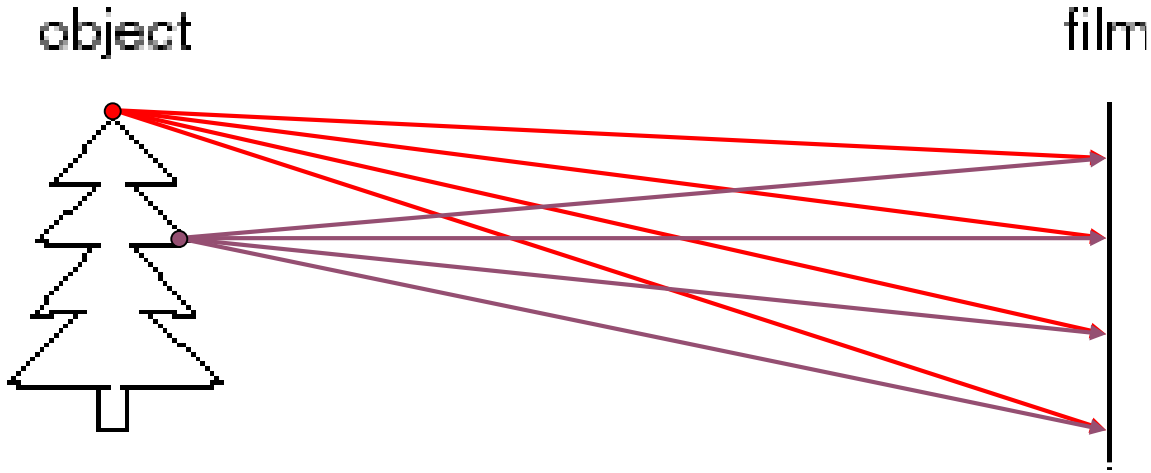
- Projection matrix





$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

Image formation

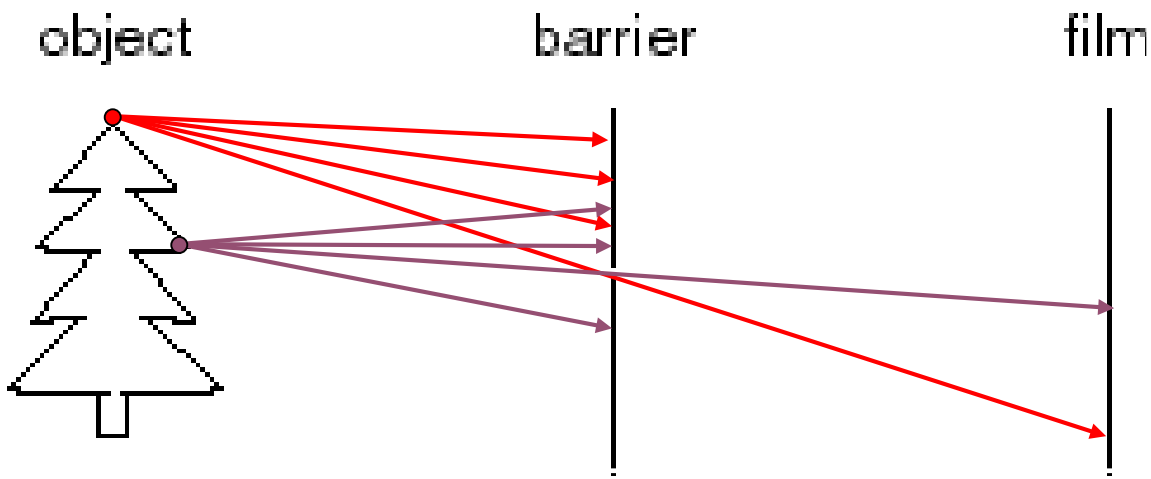


Let's design a camera

- Idea 1: put a piece of film in front of an object
- Do we get a reasonable image?

Slide source: Seitz

Pinhole camera

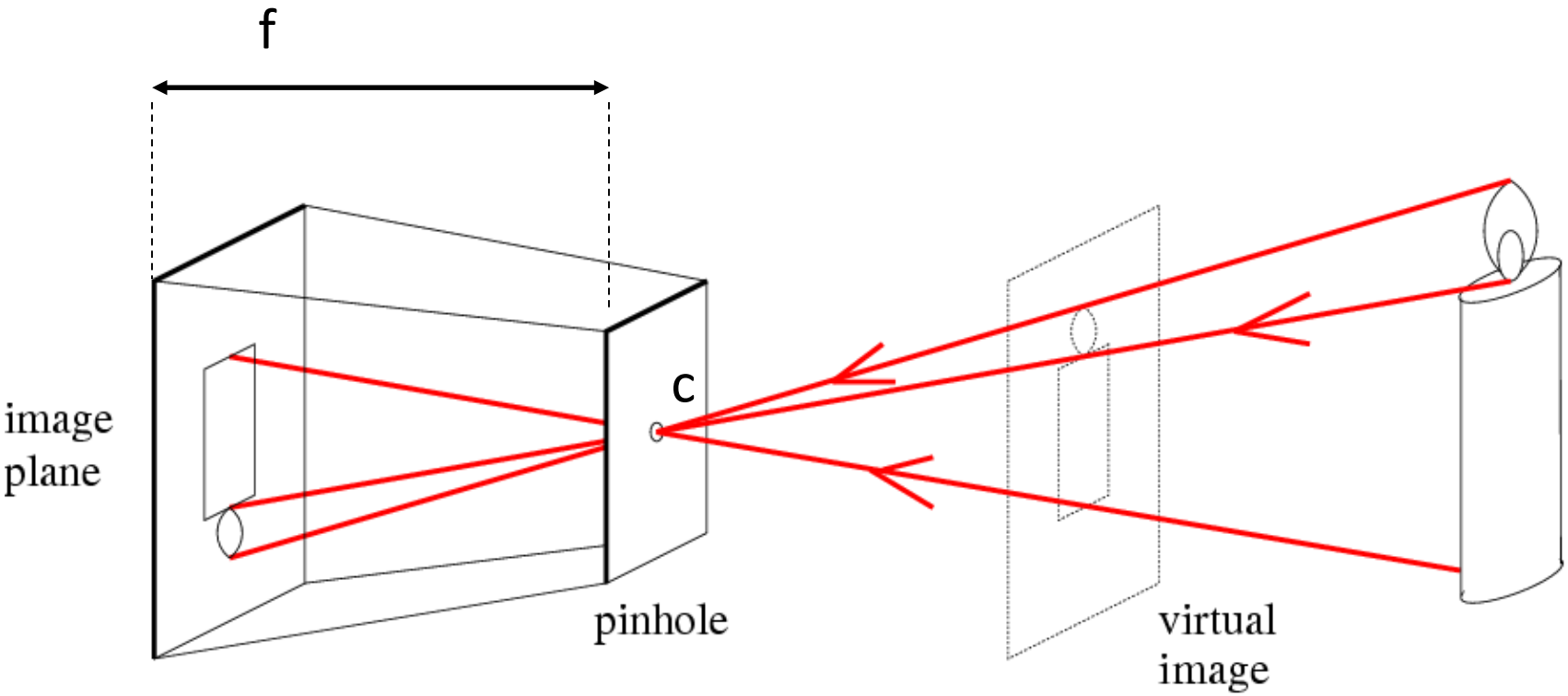


Idea 2: add a barrier to block off most of the rays

- This reduces blurring
- The opening known as the **aperture**

Slide source: Seitz

Pinhole camera



f = focal length
c = center of the camera

Figure from Forsyth

Camera obscura: the pre-camera

- First idea: Mo-Ti, China (470BC to 390BC)
- First built: Alhazen, Iraq/Egypt (965 to 1039AD)

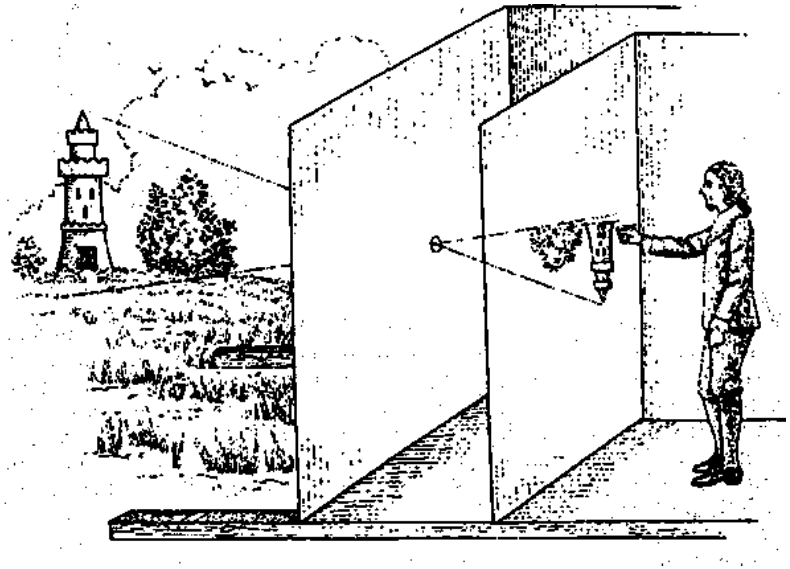
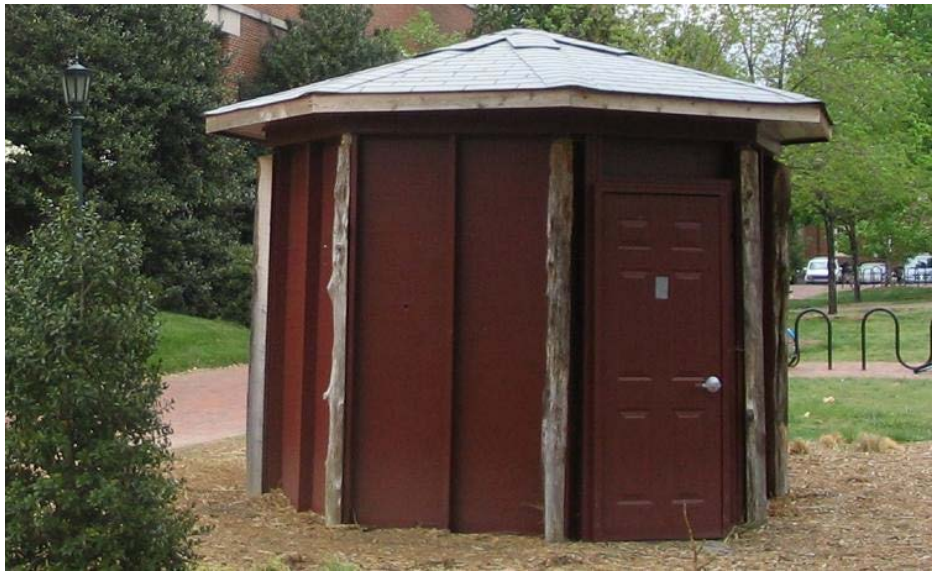


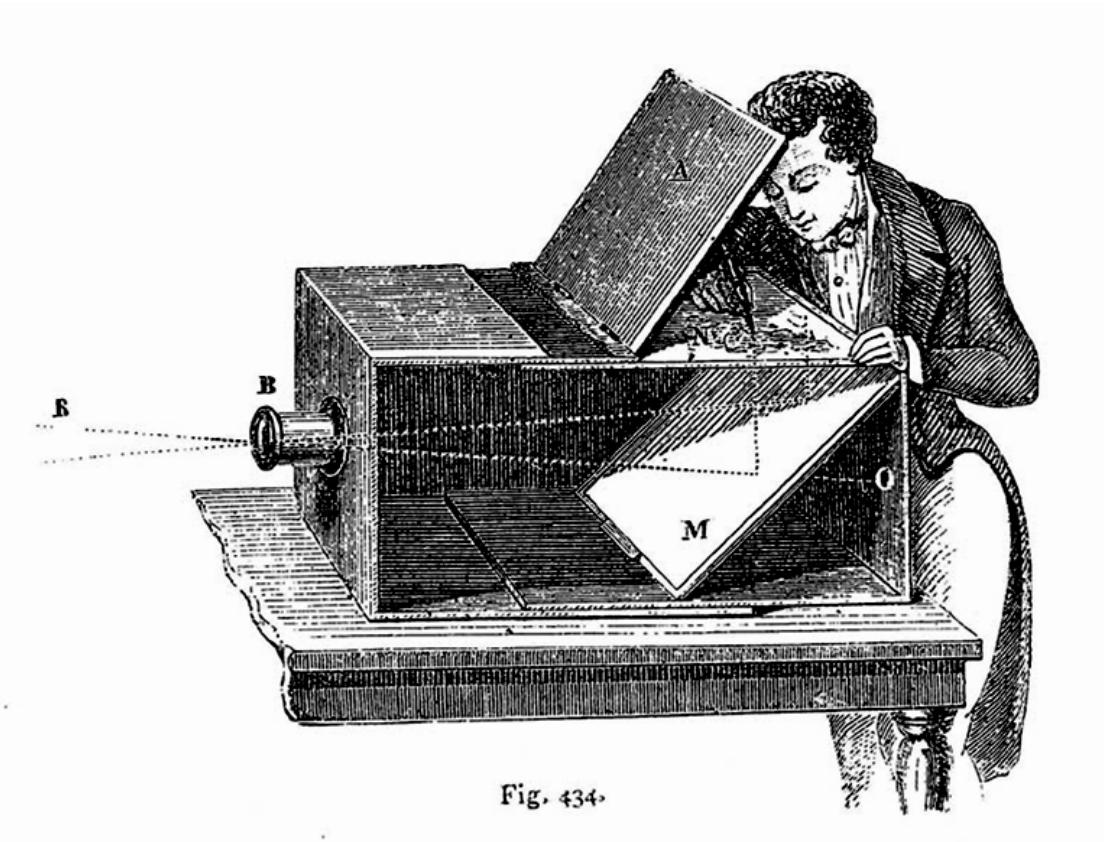
Illustration of Camera Obscura



Freestanding camera obscura at UNC Chapel Hill

Photo by Seth Ilys

Camera Obscura used for Tracing



Lens Based Camera Obscura, 1568

First Photograph

Oldest surviving photograph

- Took 8 hours on pewter plate

Joseph Niepce, 1826

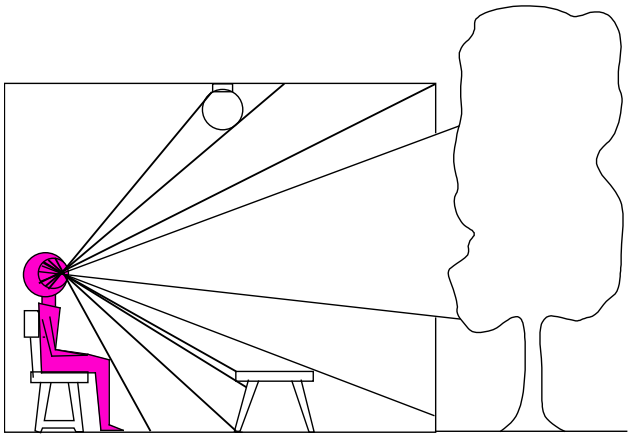
Photograph of the first photograph

Stored at UT Austin

Niepce later teamed up with Daguerre, who eventually created Daguerrotypes

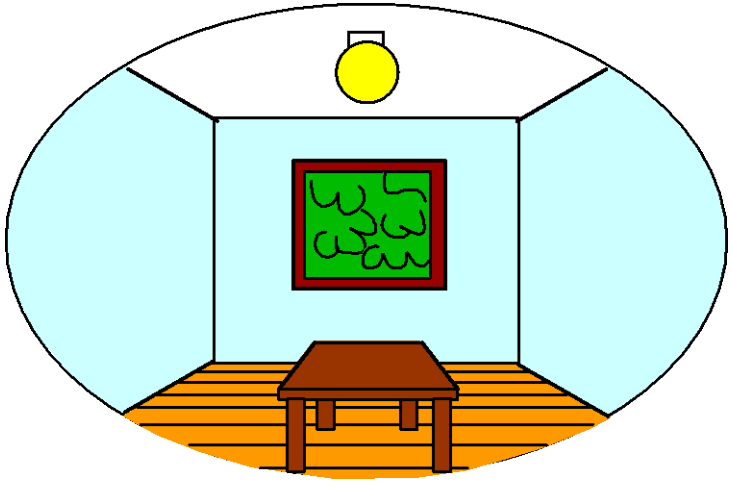
Dimensionality Reduction Machine (3D to 2D)

3D world



Point of observation

2D image



Slide source: Seitz

Projection can be tricky…

Slide source: Seitz

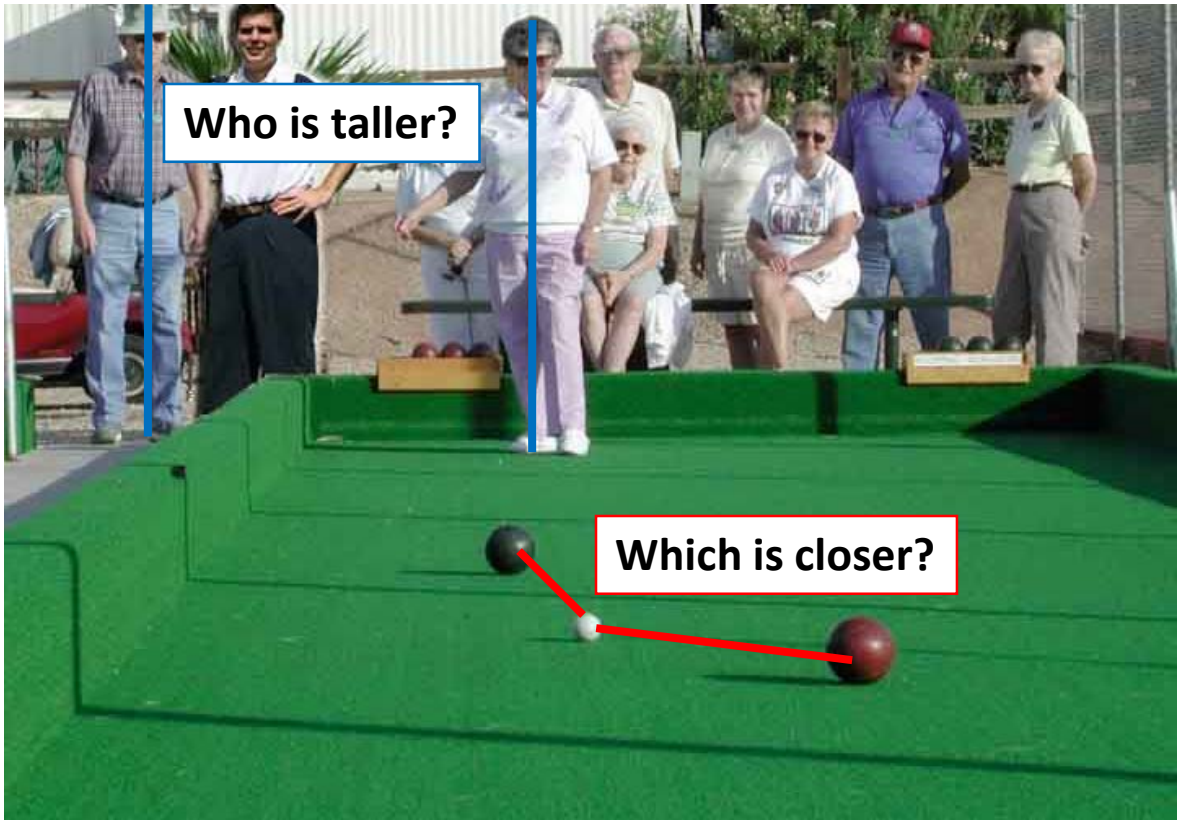
Projection can be tricky…

Making of 3D sidewalk art: http://www.youtube.com/watch?v=3SNYtd0Ayt0

Projective Geometry

What is lost?

- Length



Length is not preserved

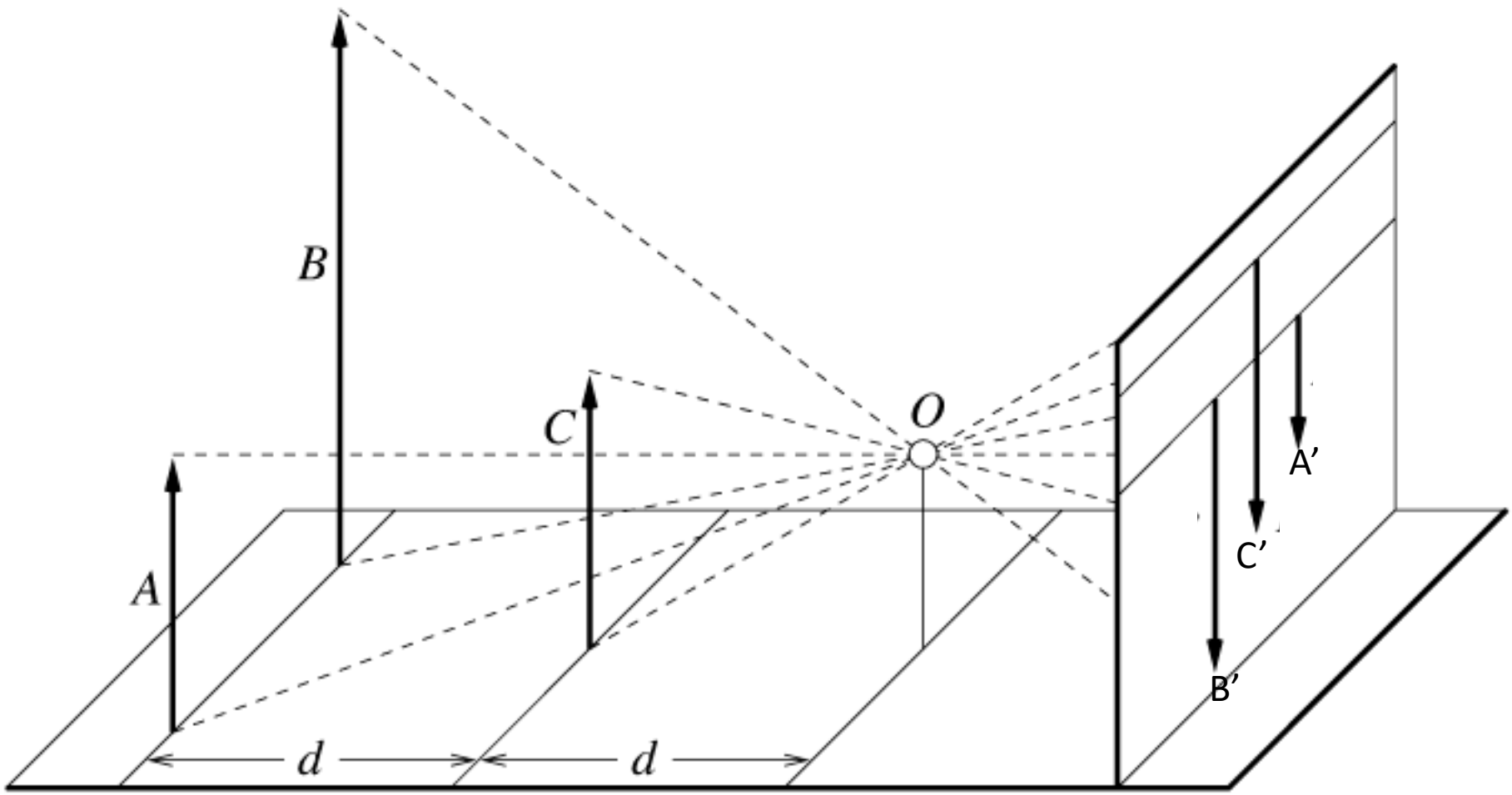
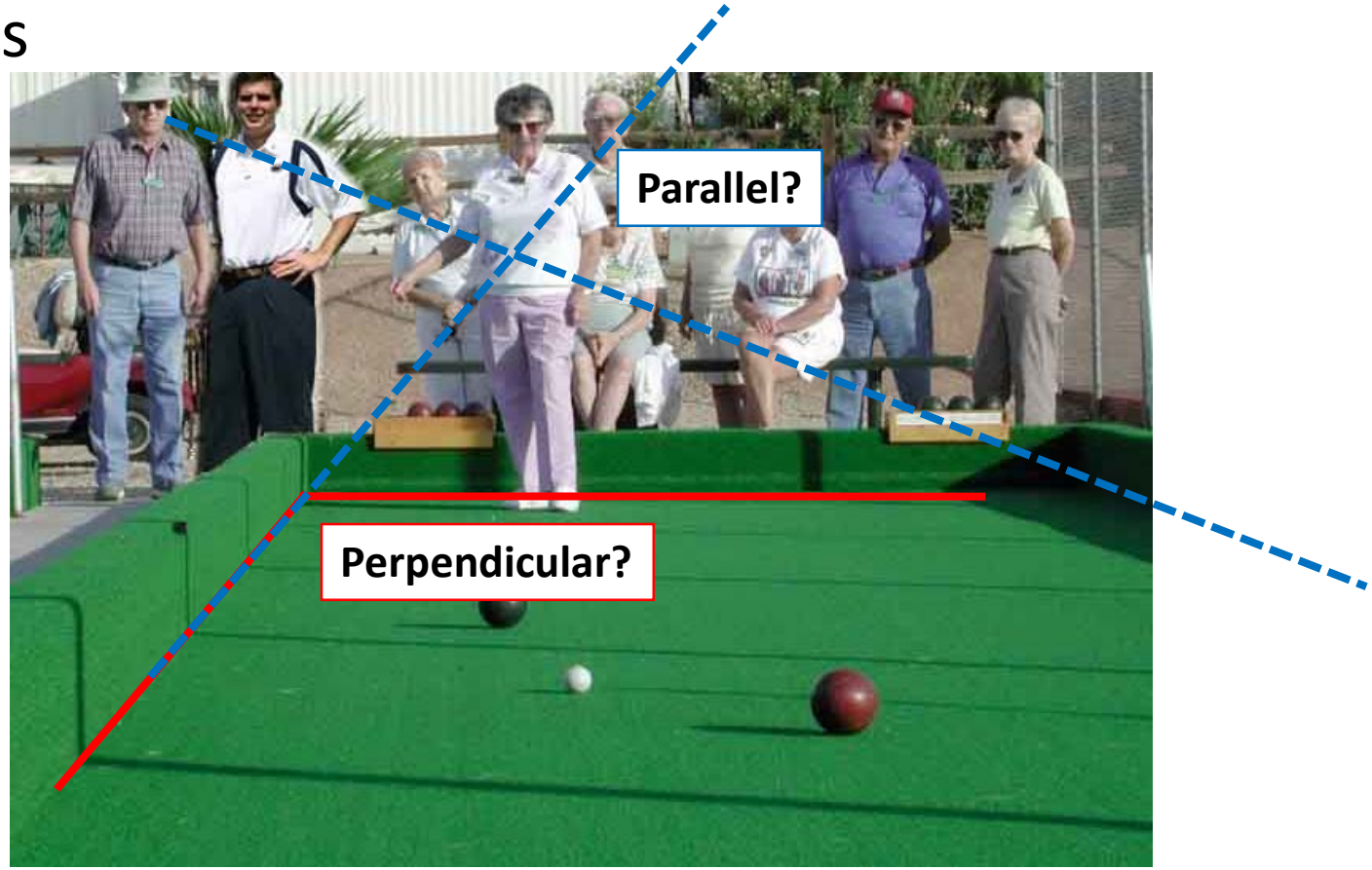


Figure by David Forsyth

Projective Geometry

What is lost?

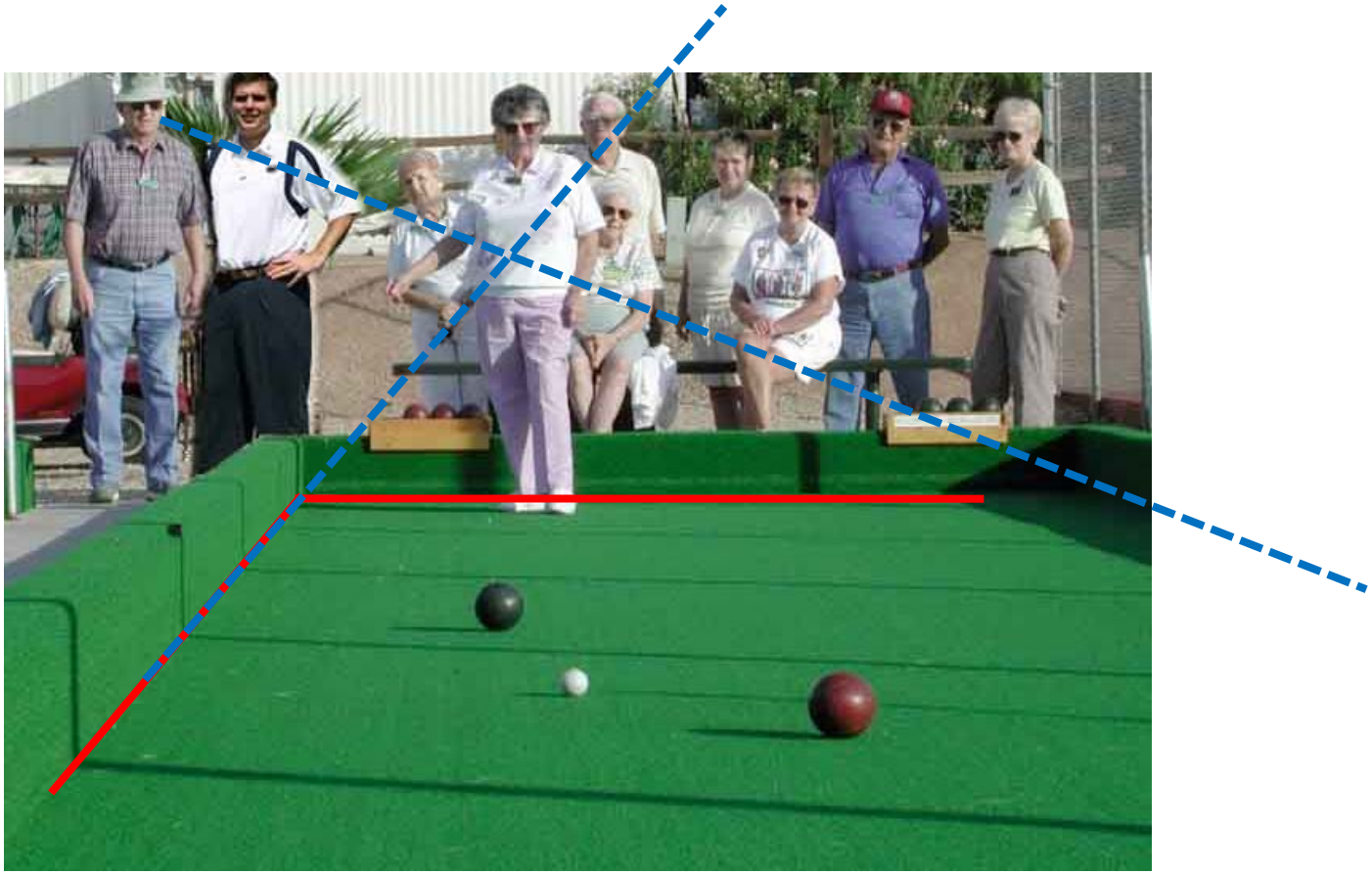
- Length
- Angles



Projective Geometry

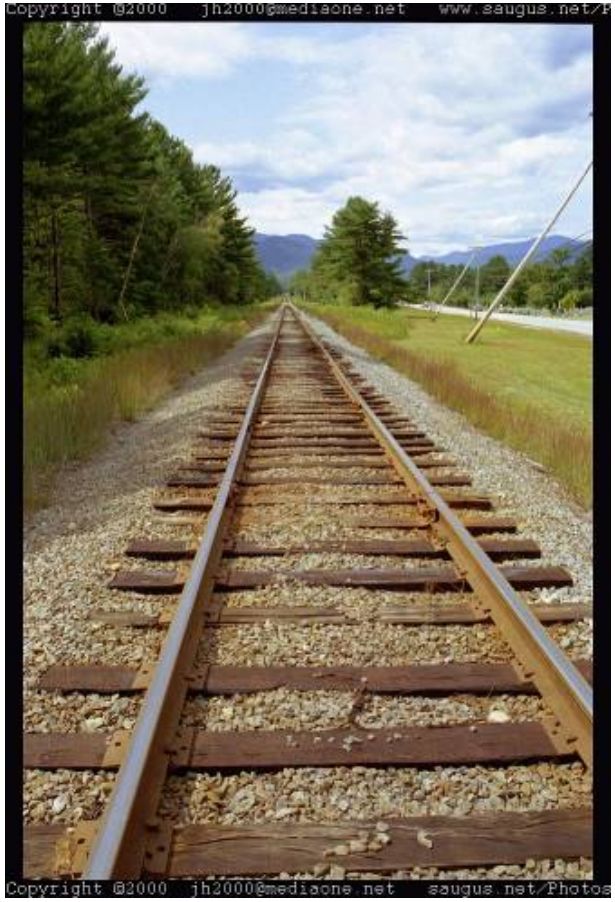
What is preserved?

- Straight lines are still straight

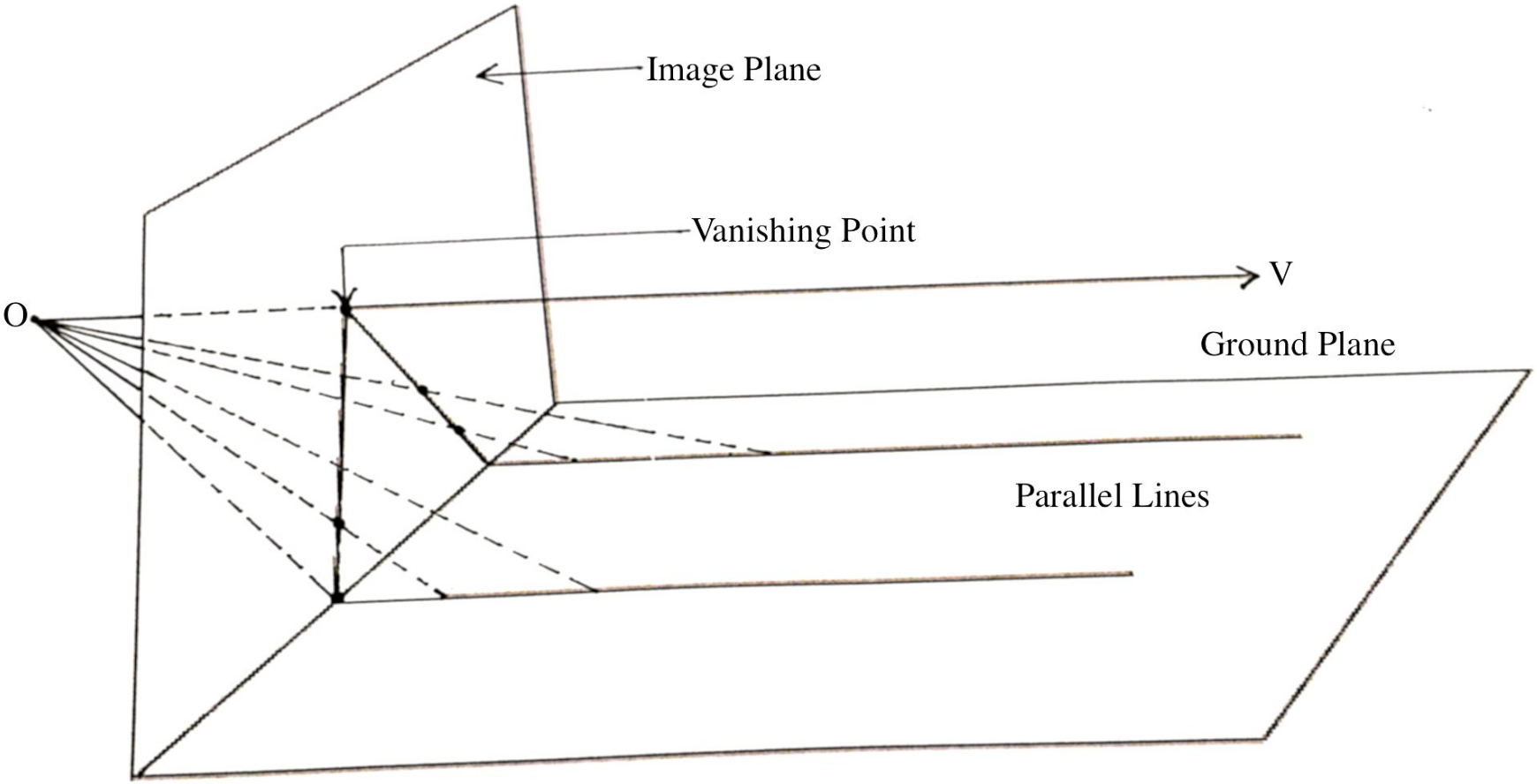


Vanishing points and lines

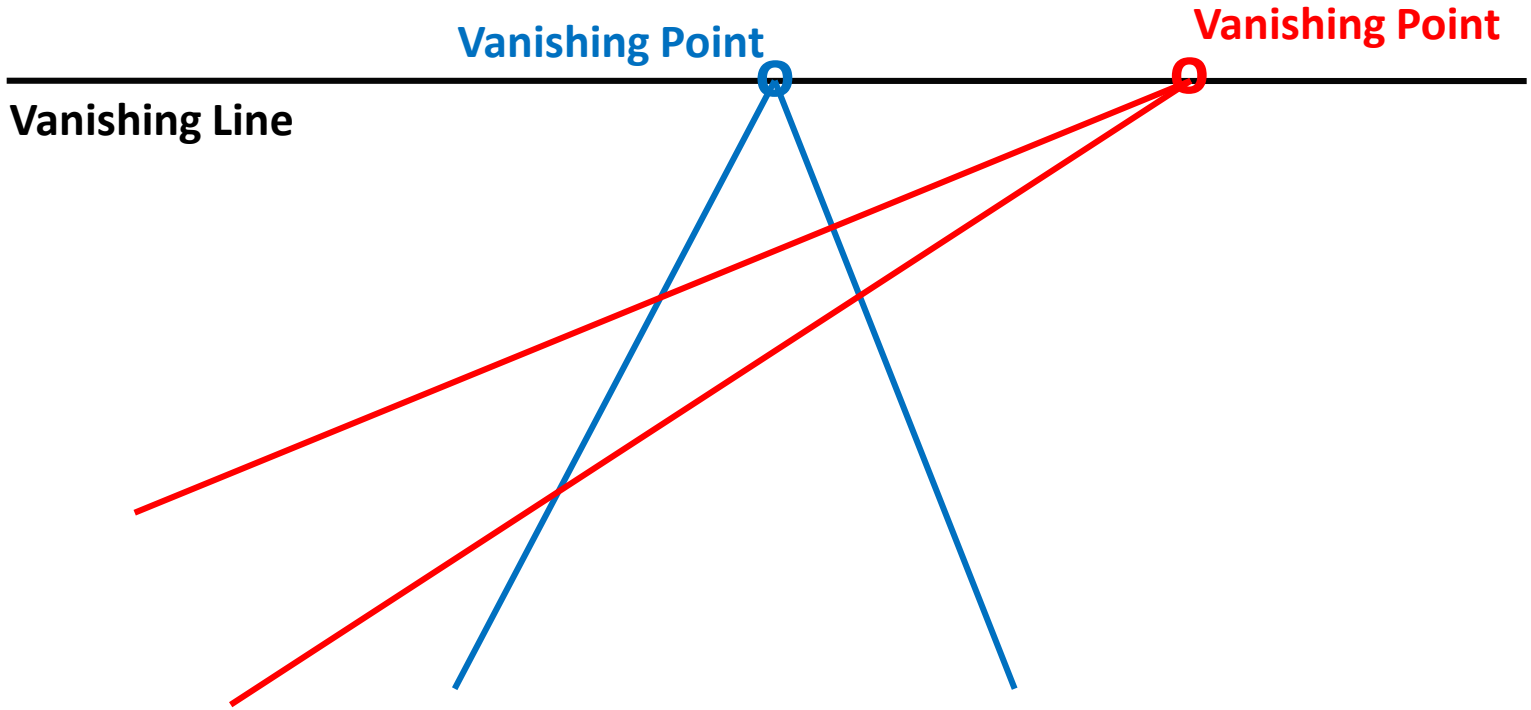
Parallel lines in the world intersect in the image at a "vanishing point"



Vanishing points

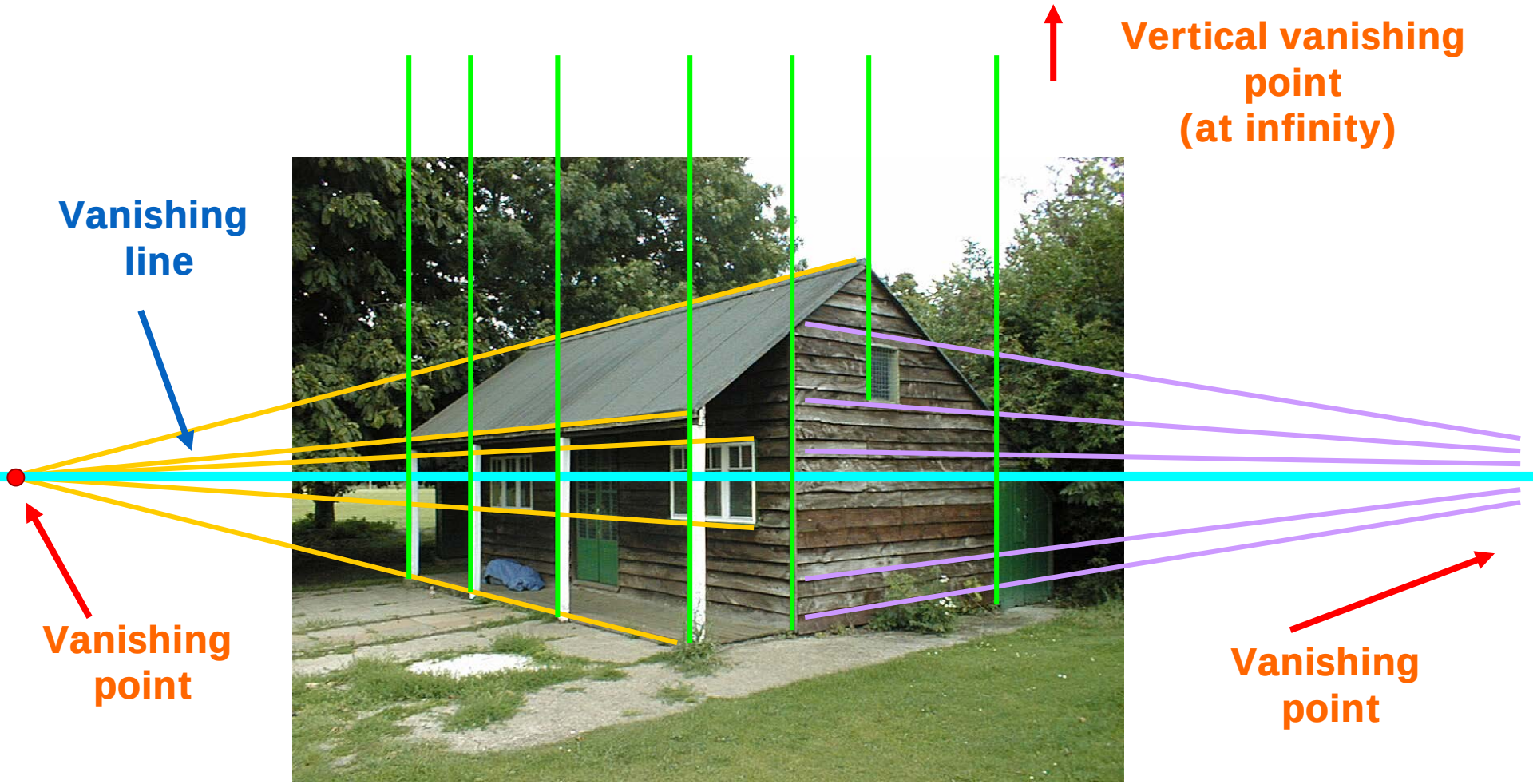


Vanishing points and lines



- The projections of parallel 3D lines intersect at a **vanishing point**
- The projection of parallel 3D planes intersect at a **vanishing line**
- If a set of parallel 3D lines are also parallel to a particular plane, their vanishing point will lie on the vanishing line of the plane
- Not all lines that intersect are parallel
- Vanishing point <-> 3D direction of a line
- Vanishing line <-> 3D orientation of a surface

Vanishing points and lines

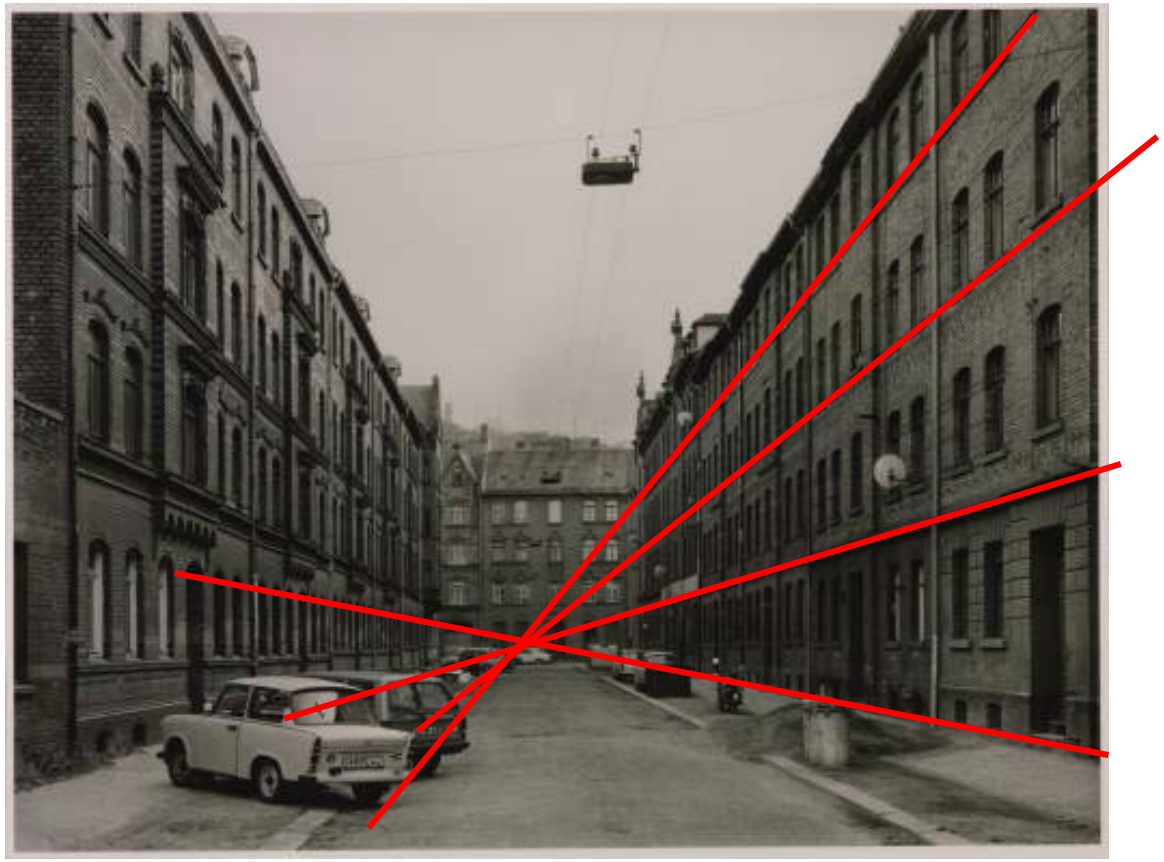


Slide from Efros, Photo from Criminisi

Vanishing points and lines

Photo from online Tate collection

Note on estimating vanishing points



Use multiple lines for better accuracy

… but lines will not intersect at exactly the same point in practice

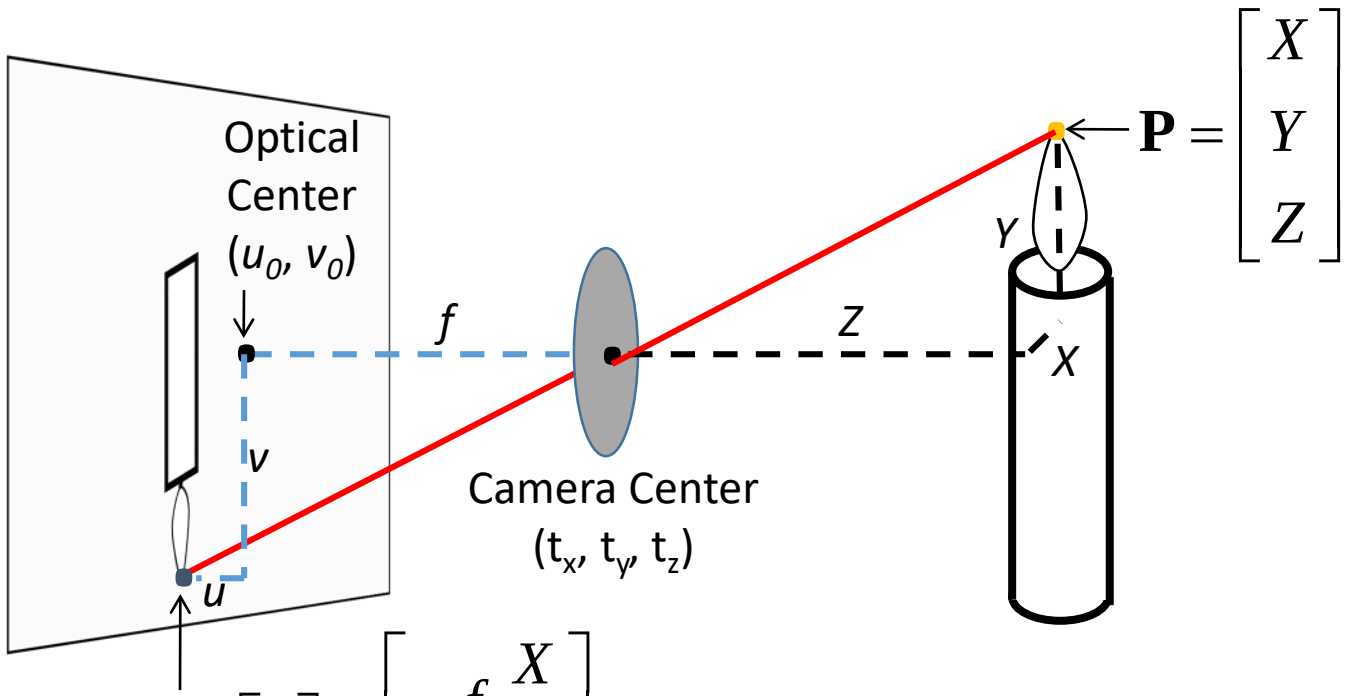
One solution: take mean of intersecting pairs

… bad idea!

Instead, minimize angular differences

Vanishing objects

Projection:
world coordinates→image coordinates



$$\mathbf{p} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -f\dfrac{X}{Z} \\ -f\dfrac{Y}{Z} \end{bmatrix}$$

Homogeneous coordinates

- Is this a linear transformation?
  - no—division by z is nonlinear

Converting to homogeneous coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Converting from homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Homogeneous coordinates

Invariant to scaling

$$k\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kw \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{kx}{kw} \\ \frac{ky}{kw} \end{bmatrix} = \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \end{bmatrix}$$

Homogeneous Coordinates

Cartesian Coordinates

Point in Cartesian is ray in Homogeneous

Basic geometry in homogeneous coordinates

- Append 1 to pixel coordinate to get homogeneous coordinate

$$p_i = \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix}$$

- Line equation:

$$line_i^\top p = 0$$

$$au + bv + c = 0 \qquad line_i = [a\ b\ c]^\top$$

- Line given by cross product of two points

$$line_{ij} = p_i \times p_j$$

- Intersection of two lines given by cross product of the lines

$$q_{ij} = line_i \times line_j$$

- Three points lies on the same line

$$p_k^\top (p_i \times p_j) = 0$$

- Three lines intersect at the same point

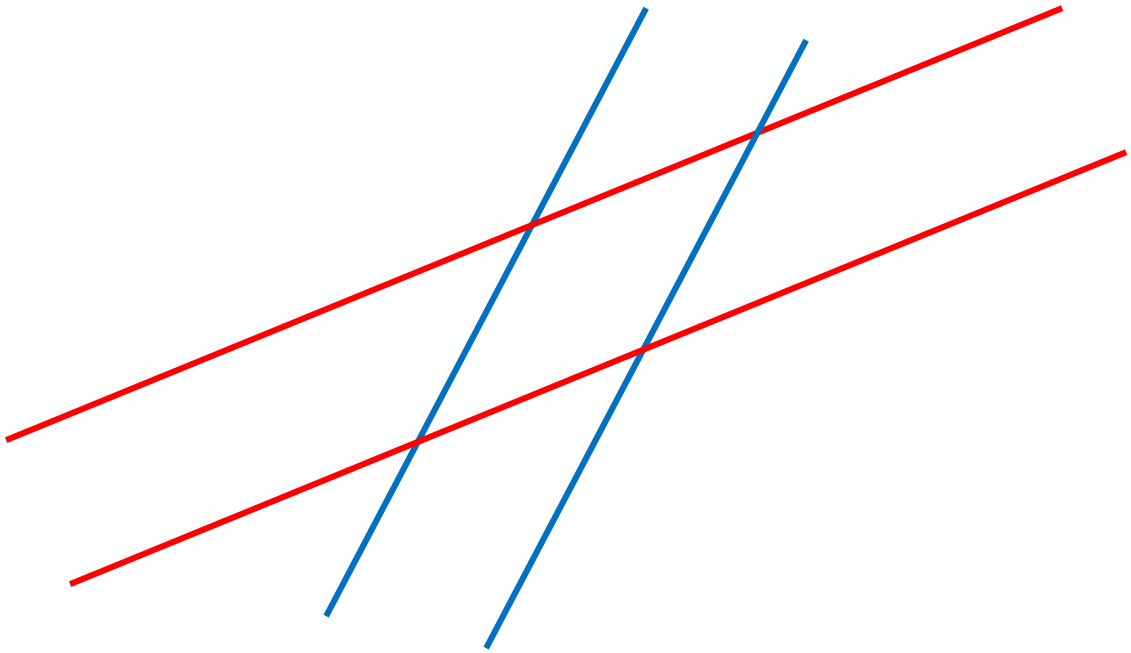
$$line_k^\top (line_i \times line_j) = 0$$

Another problem solved by homogeneous coordinates

Intersection of parallel lines

Cartesian: (Inf, Inf)
Homogeneous: (1, 1, 0)

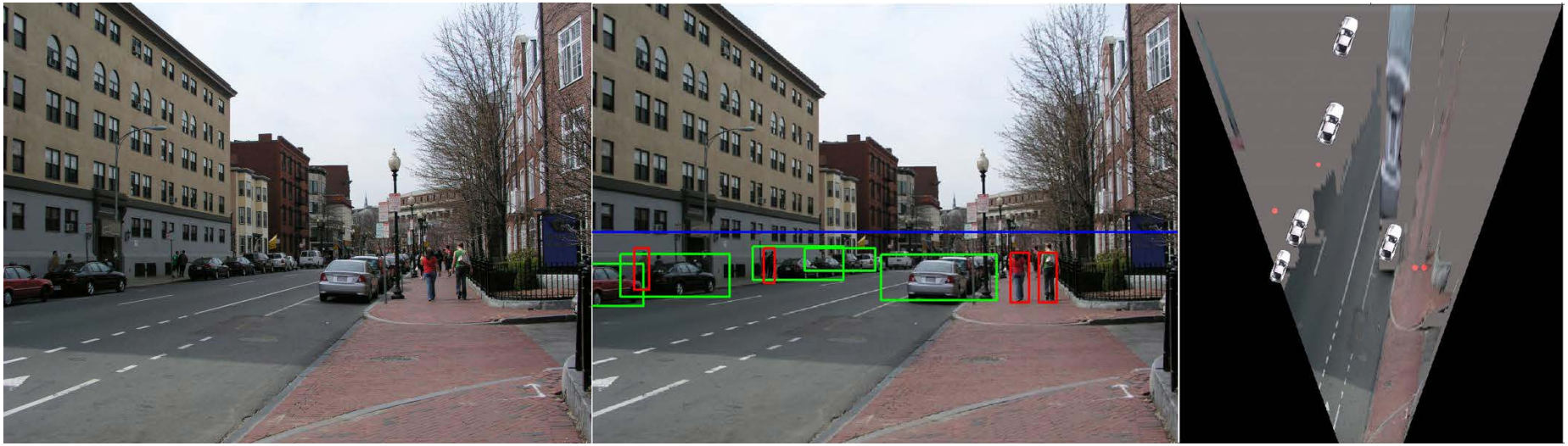
Cartesian: (Inf, Inf)
Homogeneous: (1, 2, 0)



Interlude: where can this be useful?

Applications

Object Recognition (CVPR 2006)

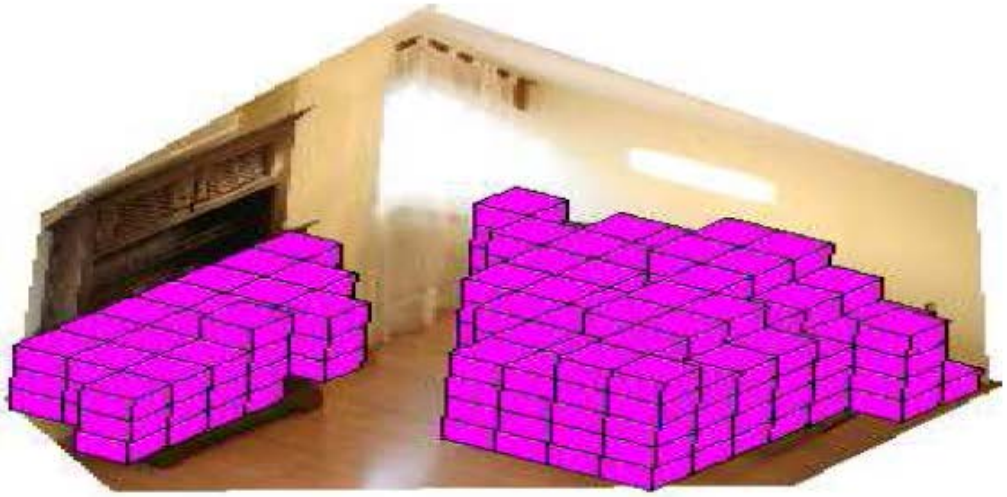


Applications

Single-view reconstruction (SIGGRAPH 2005)

Applications

Getting spatial layout in indoor scenes (ICCV 2009)



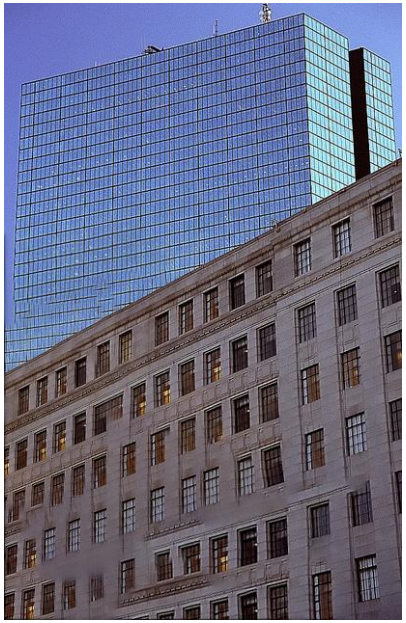
Applications




Image Completion [SIGGRAPH14]

- Revealing unseen pixels



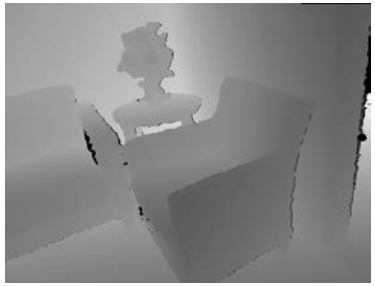


Applications

Inserting synthetic objects into images: http://vimeo.com/28962540

Applications

Creating detailed and complete 3D scene models from a single view



Two-mins break

Perspective Projection Matrix

- Projection is a matrix multiplication using homogeneous coordinates

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \Rightarrow \left(f\frac{x}{z}, f\frac{y}{z} \right)$$

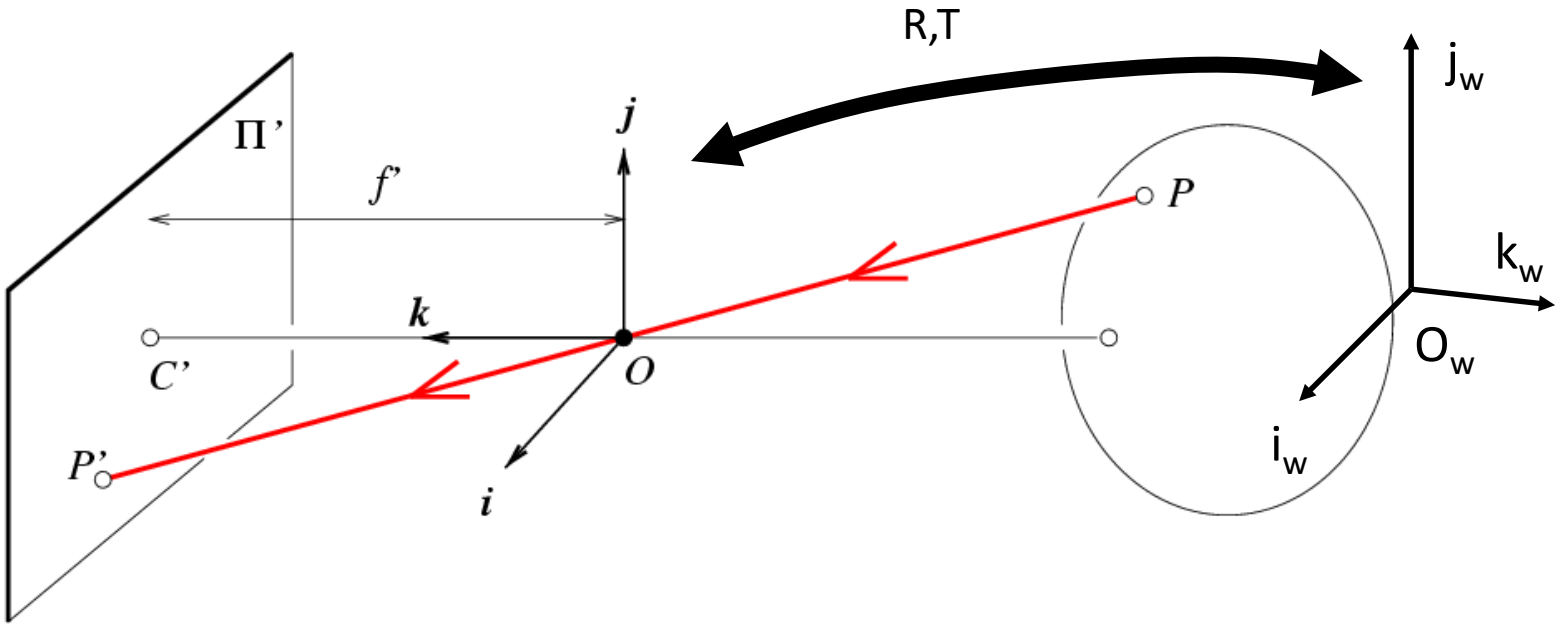
divide by the third coordinate

In practice: lots of coordinate transformations…

$$\begin{pmatrix} \text{2D point} \\ \text{(3x1)} \end{pmatrix} = \begin{pmatrix} \text{Camera to} \\ \text{pixel coord.} \\ \text{trans. matrix} \\ \text{(3x3)} \end{pmatrix} \begin{pmatrix} \text{Perspective} \\ \text{projection matrix} \\ \text{(3x4)} \end{pmatrix} \begin{pmatrix} \text{World to} \\ \text{camera coord.} \\ \text{trans. matrix} \\ \text{(4x4)} \end{pmatrix} \begin{pmatrix} \text{3D} \\ \text{point} \\ \text{(4x1)} \end{pmatrix}$$

Slide Credit: Saverese

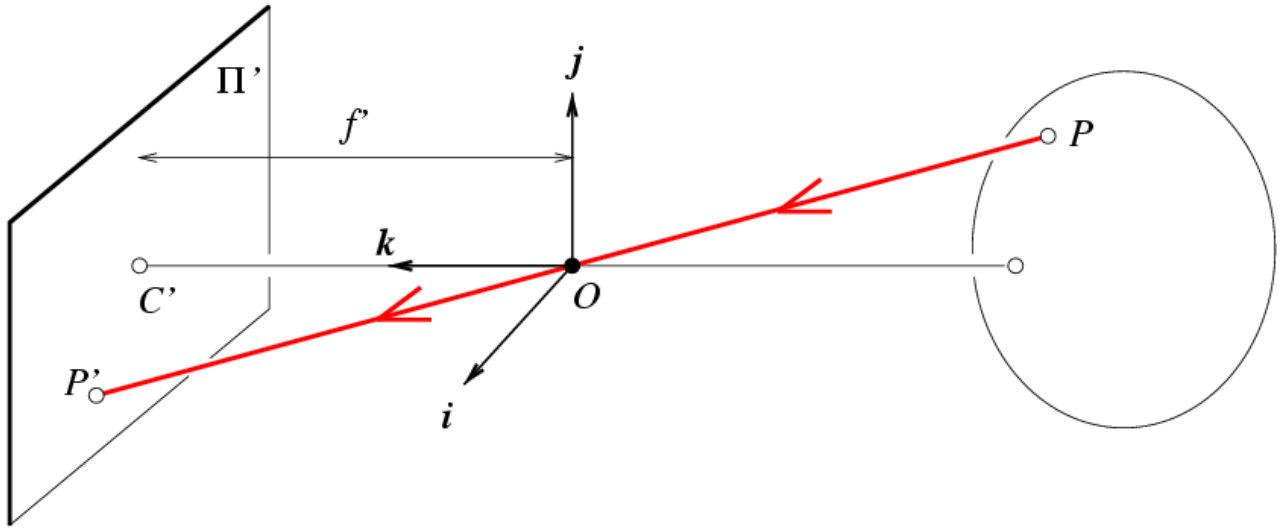
Projection matrix



$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

x: Image Coordinates: (u,v,1)
K: Intrinsic Matrix (3x3)
R: Rotation (3x3)
t: Translation (3x1)
X: World Coordinates: (X,Y,Z,1)

Projection matrix



Intrinsic Assumptions

- Unit aspect ratio
- Optical center at (0,0)
- No skew

Extrinsic Assumptions

- No rotation
- Camera at (0,0,0)

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{X} \Rightarrow w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{matrix}}^{\mathbf{K}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Slide Credit: Saverese

Remove assumption: known optical center

Intrinsic Assumptions

- Unit aspect ratio
- No skew

Extrinsic Assumptions

- No rotation
- Camera at (0,0,0)

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{X} \Rightarrow w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Remove assumption: square pixels

Intrinsic Assumptions
- No skew

Extrinsic Assumptions
- No rotation
- Camera at (0,0,0)

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{X} \Rightarrow w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Remove assumption: non-skewed pixels

Intrinsic Assumptions

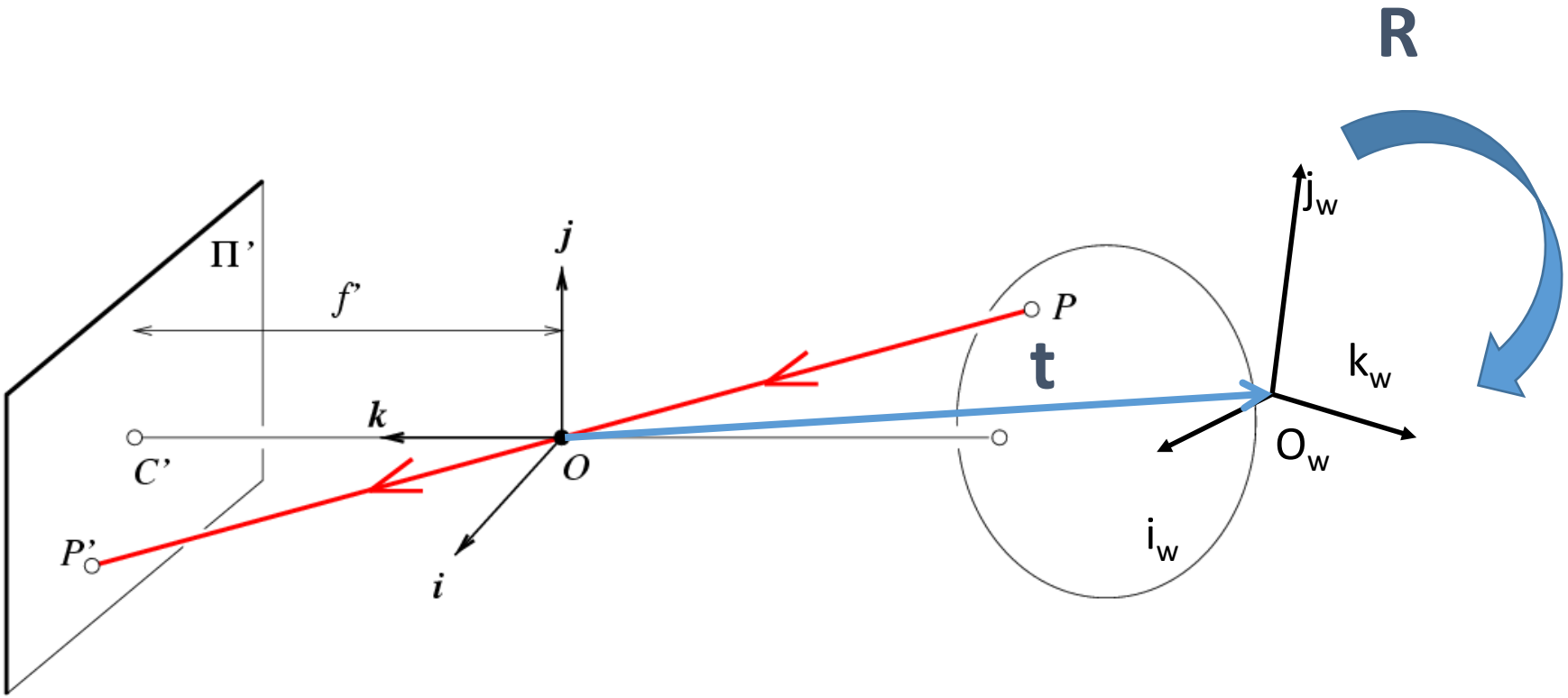
Extrinsic Assumptions

- No rotation
- Camera at (0,0,0)

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{0}\end{bmatrix}\mathbf{X} \Rightarrow w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Note: different books use different notation for parameters

Oriented and Translated Camera



Allow camera translation

Intrinsic Assumptions

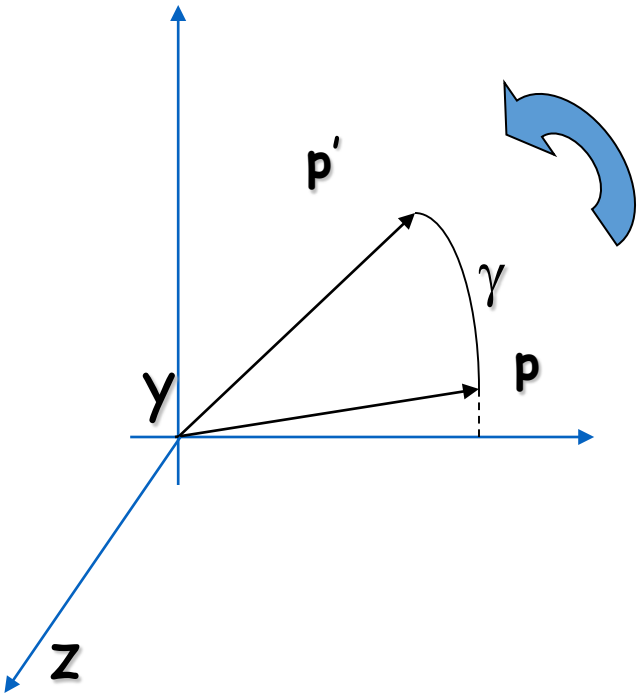
Extrinsic Assumptions

- No rotation

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{I} & \mathbf{t}\end{bmatrix}\mathbf{X} \Rightarrow w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Rotation of Points

Rotation around the coordinate axes, counter-clockwise:



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Allow camera rotation

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Degrees of freedom

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\Downarrow$$

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \overset{5}{\begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{6}{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

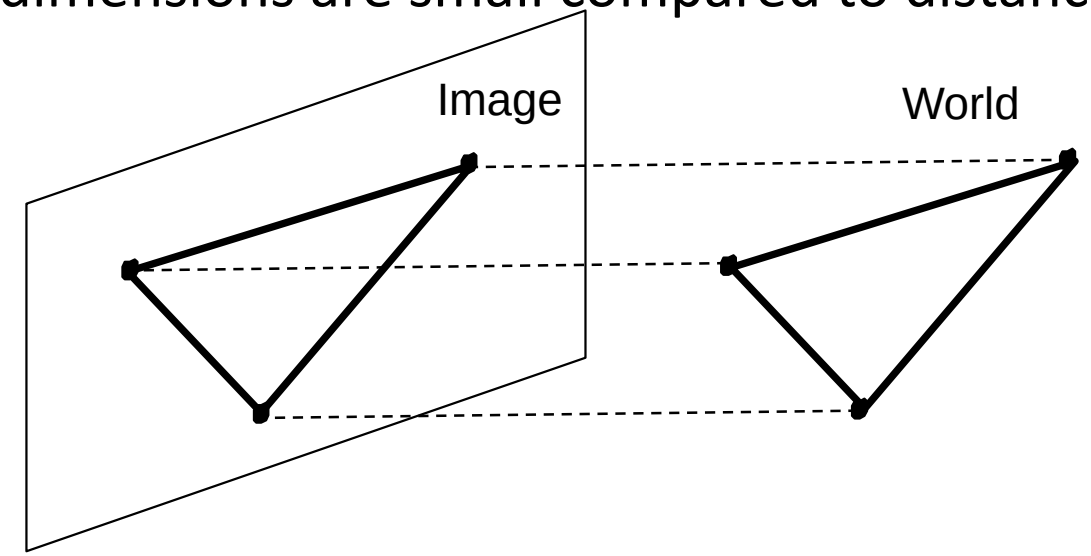
Vanishing Point = Projection from Infinity

$$\mathbf{p} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} \Rightarrow \mathbf{p} = \mathbf{KR}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \mathbf{p} = \mathbf{K}\begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix}$$

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} \Rightarrow \quad u = \frac{f x_R}{z_R} + u_0 \qquad v = \frac{f y_R}{z_R} + v_0$$

Scaled Orthographic Projection

- Special case of perspective projection
  - Object dimensions are small compared to distance to camera

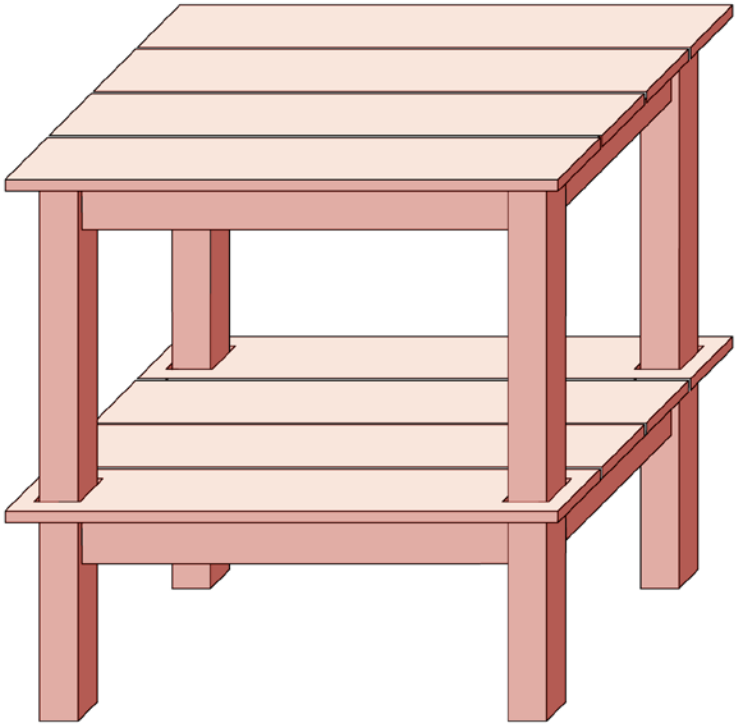


  - Also called “weak perspective”

$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Slide by Steve Seitz

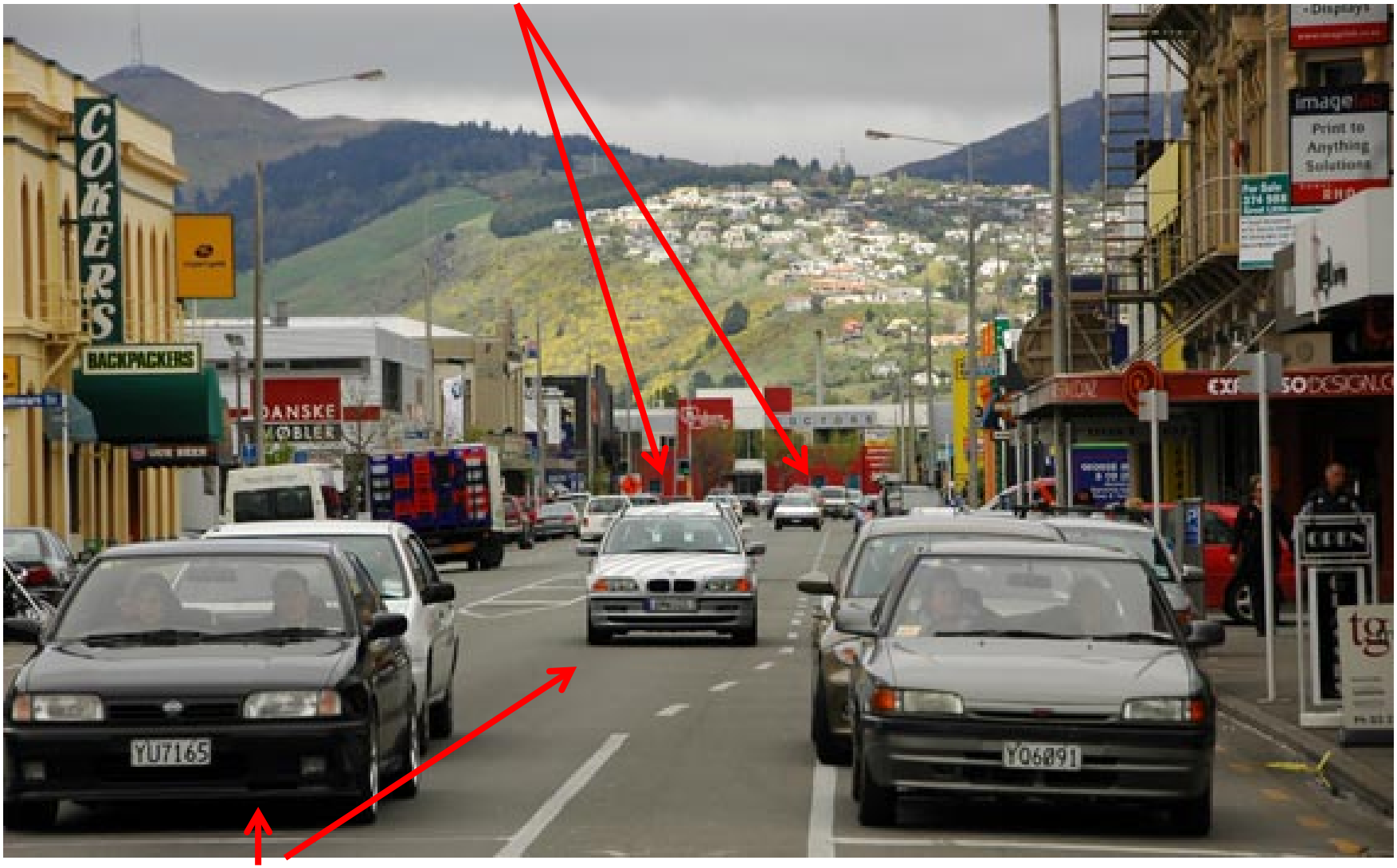
Orthographic Projection - Examples



Orthographic Projection - Examples

Applications in object detection

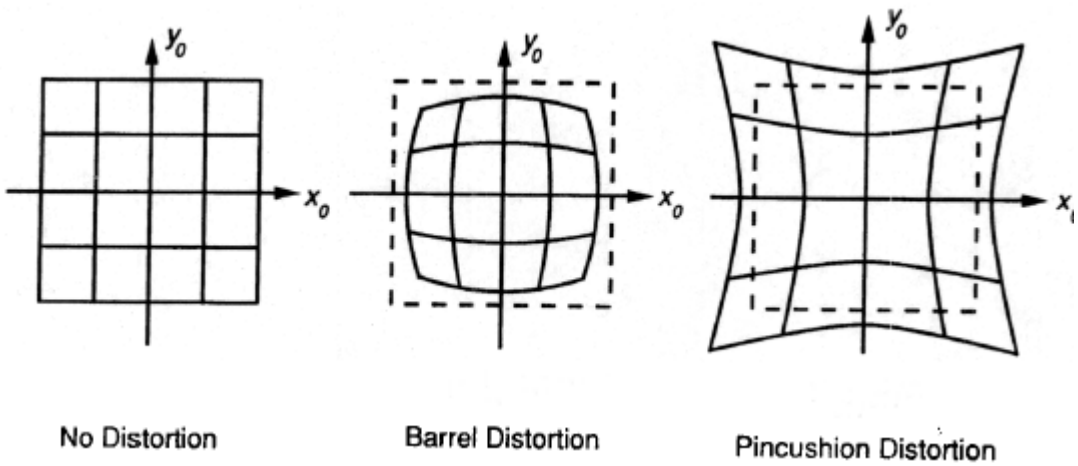
Far field: object appearance doesn't change as objects translate



Near field: object appearance changes as objects translate

Beyond Pinholes: Radial Distortion

- Common in wide-angle lenses or for special applications (e.g., security)
- Creates non-linear terms in projection
- Usually handled by through solving for non-linear terms and then correcting image

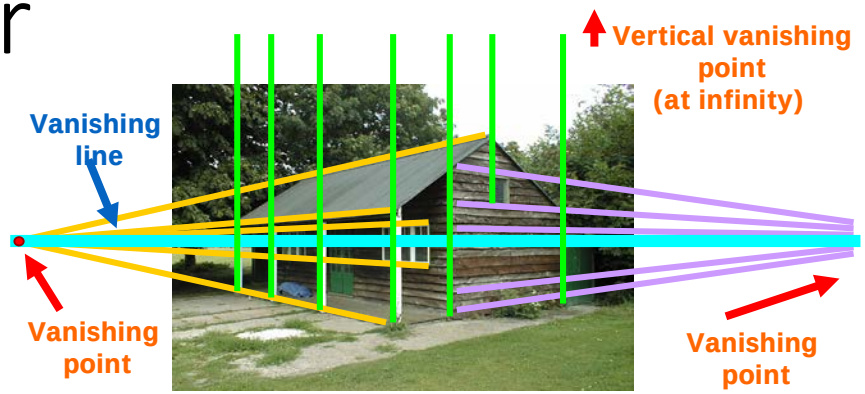


Corrected Barrel Distortion

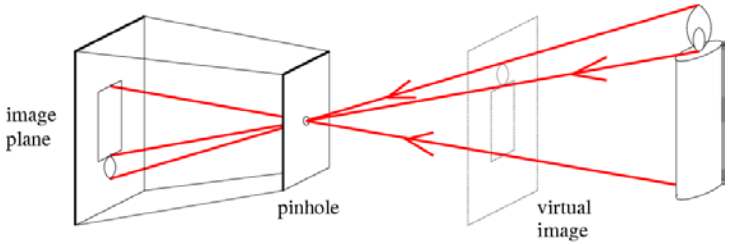
Image from Martin Habbecke

Things to remember

- Vanishing points and vanishing lines



- Pinhole camera model and camera projection matrix



$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

- Homogeneous coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Next class
Applications of camera model and projective geometry

- Recovering the camera intrinsic and extrinsic parameters from an image
- Measuring size in the world
- Projecting from one plane to another