Analysis of Clock Jitter in Switched-Capacitor Systems

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Abstract—The effect of periodic clock jitter in switched capacitor filters is analyzed extending a previously introduced state-space approach. This procedure facilitates the evaluation of the effects of internally generated noise sources upon the output spectral density. As such, one can then evaluate the sensitivity of a specific structure to its switched capacitor and operational amplifier noise sources. The application of our results to a switched capacitor integrator shows that the output noise level may be very sensitive to the jitter amplitude, and points out the need for high precision clock circuitry to minimize the effects of internally generated noise on the switched capacitor filter output.

I. INTRODUCTION

SWITCHED capacitor filters (SCF) have been of much interest in the last decade due to the extensive range of applications of these integrated circuits, especially in emulating digital filters. Although numerous SC analysis methods have been presented [1], [2], only a few among them deal with the problem of noise [3], [4], [6], [7], [9], [14]. This is essentially due to the fact that noise signals present in SC networks have bandwidths far exceeding the sampling frequencies used, which requires specific methods of analysis. This paper studies the potential sensitivity of noise levels in SC networks to periodic clock jitter as it may appear from interactions between the power supply and clock phase generator subsystems. The problem has been mentioned earlier by Knowles and Zumbado [8]; the motivation for this work is to study a more formal approach to this specific question. The “state-space” formulation derived earlier by Liou and Kuo [3] is extended in Section II to include the jitter effects. Section III applies the general derivation to a first-order SC integrator. Results show that the output noise level of this simple circuit may be very sensitive to jitter perturbations and demonstrate the need for isolating, as much as possible, the clock circuitry from any kind of periodic interference.

II. PERIODIC JITTER ANALYSIS

2.1. Introduction

This section presents the extension of the state-space formulation proposed by Liou and Kuo [3] to include the effects of clock jittering. Recall that the original approach does not have any restriction on the kind of input, which makes it extremely appealing for noise analysis of SC networks. Furthermore, recall that this state-space approach requires identification of the number of subintervals per switching period, to compute the spectral output expressions. The first-order SC integrator, as shown in Fig. 1, is used to study the effects of clock jitter. Note that two nonoverlapping clocks control the circuit, leading to four subintervals per switching period. Therefore, the following procedure is derived for this specific number of subintervals.

First-order approximations are used to derive the output spectral expressions. Furthermore, note that the introduction of jitter destroys the original cyclostationarity property of the output autocorrelation function. However, taking into account the jitter frequency, an averaged autocorrelation output expression can still be derived and this assumption will be used to identify the output spectral expressions.

2.2. SCF Description

Following the approach of Liou and Kuo [3], we use the fact that the capacitor voltage vectors $x_i(t)$ in a given subinterval $I_i$ of length $T_i$ depend on the input source vector $u(t)$ and on the final value of the capacitor voltage in the previous subinterval. We assume $T_1 = T_2 = T_4$ to simplify the derivations presented in this work. Using a state-space formulation, the behavior of the capacitor voltage in each subinterval is described by

\[ x_1(t) = F_1 x_4(nT) + G_1 u(t), \quad t \in I_1 \]
\[ x_2(t) = F_2 x_4(nT + T_1) + G_2 u(t), \quad t \in I_2 \]
\[ x_3(t) = F_3 x_4(nT + T_1 + T_2) + G_3 u(t), \quad t \in I_3 \]
\[ x_4(t) = F_4 x_4(nT + 2T_1 + T_2) + G_4 u(t), \quad t \in I_4 \]

where $I_i, \ i = 1, \cdots, 4$, represents the interval corresponding to the $i$th switching subinterval defined in the switch-

\[ 1 \] The derivation can be modified to a larger or smaller number of subintervals per switching period.

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Fig. 1. SC integrator controlled by two nonoverlapping clock signals: 4 subintervals per switching period.

The output vector $y(t)$ is then given by

$$y_k(t) = K_k x_k(t) + L_k u(t), \quad t \in I_k$$

where $F_k$, $G_k$, $K_k$, $L_k$ represent circuit matrices defined for each subinterval $I_k$, $k = 1, \ldots, 4$.

When jitter is introduced in the circuit, the switching times are not fixed to $nT + T_k$ anymore. The jitter disturbance represented by a sinusoidal function with frequency $\omega_0$, $A \sin(\omega_0 t)$, is added to the switching times. Therefore, the circuit equations (1)–(5) are modified as follows:

$$x_1(t) = F_1 \left\{ \sum_n x_n(nT + A \sin(\omega_0 nT)) \right\}$$

$$-W_f(t - nT - A \sin(\omega_0 nT)) + G_1 u(t) \cdot W_f(t)$$

$$x_2(t) = F_2 \left\{ \sum_n x_n(nT + T_1 + A \sin(\omega_0 nT + T_1)) \right\}$$

$$-W_f(t - nT - T_1 - A \sin(\omega_0 nT + T_1)) + G_2 u(t) \cdot W_f(t)$$

$$x_3(t) = F_3 \left\{ \sum_n x_n(nT + T_1 + T_2 + A \sin(\omega_0 nT + T_1 + T_2)) \right\}$$

$$-W_f(t - nT - T_1 - T_2 - A \sin(\omega_0 nT + T_1 + T_2)) + G_3 u(t) \cdot W_f(t)$$

$$x_4(t) = F_4 \left\{ \sum_n x_n(nT + 2T_1 + T_2 + A \sin(\omega_0 (nT + 2T_1 + T_2))) \right\}$$

$$-W_f(t - nT - 2T_1 - T_2 - A \sin(\omega_0 (nT + 2T_1 + T_2))) + G_4 u(t) \cdot W_f(t)$$

where the window functions $w_{T_i}$ and $W_i$, for $i = 1, \ldots, 4$, are defined by

$$W_i(t) = \sum_n w_{T_i}(t - nT - \sum_{j=1}^{i-1} T_j - A \sin(\omega_0 (nT + \sum_{j=1}^{i-1} T_j)))$$

and

$$w_{T_i}(t - t_0) = \begin{cases} 1, & \text{for } t_0 \leq t < t_0 + T_i \\ 0, & \text{elsewhere}. \end{cases}$$

Fig. 2. Definition of switching subintervals: $T$ is the SC circuit switching interval.

Evaluation of the Fourier transformed versions of the above proceeds in a manner similar to specifically evaluating $X_2(\omega)$ from (6b), which goes as follows:

$$X_2(\omega) = F_2 X_2(\omega) + G_2 X_{22}(\omega).$$

For $X_{22}(\omega)$ the Fourier transform of the corresponding term in (6b) yields

$$X_{22}(\omega) = F_4 \left\{ \sum_n x_n(nT + T_1 + A \sin(\omega_0 nT + T_1)) \right\}$$

$$-W_f(t - nT - T_1 - A \sin(\omega_0 nT + T_1)) \right\}$$

$$= \left( \frac{1 - e^{-j\omega T_1}}{j\omega} \right) \sum_n x_n(nT + T_1) + A \sin(\omega_0 nT + T_1)$$

$$-\exp(-j\omega nT + T_1 + A \sin(\omega_0 nT + T_1))).$$

With $*$ denoting convolution, the following can be found for $X_{22}(\omega)$

$$X_{22}(\omega) = F_4[u(t) \cdot W_f(t)]$$

$$= \left( \frac{1}{2\pi} \right) \cdot [U(\omega) \ast F_4[W_f(t)]]$$

$$= \left( \frac{1}{2\pi} \right) \cdot U(\omega) \ast \left[ \sum_\omega \exp(-j\omega(nT + T_1)) \right] \exp(-j\omega A \sin(\omega_0(nT + T_1))).$$
If the last term in (9) is considered as a function of \( t \) for \( nT \), then it is recognized to be a periodic function for which a Fourier series expansion can be written:

\[
\exp(-j\omega_A \sin \omega_0(t + T_1)) = \sum_m (-1)^m \cdot J_m(A\omega) \cdot e^{jm\omega_0(t + T_1)}.
\]  \( \text{(10)} \)

\( J_m(A\omega) \) represents the Bessel function of the first kind and order \( m \), as appearing in single-tone frequency modulation (FM). Using the Poisson sum formula

\[
\sum_n f_n e^{-jwnT} = \left( \frac{1}{T} \right) \cdot \sum_n F_\{f\}(\omega - 2\pi n/T) \quad \text{(11)}
\]

the term under the summation in (9) can now be written in alternative form:

\[
e^{-j\omega_1 T_1} \cdot \sum_m e^{-jn\omega_1 T_1} \cdot \sum_n (-1)^m \cdot J_m(\omega_0) \cdot e^{jn\omega_0 T} \cdot e^{jm\omega_0 T} = \sum_m (-1)^m \cdot e^{-j\omega_1 T_1} \cdot J_m(\omega_0) \cdot e^{jm\omega_0 T} \cdot e^{-jn\omega_1 T_1}
\]

\[+ \sum_n e^{-j(n - m)\omega_0 n T} \cdot (2\pi/T) \delta(\omega - m\omega_0 - 2\pi n/T). \quad \text{(12)}
\]

With \( 2\pi/T \) defined as \( \omega_1 \), the expression for \( X_{22}(\omega) \) in (9), using (12), now becomes

\[
X_{22}(\omega) = \sum_n \sum m (-1)^m \cdot e^{-jn\omega_1 T_1} \cdot J_m((m\omega_0 + n\omega_0)A) 
\]

\[\cdot U(\omega - m\omega_0 - n\omega_0) \cdot \left[ 1 - e^{-j(m\omega_0 + n\omega_0)T_1} \right] / \left[j(m\omega_0 + n\omega_0)T_1\right]. \quad \text{(13)}
\]

Using (8) and (13), the following equation is found:

\[
X_2(\omega) = F_2 \bar{X}_2(\omega) \cdot \frac{1 - e^{-j\omega_1 T_1}}{j\omega}
\]

\[+ G_2 \sum_n \sum m \theta_{2,n,m}U(\omega - n\omega_0 - m\omega_0). \quad \text{(13a)}
\]

with

\[
\bar{X}_2(\omega) = \sum_n x_2(nT + T_1 + A \sin \omega_0(nT + T_1))
\]

\[\cdot e^{-j\omega_0(nT + T_1 + A \sin \omega_0(nT + T_1))} \quad \text{(13b)}
\]

\[
\theta_{2,n,m} = (-1)^m \cdot e^{-jn\omega_1 T_1} \cdot J_m((m\omega_0 + n\omega_0)A) \left[ 1 - \exp(-j(m\omega_0 + n\omega_0)T_2) \right] / \left[j(m\omega_0 + n\omega_0)T_2\right]. \quad \text{(13c)}
\]

Note that if the jitter amplitude \( A \) vanishes, the above reaches the limit:

\[
X_2(\omega) = F_2 \frac{1 - e^{-j\omega_1 T_1}}{j\omega} \sum_n e^{-j\omega_0(nT + T_1)} \cdot x_2(nT + T_1)
\]

\[+ G_2 \sum_n \frac{1 - e^{-jn\omega_1 T_1}}{jn\omega_1 T_1} \cdot U(\omega - n\omega_0). \quad \text{(14)}
\]

which is exactly what is derived for the jitterless case.

Analogous to the above, the expressions of the other equations in (6) can be derived by direct replacement of some of the parameters. Replacing \( T_2 \) by \( T_1 \), and \( T_1 \) by \( 0 \) yields the expressions for \( X_4(\omega) \):

\[
X_4(\omega) = F_4 \bar{X}_4(\omega) \cdot \frac{1 - e^{-j\omega_1 T_1}}{j\omega}
\]

\[+ G_4 \sum_n \sum m \theta_{4,n,m}U(\omega - n\omega_0 - m\omega_0) \quad \text{(15a)}
\]

with

\[
\bar{X}_4(\omega) = \sum_n x_4(nT + A \sin \omega_0 nT) \cdot e^{-j\omega_0(nT + A \sin \omega_0 nT)} \quad \text{(15b)}
\]

\[
\theta_{4,n,m} = (-1)^m \cdot J_m((m\omega_0 + n\omega_0)A) \left[ 1 - e^{-j(m\omega_0 + n\omega_0)T_1 T_1} \right] / \left[j(m\omega_0 + n\omega_0)T_1\right]. \quad \text{(15c)}
\]

The replacement of \( T_2 \) by \( T_1(= T_1) \), and \( T_1 \) by \( T_1 + T_2 \), leads to the expression for \( X_4(\omega) \):

\[
X_4(\omega) = F_4 \bar{X}_4(\omega) \cdot \frac{1 - e^{-j\omega_1 T_1}}{j\omega}
\]

\[+ G_4 \sum_n \sum m \theta_{4,n,m}U(\omega - n\omega_0 - m\omega_0). \quad \text{(16a)}
\]

with

\[
\bar{X}_4(\omega) = \sum_n x_4(nT + T_1 + T_2)
\]

\[+ A \sin \omega_0(nT + T_1 + T_2)) \cdot e^{-j\omega_0(nT + T_1 + T_2 + A \sin \omega_0(nT + T_1 + T_2))} \quad \text{(16b)}
\]

\[
\theta_{4,n,m} = (-1)^m \cdot J_m((m\omega_0 + n\omega_0)A) \cdot e^{-jn\omega_1 T_1 T_1 T_1} \left[ 1 - e^{-j(m\omega_0 + n\omega_0)T_1} \right] / \left[j(m\omega_0 + n\omega_0)T_1\right]. \quad \text{(16c)}
\]

Replacing \( T_2 \) by \( T_1 \), and \( T_1 \) by \( 2T_1 + T_2 \), leads to the expression for \( X_4(\omega) \):

\[
X_4(\omega) = F_4 \bar{X}_4(\omega) \cdot \frac{1 - e^{-j\omega_1 T_1}}{j\omega}
\]

\[+ G_4 \sum_n \sum m \theta_{4,n,m}U(\omega - n\omega_0 - m\omega_0). \quad \text{(17a)}
\]
with
\[ \tilde{X}(\omega) = \sum_n x_n (nT + 2T_1 + T_2) + A \sin \omega_0 (nT + 2T_1 + T_2) \]
\[ \cdot \exp (-j\omega(nT + 2T_1 + T_2)) \]
\[ + A \sin \omega_0 (nT + 2T_1 + T_2) \]
\[ \cdot \exp (-j\omega(nT + 2T_1 + T_2)) \] (17b)

\[ \theta_{n,m} = (-1)^m \cdot J_m((m \omega_0 + n \omega_0)A) \cdot e^{-j(n \omega_0 + m \omega_0)T} \]
\[ \cdot \left[ 1 - e^{(m \omega_0 + n \omega_0)T} \right] \] (17c)

Next, one may solve directly for \( \tilde{X}(\omega) \), by rewriting (6) at the endpoint of the subinterval of validity and then solving for \( x_i(t) \).

\[ x_i((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1)) \]
\[ = F_1 F_2 F_3 F_4 x_i(nT + T_1 + A \sin \omega_0(nT + T_1)) \]
\[ + F_1 F_2 F_3 G_2 u(nT + T_1 + T_2) \]
\[ + A \sin \omega_0(nT + T_1 + T_2) \]
\[ + F_1 F_2 G_2 u(nT + 2T_2 + T_2) \]
\[ + A \sin \omega_0(nT + 2T_2 + T_2) \]
\[ + F_1 G_2 u(nT + T + A \sin \omega_0(nT + T)) \]
\[ + G_2 u(nT + T + T_1 + A \sin \omega_0(nT + T + T_1)) \] (18a)

For convenience, we define the following notation:
\[ f(nT + T_k + A \sin \omega_0(nT + T_k)) = f(nT + T_k + \xi) \]

Note that \( \xi \) actually depends on \( T_k \). The equations completing (18a) now can be written more compactly.

\[ x_3((n + 1)T + T_1 + T_2) \]
\[ + A \sin \omega_0((n + 1)T + T_1 + T_2) \]
\[ = F_1 F_2 F_4 x_3(nT + T_1 + T_2 + \xi) \]
\[ + F_2 F_3 F_4 G_2 u(nT + 2T_1 + T_2 + \xi) \]
\[ + F_2 F_4 G_2 u((n + 1)T + \xi) \]
\[ + F_2 G_2 u((n + 1)T + T_1 + T_2 + \xi) \]
\[ + G_3 u((n + 1)T + T_1 + T_2 + T_2) \] (18b)

\[ x_3((n + 1)T + 2T_1 + T_2 + \xi) \]
\[ = F_1 F_2 F_4 x_3(nT + 2T_1 + T_2 + \xi) \]
\[ + F_2 F_3 F_4 G_2 u(nT + 2T_1 + T_2 + T_2 + \xi) \]
\[ + F_2 F_4 G_2 u((n + 1)T + T_1 + \xi) \]
\[ + F_2 G_2 u((n + 1)T + T_1 + T_2 + \xi) \]
\[ + G_3 u((n + 1)T + 2T_1 + T_2 + \xi) \] (18c)

\[ x_4((n + 1)T + T_2 + \xi) \]
\[ = F_1 F_2 F_3 x_4(nT + \xi) \]
\[ + F_2 F_3 F_4 G_2 u(nT + T_1 + \xi) \]
\[ + F_2 F_4 G_2 u(nT + T_1 + T_2 + \xi) \]
\[ + F_2 G_3 u(nT + 2T_1 + T_2 + \xi) \]
\[ + G_3 u((n + 1)T + T_2 + \xi) \] (18d)

The equations in (18) relate the capacitor voltage vectors to the input voltage vectors. The aim is now to find a relation between \( \tilde{X}(\omega) \) and \( U(\omega) \). In the jitterless case, one would now derive Z-transform domain expressions for (18) [3], [4]. Due to the arguments involving jitter, this approach is no longer possible. Therefore, we define a modified Z-transform \( X(z, T_k) \):

\[ X(z, T_k) = \sum_n x(nT + T_k + A \sin \omega_0(nT + T_k)) \]
\[ \cdot \exp (-j\omega(nT + T_k + A \sin \omega_0(nT + T_k))) \] (19a)

with \( z \) defined as \( z = e^{j\omega T} \).

Now, note that the transform \( X(z, T_k) \) is periodic, with period \( T \), in its second argument. We now apply the above modified Z-transform to (18a) resulting in

\[ \sum_n x_i((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1)) \]
\[ \cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1))) \]
\[ - A_1 \sum_n x_i(nT + T_1 + A \sin \omega_0(nT + T_1)) \]
\[ \cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1))) \]
\[ + A_2 \sum_n u(nT + T_1 + T_2 + A \sin \omega_0(nT + T_1 + T_2)) \]
\[ \cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1))) \]
\[ + A_3 \sum_n u(nT + 2T_1 + T_2 + A \sin \omega_0(nT + 2T_1 + T_2)) \]
\[ \cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1))) \]
\[ + A_4 \sum_n u((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1)) \]
\[ \cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0((n + 1)T + T_1))) \] (19b)

where the matrices \( A_i \), proportional to the circuit matrices \( F_i \) and \( G_i \), are defined by

\[ A_1 = F_1 F_2 F_3 F_2 \]
\[ A_2 = F_1 F_2 F_3 F_2 \]
\[ A_3 = F_1 F_2 G_3 \]
\[ A_4 = F_1 G_4 \]
\[ A_2 = G_1 \]

By inspection of the above equation, we note a phase term difference in the left- and right-hand sides of the
expression. This phase term difference is produced by the different arguments present in \( x_i(\cdot) \), \( u(\cdot) \) and the exponential terms. We now use the small signal property of the jitter perturbation to simplify the above equation and carry on the computations.

Next, we expand the sinusoidal components contained in the exponential terms of (19b) in order to rewrite it in terms of the modified Z-transform terms \( X(z, T_k) \), as defined in (19a). Thus (19b) becomes

\[
\sum_n x_i ((n + 1)T + T_1 + A \sin \omega_0 ((n + 1)T + T_1)) I
\]
\[
\cdot \exp (-j\omega((n + 1)T + T_1 + A \sin \omega_0 ((n + 1)T + T_1)))
\]
\[
= A_1 \sum_n x_i (nT + T_1 + A \sin \omega_0 (nT + T_1))
\]
\[
\cdot \exp (-j\omega(nT + T + T_1 + A \sin \omega_0 ((n + 1)T + T_1)))
\]
\[
+ A_2 \sum_n u(nT + T_1 + T_2 + A \sin \omega_0 (nT + T_1 + T_2))
\]
\[
\cdot \exp (-j\omega((n + 1)T + T_1 + T_2 - T_2)
\]
\[
+ A \sin \omega_0 ((n + 1)T + T_1 + T_2 - T_2))
\]
\[
+ A_1 \sum_n u(nT + 2T_1 + T_2 + A \sin \omega_0 (nT + 2T_1 + T_2))
\]
\[
\cdot \exp (-j\omega(nT + 2T_1 + T_2 - T_1 - T_2)
\]
\[
+ A \sin \omega_0 ((n + 1)T + 2T_1 + T_2 - T_1 - T_2))
\]
\[
+ A_1 \sum_n u((n + 1)T + A \sin \omega_0 ((n + 1)T))
\]
\[
\cdot \exp (-j\omega((n + 1)T + T_1
\]
\[
+ A \sin \omega_0 ((n + 1)T + T + T_1)))
\]
\[
+ A_2 \sum_n u((n + 1)T + T_1 + A \sin \omega_0 ((n + 1)T + T_1))
\]
\[
\cdot \exp (-j\omega((n + 1)T + T_1 + T_2
\]
\[
+ A \sin \omega_0 ((n + 1)T + T + T_1)))
\]
\[
+ A_1 \sum_n u((n + 1)T + A \sin \omega_0 ((n + 1)T + T_1))
\]
\[
\cdot \exp (-j\omega((n + 1)T + T + T_1))
\]
\[
+ A \sin \omega_0 ((n + 1)T + T + T_1))
\].

(19c)

Using the fact that the jitter amplitude \( A \), the subinterval \( T_i \), and the jitter frequency \( \omega_0 \) are small, we can approximate the term \( \cos \omega_0 T_i \) for \( i = 1, \cdots, 4 \) by 1 and \( A \sin (\omega_0 T_i) \) by 0 in the above. Therefore, using the definition of the modified Z-transform given in (19a), and the periodicity of the second argument of the transform, (19c) becomes

\[
X_1(z, T + T_1) = A_1 e^{-j\omega T} \cdot X_1(z, T_1) + A_2 e^{-j\omega(T - T_2)} \cdot U(z, T_1 + T_2 + A_2 U(z, 2T_1 + T_2)
\]
\[
\cdot e^{-j\omega(T - T_1)} + A_4 U(z, 2T_1 + T_2)
\]
\[
\cdot e^{-j\omega T_1} + A_5 U(z, T_1).
\]

(20a)

Similar application of the transform to (18b) to (18d) yields

\[
X_2(z, T_1 + T_2) = B_1 e^{-j\omega T} \cdot X_2(z, T_1 + T_2)
\]
\[
+ B_2 e^{-j\omega(T - T_1)} \cdot U(z, 2T_1 + T_2)
\]
\[
+ B_3 U(z, T_1 + T_2) + B_4 U(z, T_1)
\]
\[
\cdot e^{j\omega T_1} + B_5 U(z, 0) \cdot e^{j\omega(T + T_2)}
\]
\[
X_2(z, 2T_1 + T_2)
\]
\[
= C_1 e^{-j\omega T} X_2(z, 2T_1 + T_2)
\]
\[
+ C_2 e^{j\omega(T - T_1)} \cdot U(z, 0)
\]
\[
+ C_3 U(z, T_1) \cdot e^{j\omega(T - T_2)}
\]
\[
+ C_4 U(z, T_1 + T_2) \cdot e^{j\omega T_1}
\]
\[
+ C_5 U(z, 2T_1 + T_2)
\].

(20b)

\[
X_4(z, 0)
\]
\[
= D_1 e^{-j\omega T} X_4(z, 0) + D_2 e^{-j\omega(T - T_1)} \cdot U(z, T_1)
\]
\[
+ D_3 U(z, T_1 + T_2) \cdot e^{j\omega(T - T_2)}
\]
\[
\cdot e^{-j\omega T} + D_4 U(z, 2T_1 + T_2)
\]
\[
\cdot e^{j\omega(2T_1 + T_2)} - e^{-j\omega T} + D_5 U(z, 0)
\]

(20d)

where the matrices \( A_1, B_2, C_4, D_1 \) are defined as follows

\[
A_1 = F_1 F_4 F_5 F_2 \quad B_1 = F_2 F_4 F_5 F_3
\]
\[
A_2 = F_1 F_4 F_5 G_2 \quad B_2 = F_2 F_4 F_5 G_3
\]
\[
A_3 = F_1 F_4 G_5 \quad B_3 = F_2 F_4 G_4
\]
\[
A_4 = F_4 F_4 G_4 \quad B_4 = F_4 F_4 G_1
\]
\[
A_5 = G_1 \quad B_5 = G_2
\]
\[
C_1 = F_3 F_5 F_4 F_5 \quad D_1 = F_3 F_5 F_2 F_1
\]
\[
C_2 = F_3 F_5 G_4 \quad D_2 = F_3 F_5 F_5 G_1
\]
\[
C_3 = F_3 F_5 G_4 \quad D_3 = F_3 F_5 G_2
\]
\[
C_4 = F_4 G_2 \quad D_4 = F_4 G_3
\]
\[
C_5 = G_3 \quad D_5 = G_4.
\]

Let us review the process to date. First, Fourier transforming the state vectors \( x_i(\cdot) \) (see (13a), (15a), (16a), (17a)) gave us expressions for \( X_i(\omega) \) as a function of \( \bar{X}_i(\omega) \), which themselves depend on the time domain expressions for \( x_i(\cdot) \). Our intention is now to directly relate \( X_i(\omega) \) to the Fourier transform of the input vectors in order to get expressions of the state vectors in terms of the input spectrum. With this intent, we defined the modified Z-transform, defined in (19a), which will enable us to relate \( x_i(\cdot) \) to \( \bar{X}_i(\omega) \).

Comparing successively the expressions for \( \bar{X}_i(\omega) \) given in (13b), (15b), (16b), (17b) to the corresponding modified Z-transforms of the state vectors, we get the following relations (recall (19a) that \( z = e^{j\omega T} \)):

\[
\bar{X}_1(\omega) = X_1(z, T_1)
\]
\[
\bar{X}_2(\omega) = X_2(z, T_1 + T_2)
\]
\[
\bar{X}_3(\omega) = X_3(z, 2T_1 + T_2)
\]
\[
\bar{X}_4(\omega) = X_4(z, 0).
\]

The above relations show that there exists a direct correspondence between \( \bar{X}_i(\omega) \) and \( X_i(z, T_i) \). The next
step is to express \( u(t) \) in terms of its Fourier Transform in order to relate the input and output spectra of the system. Let us first compute \( U(z, T_i) \) which is defined by

\[
U(z, T_i) = \sum_n u(nT + T_i + A \sin \omega_0(nT + T_i))
\]

\[
\cdot \exp(-j\omega(nT + T_i + A \sin \omega_0(nT + T_i)))
\]

\[
= \sum_n f(t + v(t)) e^{-j\omega T_i} \tag{21e}
\]

where

\[
f(t) \triangleq u(t) \cdot e^{-j\omega t}
\]

\[
v(t) \triangleq A \sin \omega_0 t.
\]

Assuming the jitter to be a small perturbation, a first-order approximation is made. Therefore,

\[
f(t + v(t)) = f(t) + v(t) \cdot \frac{df}{dt}
\]

\[
= u(t)e^{-j\omega t} + v(t) \cdot \frac{d}{dt}(u(t) \cdot e^{-j\omega t})
\]

\[
= u(t)e^{-j\omega t} + v(t) \cdot (u'(t) - j\omega u(t)) \cdot e^{-j\omega t} \tag{22a}
\]

Assuming that the function \( u(t) \) has continuous first-order derivatives, \( U(z, T_i) \) becomes

\[
U(z, T_i) = \sum_n u(t)e^{-j\omega T_i} e^{j\omega(nT + T_i) - j\omega T_i}
\]

\[
+ \sum_n v(t) \cdot u'(t) e^{-j\omega T_i} |_{-nT + T_i} - j\omega \sum_n v(t) \cdot u(t) e^{-j\omega T_i} |_{-nT + T_i} \tag{22b}
\]

Each of the summations present in (22b) is now computed separately. For the first term, by direct application of the Poisson sum formula, we obtain

\[
\sum_n u(nT + T_i) e^{-j\omega_0(nT + T_i)} = \frac{1}{T} \sum_n U(\omega - n\omega_0) \cdot e^{-j\omega_0 T_i} \tag{23a}
\]

where \( \omega_0 = 2\pi/T \). The second term is defined by

\[
\sum_n v(t) u'(t) e^{-j\omega T_i} |_{-nT + T_i} = \sum_n [v(t) \cdot u'(t)] |_{-nT + T_i} \cdot e^{-j\omega T_i} \tag{23b}
\]

Application of the Poisson sum formula (PSF) to the above relation yields

\[
\sum_n \sum_{-T_i + T_i} v(t) u'(t) e^{-j\omega T_i} = \frac{1}{T} \sum_n \sum_{-T_i + T_i} F(v \cdot u')(\omega - n\omega_0) \cdot e^{-j\omega_0 T_i}
\]

\[
= \frac{1}{2\pi T} \sum_n \sum_{-T_i + T_i} \{V(\omega - n\omega_0)
\]

\[
\cdot \{j(\omega - n\omega_0)U(\omega - n\omega_0)\} \cdot e^{-j\omega_0 T_i} \tag{23c}
\]

where \( \ast \) represents the convolution operation. Recall that \( v(t) = A \sin (\omega_0 t) \), therefore, \( V(\omega) \) is defined by

\[
V(\omega) = AJ \pi (\delta(\omega + \omega_0) - \delta(\omega - \omega_0)). \tag{23d}
\]

Now, substituting (23d) into (23c) yields

\[
\sum_n \sum_{-T_i + T_i} v(t) u'(t) e^{-j\omega T_i} = \frac{A}{2T} \sum_n \{(\omega - n\omega_0 - \omega_0)U(\omega - n\omega_0 - \omega_0)
\]

\[
- (\omega - n\omega_0 + \omega_0)U(\omega - n\omega_0 + \omega_0) \cdot e^{-j\omega_0 T_i} \tag{23c}
\]

The evaluation of the third term of (22b) proceeds similarly and yields

\[
-j\omega \sum_n v(t) \cdot u'(t) e^{-j\omega T_i} |_{-nT + T_i}
\]

\[
= \frac{\omega A}{2T} \sum_n \{U(\omega - n\omega_0 + \omega_0)
\]

\[
- U(\omega - n\omega_0 - \omega_0) \} \cdot e^{-j\omega_0 T_i} \tag{24}
\]

Therefore, (22b) becomes

\[
U(z, T_i) = \left( \frac{1}{T} \right) \sum_n [U(\omega - n\omega_0) + \left( \frac{A}{2} \right)
\]

\[
\cdot [U(\omega - n\omega_0 - \omega_0) \cdot (\omega_0 - \omega_0)
\]

\[
- U(\omega - n\omega_0 + \omega_0) \cdot (-\omega_0 + \omega_0)]
\]

\[
\cdot e^{-j\omega_0 T_i} \tag{25}
\]

For convenience, let us define the following quantity:

\[
U(\omega, n\omega_0, \omega_0) = T \cdot U(z, 0).
\]

The next step is to express \( X_1(z, T_i) \) in terms of shifted versions of \( U(\omega) \) by substituting (25) into the expressions for \( X_1(z, T_m) \), for different \( k \) and \( m \), as defined in (20a)
to (20d). The following expressions result:

\[
X_i(z, T_i) = \left(zI - A_i\right)^{-1} \cdot \left[\frac{e^{j\omega T_i}}{T_i}\right] \\
\cdot \sum_n \left[A_2 e^{(\omega - n\omega_s)T_i} + A_3 e^{(\omega - n\omega_s)T_i + T_2}\right] \\
+ A_4 e^{j\omega T_i} + A_5 e^{(\omega - n\omega_s)T_i + j\omega T_i}\cdot U(\omega, n\omega_s, \omega_0)
\] (26a)

where

\[
U(\omega, n\omega_s, \omega_0) = U(\omega - n\omega_s) + \left(\frac{A}{2}\right)\cdot U(\omega - n\omega_s - \omega_0) \left[- n\omega_s - \omega_0\right] \\
- \left(\frac{A}{2}\right)\cdot U(\omega - n\omega_s + \omega_0) \left[- n\omega_s + \omega_0\right]
\] (26b)

\[
X_2(z, T_1 + T_2) = \left(zI - B_2\right)^{-1} \cdot \left[\frac{e^{j\omega (T_1 + T_2)}}{T_1 + T_2}\right] \\
\cdot \sum_n \left[B_2 e^{(\omega - n\omega_s)(T_1 + T_2)} + B_3 e^{(\omega - n\omega_s)T_1} e^{j\omega T_2}\right. \\
+ B_4 e^{(\omega - n\omega_s)T_1} e^{j\omega T_2} + B_5 e^{j\omega T_2}\cdot U(\omega, n\omega_s, \omega_0)
\] (26c)

\[
X_3(z, 2T_1 + T_2) = \left(zI - C_3\right)^{-1} \cdot \left[\frac{e^{j\omega (2T_1 + T_2)}}{2T_1 + T_2}\right] \\
\cdot \sum_n \left[C_2 e^{j\omega T_1} + C_3 e^{(\omega - n\omega_s)T_1} e^{j\omega T_2}\right. \\
+ C_4 e^{(\omega - n\omega_s)T_1} e^{j\omega T_2} + C_5 e^{j\omega T_2}\cdot U(\omega, n\omega_s, \omega_0)
\] (26d)

Finally, recall the output equations:

\[
y_k(t) = K_k x_k(t) + L_k u(t) \cdot w_k(t), \quad \text{for } k = 1, \ldots, 4
\] (27a)

which transform to

\[
Y_i(\omega) = K_i X_i(\omega) + L_i \cdot F_i \left[u(t) \cdot w_i(t)\right],
\] (27b)

for \( k = 1, \ldots, 4 \).

We now substitute (21a)-(21d) and (26b)-(26e) in (20a)-(20d), and then substitute the result in (13b), (15b), (16b), and (17b). We next substitute the latter expressions for \( \tilde{X}_i(\omega) \), for \( i = 1, \ldots, 4 \), in (13a), (15a), (16a), and (17a) to get the final expressions for the state vectors \( X_i(\omega) \) in terms of Fourier transforms of the input vector \( u(\cdot) \). Finally, Fourier transforming the expression of the output vector \( y(t) \) leads to the following expression:

\[
Y(\omega) = \sum_{n} \sum_{k=1}^{4} \left[1 - \frac{\exp(-j\omega k)}{j\omega}\right] K_k F_k P_{k,n}(\omega) \\
\cdot U(\omega, n\omega_s, \omega_0) + \sum_{n} \sum_{k=1}^{4} \sum_{m} \theta_{k,n,m} (K_k G_k + L_k) \\
\cdot U(\omega - n\omega_s - m\omega_0)
\] (27c)

with \( U(\omega, n\omega_s, \omega_0) \) defined as in (26b). A general expression for \( \theta_{k,n,m} \) identified directly from (13c), (15c), (16c), and (17c) is given by

\[
\theta_{k,n,m} = (-1)^m \cdot \phi_m (m\omega_0 + \omega_n A)
\] (27d)

The expressions for \( P_{k,n}(\omega) \) are defined as follows:

\[
P_{1,n}(\omega) = \phi_1(e^{(\omega T)}) \cdot \left[D_2 e^{(\omega - n\omega_s)T_1 + T_2} + D_3 e^{(\omega - n\omega_s)T_1 + T_2} + D_4 e^{(\omega - n\omega_s)T_1 + j\omega T_2}\right]
\] (28a)

\[
P_{2,n}(\omega) = \phi_2(e^{(\omega T)}) \cdot \left[D_2 e^{(\omega - n\omega_s)T_1 + T_2} + D_3 e^{(\omega - n\omega_s)T_1 + T_2} + D_4 e^{(\omega - n\omega_s)T_1 + j\omega T_2}\right]
\] (28b)

\[
P_{3,n}(\omega) = \phi_3(e^{(\omega T)}) \cdot \left[D_2 e^{(\omega - n\omega_s)T_1 + T_2} + D_3 e^{(\omega - n\omega_s)T_1 + T_2} + D_4 e^{(\omega - n\omega_s)T_1 + j\omega T_2}\right]
\] (28c)

\[
P_{4,n}(\omega) = \phi_4(e^{(\omega T)}) \cdot \left[D_2 e^{(\omega - n\omega_s)T_1 + T_2} + D_3 e^{(\omega - n\omega_s)T_1 + T_2} + D_4 e^{(\omega - n\omega_s)T_1 + j\omega T_2}\right]
\] (28d)

where

\[
\phi_1(z) = (zI - F_1 F_2 F_3)^{-1} \\
\phi_2(z) = (zI - F_2 F_3 F_4)^{-1} \\
\phi_3(z) = (zI - F_3 F_4 F_5)^{-1} \\
\phi_4(z) = (zI - F_4 F_5 F_6)^{-1}
\]
extra terms $U(\omega - n \omega_b - \omega_0)$ and $U(\omega - n \omega_b + \omega_0)$ may be thought of as the first two sidebands of an angle modulation introduced by the jitter. The summation index $m$ is due to the modulation introduced by the jitter. Furthermore, as no approximation has yet been made to perform the computation of the second part of (27c), the triple summation operation represents the complete modulation effect.

We now compute the autocorrelation of the output vector $y(t)$. To this end, we return to the time-domain expression for $y(t)$. For convenience, we will define the following:

$$
\tilde{P}_k(\omega) = \left(1 - \frac{\exp^{-j\omega T_b}}{j\omega T_b}\right) \cdot K_k F_k P_k(\omega)
$$

(29a)

$$
P_k(\omega) = \sum_{k=1}^{4} \tilde{P}_k(\omega)
$$

(29b)

$$
Q_{n,m} = \sum_k \left(K_k G_k + L_k\right) \cdot \theta_{k,n,m}.
$$

(29c)

From (27c), we have with these definitions

$$
Y(\omega) = \sum_{n} \left[P_n(\omega) \cdot U(\omega, \omega_b + \omega_0) \right. \\
+ \left. \sum_{m} Q_{n,m} \cdot U(\omega - n \omega_b - m \omega_0) \right].
$$

(30)

An inverse Fourier transform of (30) yields

$$
y(t) = \sum_{n} \tilde{P}_n(t) * \left[u(t) e^{j\omega_0 t} \right. \\
+ \left. (1 - A(jn \omega_b \sin(\omega_0 t)) \right] + \sum_{n,m} Q_{n,m} \cdot e^{j\omega_0 t} \cdot e^{j\omega_0 T_b} \cdot u(t)
$$

(31)

where $\tilde{P}(t)$ represents the inverse Fourier transform of $P_n(\omega)$ and $*$ represents the convolution operation.

The second part of the above expression involves a double summation operation. We will now investigate the relative effect of the term $Q_{n,m} e^{j\omega_0 t} \cdot e^{j\omega_0 t}$ as a function of the arguments $n$ and $m$. Let us define

$$
y_{\tilde{Q}}(t) = \sum_{n,m} Q_{n,m} e^{j\omega_0 t} \cdot e^{j\omega_0 t}. 
$$

(32)

Therefore, $y_{\tilde{Q}}(t)$ can be rewritten as

$$
y_{\tilde{Q}}(t) = \sum_{k} \left(K_k G_k + L_k \right) \theta_{k}(t)
$$

(33)

where $\theta_{k}(t)$ is defined as

$$
\theta_{k}(t) = \sum_{n,m} \theta_{k,n,m} e^{j\omega_0 t} \cdot e^{j\omega_0 t}. 
$$

(34)

$\theta_{k,n,m}$, given in (27d), can be rewritten as

$$
\theta_{k,n,m} = (-1)^m \cdot J_m((m \omega_0 + n \omega_b) A) \\
\cdot \left(\frac{2T_k}{T} \right) \sin \left(\frac{(n \omega_b + m \omega_0) T_k}{2} \right) \\
\cdot e^{-j(m \omega_0 + n \omega_b) T_k/2} \exp \left[-j n \omega_b \sum_{r=1}^{k-1} T_r \right]. 
$$

(35)

Let us take a closer look at $\theta_{k,n,m}$ first. The sampling functions attenuate the Bessel coefficients for large values of $n$ and $m$ due to the usually high frequencies $\omega_b$ used in SC filters. On the other hand, when $m$ is small and $n$ nonzero, the argument of the Bessel coefficient can be approximated by $n \omega_b / A$. In this case, $J_m((m \omega_0 + n \omega_b) A)$ is small due to the usually high $\omega_b$ used and tends to zero. The variation of $\theta_{k,n,m}$ for the SC integrator parameters defined in Table 1, in terms of $n$ and $m$ is presented in Fig. 3. The index $k$ has been chosen equal to 1; the behavior for $k$ equal to 2, 3, or 4, is similar. We find, as expected, that the major contribution of $\theta$ is obtained for $m$ and $n$ equal to zero.

We then investigate the behavior of the term $\theta_{k}(t)$, by defining an approximation to all but the $(0,0)$ term in the double summation. Let us define

$$
\theta_{k,n,m} = \sum_{n,-m \leq n \leq m, n \neq m} \theta_{k,n,m} e^{j\omega_0 t} \cdot e^{j\omega_0 t} 
$$

(36)

with $(n,m) \neq (0,0)$.

The function $J_m((m \omega_0 + n \omega_b) A) \sin ((n \omega_b + m \omega_0) T_k/2)$ exhibits a damped oscillatory behavior around 0 for increasing values of $n$ and $m$. Therefore, in order to study the trend for $\theta_{k}(t)$, we have chosen to approximate the double infinite summation indices present in that function with reasonably high indexes $n_{\text{max}}$ and $m_{\text{max}}$ chosen to be equal to 200. Fig. 4 presents the variations of $\theta_{k,n,m}$ for $n_{\text{max}} = m_{\text{max}} = 200$, $k = 1$, and $t$ varying from 0 to $2\pi$ seconds. This graph shows that the approximation to $\theta_{k}(t)$ oscillates around 0 and is much smaller than $\theta_{k,0,0} = 1/2$. Increasing the values of $n_{\text{max}}$ and $m_{\text{max}}$ will not affect the global behavior of the double summation present in $y_{\tilde{Q},n_{\text{max}},m_{\text{max}}}(t)$ because the terms corresponding to nonzero $m$ and $n$ are oscillatory and tend to cancel each other in the double summation. Therefore, we can reasonably approximate $\theta_{k}(t)$ in (33) by $\theta_{k,0,0}$.

The spectrum of a stochastic signal is defined as the Fourier transform of its autocorrelation function, which is derived from the above indicated approximation to (31) in Appendix A. Some comments related to the autocorrelation function are appropriate here. Due to the introduc-
Fig. 3. Isometric plotting of \(|\theta_{k,n,m}|: T_c = T/4\) for \(k = 1, \ldots, 4\).

Note that when we can define the period \(T_c\), where \(T_c\) is a common integer multiple of \(T\) the switching period, and \(T_0\) the jitter period, the autocorrelation function \(R_{s}(t + \tau, t)\) becomes cyclostationary with period \(T_c\). In such a case, the cyclostationary property is used to define the autocorrelation \(R_{s,f,T_c}\) averaged over the period \(T_c\) given by

\[
R_{s,f,T_c}(\tau) = \left( \frac{1}{T_c} \right) \int_{T_c} R_{s}(t + \tau, t) \, dt. \tag{39}
\]

Next, note that when \(R_{s}(t + \tau, t)\) is cyclostationary, with period \(T_c = mT_0 = nT\), then the long-term average limit \(\bar{R}_{s}(\tau)\) becomes equivalent to the time-averaged correlation function \(R_{s,f}(\tau)\). Thus using the small signal approximation and the periodicity of the jitter disturbance as shown in Appendix A, and assuming that either the observation interval \(T_c\) is large compared with the sampling period \(T_c\), or \(T_c\) is a common multiple of \(T\) and \(T_0\), the final expression derived from (31) using (38) or (39) is given by the following expression:

\[
\bar{S}_{s}(\omega) = Q_{0,0} \cdot Q_{0,0}^* + P_0(\omega) \cdot S_s(\omega) \cdot P_0^*(\omega) + Q_{0,0} \cdot S_s(\omega) \cdot P_0^*(\omega) + \sum_{n} P_n(\omega) \cdot S_s(\omega - n\omega_j)
\]

\[
\cdot P_n^*(\omega) + A^2 \left\{ \sum_{n} \left( (n\omega_j/2)^2 \cdot P_n(\omega) \right) \right\} \left[ S_s(\omega - n\omega_j - \omega_0) - S_s(\omega - n\omega_j + \omega_0) \right]
\]

\[
\cdot P_s^*(\omega) + (n\omega_j\omega_0/2) \cdot P_s(\omega)
\]

\[
\left[ S_s(\omega - n\omega_j - \omega_0) - S_s(\omega - n\omega_j + \omega_0) \right]
\]

\[
\cdot P_s^*(\omega) + (\omega_0^2/4) \cdot P_s(\omega)
\]

\[
\left[ S_s(\omega - n\omega_j - \omega_0) + S_s(\omega - n\omega_j + \omega_0) \right]
\]

\[
\cdot P_s^*(\omega) \right) \right]. \tag{40}
\]

where \(S^*\) is defined as the Hermitian transpose of \(S\).
For scalar inputs and outputs, the expression for the spectrum in (40) becomes
\[
\tilde{S}_s(\omega) = \{(Q_{0,0})^2 + 2\text{real}(Q_{0,0} \cdot P_0(\omega) + Q_{0,0} \cdot P_0(\omega))\} \\
\times S_n(\omega) + \sum_n |P_n(\omega)|^2 \cdot S_n(\omega - n\omega_s) \\
+ A^2 \left( \sum_n |P_n(\omega)|^2 \cdot (n\omega_s/2)^2 \cdot [S_n(\omega - n\omega_s - \omega_0)] \right. \\
\left. + (n\omega_s\omega_0/2) \cdot |P_0(\omega)|^2 \cdot [S_0(\omega - n\omega_s) - \omega_0) \right. \\
\left. - S_0(\omega - n\omega_s + \omega_0) (\omega_0^2/4) \right. \\
\left. + P_0(\omega)|^2 \cdot [S_0(\omega - n\omega_s) - \omega_0) \right. \\
\left. + S_0(\omega - n\omega_s + \omega_0) \right) \right]. \quad (41)
\]

The above result theoretically indicates that \(\tilde{S}_s(\omega)\) is unbounded, due to the infinite summations. However, as was noted by Liou and Kuo [3], the circuit components present in SCF networks have bandlimited characteristics, and the input noise spectra are valid only for \(\omega_{\text{min}} \leq |\omega| \leq \omega_{\text{max}}\) where \(\omega_{\text{min}}\) and \(\omega_{\text{max}}\) are some lower and upper bound frequencies. Thus, the infinite summations present in the final expression for \(\tilde{S}_s(\omega)\) reduce to finite summations.

III. SIMULATION RESULTS

The above results are now applied to the first-order SC integrator shown in Fig. 1. The output noise source is assumed to have three origins; the thermal noise sources created by the ON resistances of the two switches and the noise generated by the finite gain amplifier. The three noise sources are assumed uncorrelated, so their effects can be evaluated separately. Thus three different sets of circuit matrices \(F_i, G_i, K_i\), and \(L_i\), for \(i = 1, \ldots, 4\), are used to compute the terms \(P_i(\omega)\), as defined in (29b), and \(Q_{0,0}\), as defined in (29c). The coefficients \(P_i(\omega)\) and \(Q_{0,0}\) are then used to compute the average noise spectrum \(\bar{S}_s(\omega)\), as defined in (40). A list of the circuit matrices defined for the first-order SC integrator is given in Appendix B. The parameter values of the circuit are presented in Table I. Note that the total averaged output noise spectrum obtained depends on the jitter amplitude squared, \(A^2\). Fig. 5 shows the averaged noise spectrum of the integrator for different jitter amplitudes \((A)\) varying from 1 to 0.001\% of the switching period. It shows that the noise spectrum may increase drastically with increasing jitterer for this particular system. This is due to the terms \((nA\omega_s)^2\) present in (39). Note, that as expected from this expression, an increase in jitter amplitude by a factor of 10 increases the noise spectrum by 20 dB. However, variations of the jitter frequency, for a fixed jitter amplitude, seem to have no effect on the level of the noise spectrum. The latter is clearly shown in Fig. 6 where the jitter amplitude is fixed to 0.1\% of the switching period, and where the jitter frequency varies from 60 Hz to 1 kHz.

This example shows that SC circuits may be very sensitive to jitter perturbation. Therefore, it shows the need for isolating, as much as possible, the clock signal circuitry from any kind of periodic interference. This seems especially important with regard to the effects of noise generated internal to the SCF system. The effect of these wideband noise signals becomes very noticeable due to the sampling process in connection with the use of relatively wideband operational amplifiers as SCF components.

IV. SUMMARY

This paper has presented a time-domain approach for analyzing the effects of periodic jitter perturbation in SC filters. First-order and small-signal approximations were made to facilitate evaluation of the approximate final
expression. This result makes it now possible to evaluate the effects of duty cycle, nonoverlapping clock signals, operational amplifier transfer functions, and a variety of internally generated noise spectra, on the SCF output noise spectrum.

Finally, a numerical evaluation of jitter amplitude and frequency effects was conducted for a SC integrator. We found here, that the presence of a periodic jitter perturbation in the clock circuitry may increase the output noise spectrum drastically. Furthermore, the noise level is very sensitive to jitter amplitude variations due to the dependence on the squared amplitude, but it is rather insensitive to jitter frequency changes. These results show the need for high precision clock circuitry, in order to minimize the effect of internal noise on the SCF output noise.

The quantitative effects seen here do not necessarily show up in the same way for other SCF circuits, as they are influenced nonlinearly by the particular circuit matrices corresponding to a specific filter, its structure, and operational amplifier properties. Upon the derivation of the circuit matrices for alternative structures, however, we can now evaluate their respective sensitivities to the internally generated noise.

APPENDIX A

DERIVATION OF EQUATION (40)

Using the first-order and small-signal approximations presented earlier, for $y(t)$ given by

$$y(t) = \sum_{n} \mathcal{P}_{n}(t) \ast [u(t) \cdot e^{j n \omega_{c} t}] \cdot [1 - A$$

$$\cdot (j n \omega_{c} \sin(\omega_{c} t) + \omega_{c} \cos(\omega_{c} t))] + Q_{0,0} \cdot u(t) \quad (A.1)$$

the following autocorrelation function is evaluated:

$$R_{y}(t + \tau, t) = R_{y}(t + \tau, t) + R_{y}(t + \tau, t)$$

$$+ R_{y,y}(t + \tau, t) + R_{y,y}(t + \tau, t) \quad (A.2)$$

where

$$y(t) = \sum_{n} \mathcal{P}_{n}(t) \ast [u(t) \cdot e^{j n \omega_{c} t}] + Q_{0,0} \cdot u(t)$$

$$= \sum_{n} \mathcal{P}_{n}(t) \ast [u(t) \cdot e^{j n \omega_{c} t}] + Q_{0,0} \cdot u(t) \quad (A.3)$$

and

$$y(t) = \sum_{n} \mathcal{P}_{n}(t) \ast \mathcal{P}_{n}(t) \ast [u(t) \cdot e^{j n \omega_{c} t}] \cdot u(t)$$

$$\cdot \mathcal{P}_{n}(t) \ast [u(t) \cdot e^{j n \omega_{c} t}] \cdot u(t) \quad (A.4)$$

Now, assuming $u(t)$ to be a stationary signal, the evaluation of $R_{y}$ yields

$$R_{y}(t + \tau, t) = \sum_{n,p} \int ds \int dv \mathcal{P}_{n}(s) \cdot R_{y}(\tau - s + v)$$

$$\cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t + \tau - s - v)}$$

$$+ \sum_{n} \int ds \mathcal{P}_{n}(s) \cdot R_{y}(\tau - s)$$

$$\cdot Q_{0,0} e^{j n \omega_{c} (t + \tau - s)} + \sum_{p} \int dv Q_{0,0}$$

$$\cdot R_{y}(\tau + v) \cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t - v)}$$

$$+ Q_{0,0} \cdot R_{y}(\tau) \cdot Q_{0,0}^{*} \quad (A.5)$$

where $S^{*}$ is defined as the Hermitian transpose of the matrix $S$.

Similarly, for $u(t)$ a stationary process, the autocorrelation function associated with $y_{2}(t)$ is evaluated as follows:

$$R_{y_{2}}(t + \tau, t)$$

$$= A^{2} \sum_{n,p} \int ds \int dv$$

$$\cdot \mathcal{P}_{n}(s) \cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t + \tau - s)}$$

$$\cdot R_{y}(\tau - s + v)$$

$$\cdot \mathcal{P}_{n}(s) \cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t - v)} \cdot R_{y}(\tau - s) \cdot Q_{0,0} \cdot R_{y}(\tau) \cdot Q_{0,0}^{*} \quad (A.6)$$

The cross-correlation function associated with $y_{1}(t)$ and $y_{2}(t)$ is given by

$$R_{y_{1},y_{2}}(t + \tau, t)$$

$$= (-A) \sum_{n,p} \int ds \int dv \mathcal{P}_{n}(s) + Q_{0,0} \cdot \delta_{s} \cdot \mathcal{P}_{p}^{*}(v) \cdot \mathcal{P}_{n}(s) \cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t + \tau - s)} \cdot e^{j n \omega_{c} (t - v)} \cdot \mathcal{P}_{n}(s) \cdot \mathcal{P}_{p}^{*}(v) \cdot e^{j n \omega_{c} (t - v)} \cdot R_{y}(\tau - s) \cdot Q_{0,0} \cdot R_{y}(\tau) \cdot Q_{0,0}^{*} \quad (A.7)$$

Equations (A.5)-(A.7) show that the autocorrelation function $R_{y}(t + \tau, t)$ given in (A.2) is not stationary. However, assuming that the interval $T_{c}$ is large compared with the sampling period $T_{s}$, we define the long-term average limit $\overline{R}_{y}(\tau)$ expression as

$$\overline{R}_{y}(\tau) = \lim_{T_{c} \to \infty} \left( \frac{1}{T_{c}} \right) \int_{T_{c}} R_{y}(t + \tau, t) \, dt. \quad (A.8)$$

Individual expressions for $\overline{R}_{y_{1}}(\tau)$, $\overline{R}_{y_{2}}(\tau)$, and $\overline{R}_{y_{1},y_{2}}(\tau)$ may be found by, respectively, replacing $R_{y}(t + \tau, t)$ with $R_{y_{1}}(t + \tau, t)$, $R_{y_{2}}(t + \tau, t)$, and $R_{y_{1},y_{2}}(t + \tau, t)$ in (A.8). Note that the correlation function $\overline{R}_{y}(t + \tau, t)$ is cyclos-
tionary with period $T$, when there exists a period $T$ which is a common multiple of the switching period $T$ and the jitter period $T_0$. In such a case the long-term limit correlation function becomes equal to the time-averaged correlation function defined as

$$
\bar{R}_{s,t}(\tau) = \left( \frac{1}{T_c} \right) \int_{T_c} R_{s,t}(t + \tau, t) \, dt.
$$

(A.9)

In the following we present two of the main steps present in the computation of the long-term correlation function $\bar{R}(\tau)$ or the time-averaged correlation function $R_{s,t}(\tau)$, when the cyclostationarity property is satisfied. The first property used in the derivation of the long-term average limit $\bar{R}(\tau)$ is the fact that

$$
\lim_{T_c \to +\infty} \left( \frac{1}{T_c} \right) \int_{T_c} e^{(n-p)\omega_0 t} \, dt = \delta(n-p)
$$

(A.10)

where $\delta(n)$ denotes the Kronecker delta. Note that when the cyclostationarity is satisfied, $\bar{R}_{s,t}(\tau)$ is computed by integrating over a finite common period $T_c = mT = nT_0$. This leads to the following type of integration operation:

$$
\alpha = \lim_{T_c \to +\infty} \left( \frac{1}{T_c} \right) \int_{T_c} \cos(2\omega_0 t + \phi) e^{i\omega_0(n-p) t} \, dt.
$$

(A.11)

which yields a result similar to that obtained for the long-term average limit operation. The second main step used in the computation of $\bar{R}(\tau)$ involves computations of the following type:

$$
\alpha = \lim_{T_c \to +\infty} \left( \frac{1}{T_c} \right) \int_{T_c} \cos(2\omega_0 t + \phi) e^{i\omega_0(n-p) t} \, dt = 0
$$

(A.12)

Note that $\alpha$ goes to 0 as $T_c$ goes to infinity when $\omega_0 \neq (n - p)\omega_0/2$, which is always realized when the jitter frequency is much lower than the sampling frequency. In addition, when the cyclostationarity property is satisfied, $R_{s,t}(\tau)$ involves computations similar to that of (A.12), where the integration to the limit operation is replaced with integration over a finite interval $T_c$. In such a case:

$$
\alpha = \left( \frac{1}{T_c} \right) \int_{T_c} \cos(2\omega_0 t + \phi) e^{i\omega_0(n-p) t} \, dt = 0
$$

(A.13)

as the integration is carried out over the finite period $T_c$, defined as a multiple number of periods $T_c = mT = nT_0$. Thus, computing the long-term average correlation function associated with $R_{s,t}(t + \tau, t)$ in (A.5) leads to

$$
\bar{R}_{s,t}(\tau) = \lim_{T_c \to +\infty} \left( \frac{1}{T_c} \right) \int_{T_c} R_{s,t}(t + \tau, t) \, dt
$$

$$
= \sum_n \int ds \, dv \bar{P}_n(s) \cdot R_n(t + \tau - s + v) + Q_{0,0} \cdot R_n(\tau + \tau) + P_{0,0} \cdot R_n(\tau) \cdot \bar{P}_0(\tau)
$$

We used $\bar{P}_n(\cdot)$ to denote the inverse Fourier transform of $P_n(\cdot)$ again, note that when the time-varying correlation function is cyclostationary with period $T$, we have $R_{s,t}(\tau) = R_{s,t}(\tau)$. The averaged spectral density associated with $y(t)$ may be found by Fourier transforming the above expression.

$$
\bar{S}_y(\omega) = \sum_n \left[ P_n(\omega) \cdot S_n(\omega - n\omega_s - \omega_0) \right]
$$

$$
+ Q_{0,0} \cdot S_n(\omega) \cdot Q_{0,0} - Q_{0,0} \cdot S_n(\omega) \cdot P_{0,0} \cdot (\omega)
$$

$$
+ P_{0,0} \cdot S_n(\omega) \cdot Q_{0,0}.
$$

(A.15)

Similarly, a time averaging operation is then performed on the nonstationary autocorrelation function $R_{s,t}(t + \tau, t)$. Fourier transforming the resulting expression gives the following time averaged spectral density associated with $y(t)$:

$$
\bar{S}_{\bar{y}}(\omega) = A^2 \left( \sum_n \left( (n\omega_s/2) \cdot P_n(\omega) \cdot \left[ S_n(\omega - n\omega_s - \omega_0) \right] \right)
$$

$$
+ S_n(\omega) \cdot \left[ S_n(\omega - n\omega_s - \omega_0) \right] \cdot \left[ P_n(\omega) + (n\omega_s/2) \right]
$$

$$
+ S_n(\omega) \cdot \left[ S_n(\omega - n\omega_s - \omega_0) \right] \cdot \left[ P_n(\omega) + (\omega_0^2/4) \right]
$$

Finally, we note that the effects of $R_{s,t}(t + \tau, t)$ vanish upon time averaging of the nonstationary cross-correlation function. A similar result is obtained for $R_{s,t}(t + \tau, t)$ Therefore, the total averaged spectral density associated with $y(t)$ is given by the summation of the spectra respectively associated with $y(t)$ and $y(t)$.

$$
\bar{S}_y(\omega) = \sum_n \left[ P_n(\omega) \cdot S_n(\omega - n\omega_s - \omega_0) \right]
$$

$$
+ Q_{0,0} \cdot S_n(\omega) \cdot Q_{0,0} + Q_{0,0} \cdot S_n(\omega) \cdot P_{0,0} \cdot (\omega)
$$

$$
+ P_{0,0} \cdot S_n(\omega) \cdot Q_{0,0}.
$$

(A.16)
APPENDIX B
SETS OF CIRCUIT MATRICES DEFINED FOR THE FIRST-ORDER SC INTEGRATOR

Due to the nonoverlapping clock signals \( \phi_1 \) and \( \phi_2 \), four different configurations of the SC integrator circuit, as shown in Fig. 1, need to be studied.

- subinterval 1, corresponding to \( \phi_1 \) open, \( \phi_2 \) closed,
- subinterval 2, corresponding to \( \phi_1 \) open, \( \phi_2 \) open,
- subinterval 3, corresponding to \( \phi_1 \) closed, \( \phi_2 \) open,
- subinterval 4, corresponding to \( \phi_1 \) closed, \( \phi_2 \) closed.

Three different sets of circuit matrices \( F_i, G_i, K_i, L_i \), for \( i = 1, \ldots, 4 \) are found, one associated with the left switch \( (\phi_1) \) noise source, one associated with the right switch \( (\phi_2) \) noise source, and one associated with the amplifier noise source. The details of the computations leading to these sets are presented in [4].

1) The set of circuit matrices associated with the left switch noise source is given by

\[
F_1 = \begin{bmatrix}
C_1/\Delta & C_2/\Delta \\
C_1(A_0 + 1)/\Delta & C_2(A_0 + 1)/\Delta
\end{bmatrix},
G_1 = [0, 0]^T,
L_1 = 0, K_1 = [0, -A_0/(A_0 + 1)]
\]

2) The set of circuit matrices associated with the right switch noise source is given by

\[
F_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
G_2 = [0, 0]^T, L_2 = 0,
K_2 = [0, -A_0/(A_0 + 1)]
\]

3) The set of circuit matrices associated with the amplifier noise source is given by

\[
F_3 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix},
G_3 = [0, 0]^T, L_3 = -A_0/(A_0 + 1),
K_3 = [0, -A_0/(A_0 + 1)]
\]

where \( A_0 \) and \( \Delta \) are defined as in Section 1 of this Appendix.

REFERENCES

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