Supplementary Notes for
ECE 2714: Signals and Systems

C.L. Wyatt

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Commit `901467a`

Preface

To the student:

This is a set of supplementary notes and examples for ECE 2714. It is not a replacement for the textbook, but can act as a reference and guide your reading. These notes are not comprehensive – often additional material and insights are covered during class.

This material is well covered in the official course text "Oppenheim, A. V., Willsky, A. S., and Nawab, S. H. Signals and Systemsii, Essex UK: Prentice Hall Pearson, 1996." [1] (abbreviated OW). This is an older, but very good book. However there are many, many texts that cover the same material. *Engaged* reading a textbook is one of the most important things you can do to learn this material. Again, these notes should **not** be considered a replacement for a textbook.

To the instructor:

These notes are simply a way to provide some consistency in topic coverage and notation between and within semesters. Feel free to share these with your class but you are under no obligation to do so. There are many alternative ways to motivate and develop this material and you should use the way that you like best. This is just how I do it.

Each chapter corresponds to a "Topic Learning Objective" and would typically be covered in one class meeting on a Tuesday-Thursday or Monday-Wednesday schedule. Note CT and DT topics are taught interleaved rather than in separate blocks. This gets the student used to going back and forth between the two signal and system types. We introduce time-domain topics first, followed by (real) frequency domain topics, using complex frequency domain for sinusoidal analysis only and as a bridge. Detailed analysis and application of Laplace and Z-transforms is left to ECE 3704.

Acknowledgements

The development of this course has been, and continues to be, a team effort. Dr. Mike Buehrer was instrumental in the initial design and roll-out of the course. Dr. Mary Lanzerotti has helped enormously with the course organization and academic integrity. All the instructors thus far: Drs. Buehrer, Safaai-Jazi, Lanzerotti, Kekatos, Poon, Xu, and Talty, have shaped the course in some fashion.

C.L. Wyatt
May 7, 2024

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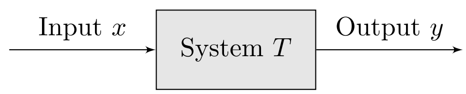
Chapter 1

Course Introduction

The concepts and techniques in this course are probably the most useful in engineering. A *signal* is a function of one or more independent variables conveying information about a physical (or virtual) phenomena. A *system* may respond to signals to produce other signals, or produce signals directly.

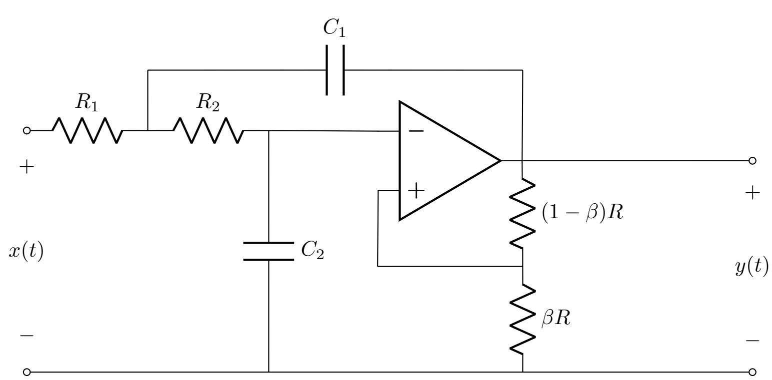


This course is about the mathematical models and related techniques for the design and understanding of systems as signal transformations. We focus on a broadly useful class of systems, known as *linear, time-invariant systems*. You will learn about:

- the representation and analysis of signals as information carrying channels
- and how to analyze and implement linear, time-invariant systems to transform those signals.

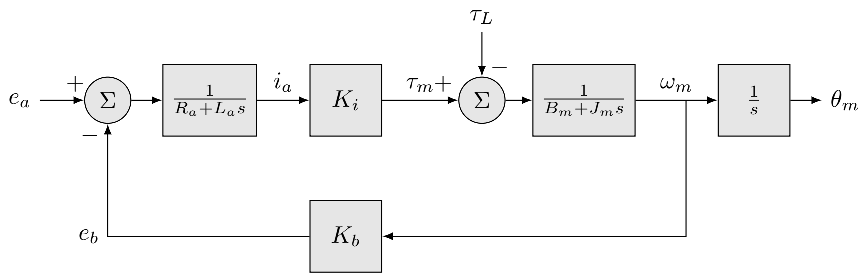
1.1 Example Signals and Systems

Example 1.1.1. Electrical Circuits. This is a Sallen-Key filter, a second-order system commonly use to select frequencies from a signal:



There are two signals we can easily identify, the input signal as the voltage applied across $x(t)$, and the output voltage measured across $y(t)$. We build on your circuits course by viewing this circuit as an implementation of a more abstract linear system. We see how it can be viewed as a frequency selective filter. We will see how to answer questions such as: how do we choose the values of the resistors and capacitors to select the frequencies we are interested in? and how do we determine what those frequencies are?

Example 1.1.2. Robotic Joint. This is a Linear, Time-Invariant model of a DC motor, a mixture of electrical and mechanical components.

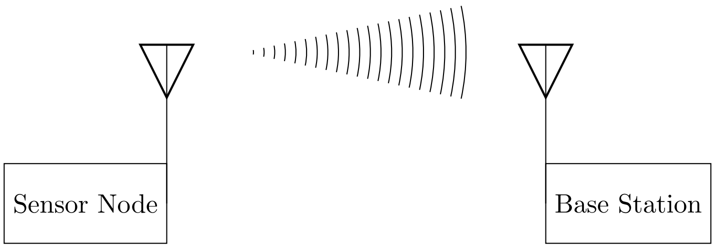


How do we convert the motor into a servo for use in a robotic joint? What are its characteristics (e.g. how fast can it move)?

Example 1.1.3. Audio Processing. Suppose you record an interview for a podcast, but during an important part of the discussion, the HVAC turns on and there is an annoying noise in the background.

How could you remove the noise minimizing distortion to the rest of the audio?

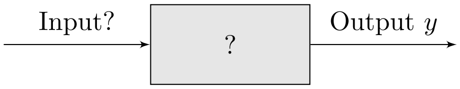
Example 1.1.4. Communications. Consider a wireless sensor, that needs to transmit to a base station, e.g. a wireless mic system.



How should the signal be processed so it can be transmitted? How should the received signal be processed?

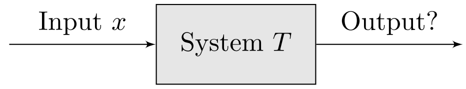
1.2 Types of Problems

Applications of this material occur in all areas of science and engineering. When we have a measured output but are unsure what combination of inputs and system components could have produced it, we have a *modeling* problem.



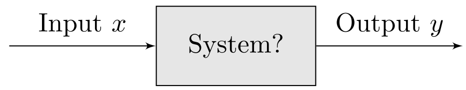
Models are the bedrock of the scientific method and are required to apply the concepts of this course to engineering problems.

When we know the input and the system description and desire to know the output we have an *analysis* problem.

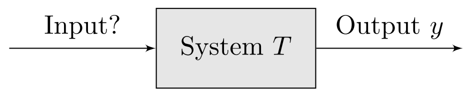


Analysis problems are the kind you have encountered most often already. For example, given an electrical circuit and an applied voltage or current, what are the voltages and currents across and through the various components.

When we know either the input and desired output and seek the system to perform this transformation,



or we know the system description and output and desire the input that would generate the output,



we have a *design problem*.

This course focuses on modeling and analysis with applications to electrical circuits and devices for measurement and control of the physical world and is broadly applicable to all ECE majors. Some Examples:

- Controls, Robotics, & Autonomy: LTI systems theory forms the basis of perception and control of machines.
- Communications & Networking: LTI systems theory forms the basis of transmission and reception of signals, e.g. AM and FM radio.
- Machine Learning: LTI systems are often used to pre-process samples or to create basis functions to improve learning.
- Energy & Power Electronic Systems: linear circuits are often modeled as LTI systems.

Subsequent courses, e.g. ECE 3704, focus more on analysis and design.

1.3 Learning Objectives

The learning objectives (LOs) for the course are:

LO-1 Describe a given system using a block-level description and identify the input/output signals.

LO-2 Mathematically model continuous and discrete linear, time-invariant systems using differential and difference equations respectively.

LO-3 Analyze the use of filters and their interpretation in the time and frequency domains and implement standard filters in hardware and/or software.

LO-4 Apply computations of the four fundamental Fourier transforms to the analysis and design of linear systems.

LO-5 Communicate solutions to problems and document projects within the domain of signals and systems through formal written documents.

These are broken down further into the following topic learning objectives (TLOs). The TLOs generally map onto one class meeting but are used extensively in later TLOs.

TLO-1 Course introduction (OW Forward and §1.0)

(a) Signals as models
(b) Systems as transformation of signals
(c) Prerequisites

TLO-2 Continuous-time (CT) signals (OW §1.1 through 1.4 and 2.5): A continuous-time (CT) signal is a function of one or more independent variables conveying information about a physical phenomena. This lecture gives an introduction to continuous-time signals as functions. You learn how to characterize such signals in a number of ways and are introduced to two very important signals: the unit impulse and the complex exponential.

(a) Continuous-time signals as functions $\mathbb{R} \mapsto \mathbb{C}$
(b) Transformations of time
(c) Characterizing signals
   i. periodic/aperiodic
   ii. even/odd
   iii. energy or/nor power
(d) Impulse function
(e) Step function
(f) Complex exponential

TLO-3 Discrete-time (DT) signals (OW §1.1 through 1.4)

(a) Discrete-time signals as functions $\mathbb{Z} \mapsto \mathbb{C}$
(b) Transformations of time index
(c) Characterizing signals
   i. periodic/aperiodic
   ii. even/odd
   iii. energy or/nor power
(d) Impulse function
(e) Step function
(f) Complex exponential

TLO-4 CT systems as linear constant coefficient differential equations (OW §2.4.1)

(a) LCCDE and their solution (1st and 2nd order)
(b) impulse response from LCCDE

TLO-5 DT systems as linear constant coefficient difference equations (OW §2.4.2)

(a) LCCDE and their solution (1st and 2nd order)
(b) impulse response from LCCDE

TLO-6 Linear time invariant CT systems (OW §1.5, 1.6, 2.3)

(a) Memory
(b) Invertability
(c) Causality

(d) Stability
(e) Time-invariance
(f) Linearity
(g) Define LTI system

TLO-7 Linear time invariant DT systems (OW §1.5, 1.6, 2.3)

(a) Memory
(b) Invertability
(c) Causality
(d) Stability
(e) Time-invariance
(f) Linearity
(g) Define LTI system

TLO-8 CT convolution (OW §2.2)

(a) Review CT LTI systems and superposition property
(b) CT Convolution Integral
(c) Properties of convolution
   i. communative
   ii. distributive
   iii. associative
(d) Determining system response using convolution with impulse response

TLO-9 DT convolution (OW §2.1)

(a) Review DT LTI systems and superposition property
(b) DT Convolution Sum
(c) Properties of convolution
   i. communative
   ii. distributive
   iii. associative
(d) Determining system response using convolution with impulse response

TLO-10 CT block diagrams (OW §1.5.2 and 2.4.3)

(a) blocks represented by impulse response
(b) series and parallel connections, reductions
(c) scale, sum, and integrator blocks
(d) equivalence of LCCDE's and block diagrams
(e) first-order differential equation as feedback motif
(f) second-order differential equation as a feedback motif
(g) implementing a LCCDE using adders, multipliers, and integrators

TLO-11 DT block diagrams (OW §1.5.2 and 2.4.3)

(a) blocks represented by impulse response

(b) series and parallel connections, reductions
(c) scale, sum, and unit delay blocks
(d) equivalence of LCCDE's and block diagrams
(e) first-order difference equation as feedback motif
(f) second-order difference equation as a feedback motif
(g) implementing a LCCDE using adders, multipliers, and delays

TLO-12 Eigenfunctions of CT systems (OW §3.2 and 3.8)

(a) Eigenfunction e^{st}
(b) Transfer Function $H(s)$
(c) Stability and Frequency Response (FR) $H(j\omega)$
(d) How this is useful - decomposition of input signal into complex exp
(e) What signals can be decomposed this way, foreshadow Fourier Analysis

TLO-13 Eigenfunctions of DT systems (OW §3.2 and 3.8)

(a) Eigenfunction z^n
(b) Transfer Function $H(z)$
(c) Stability and Frequency Response (FR) $H\left(e^{j\omega}\right)$
(d) How this is useful - decomposition of input signal into complex exp
(e) What signals can be decomposed this way, foreshadow Fourier Analysis

TLO-14 CT Fourier Series representation of signals (OW §3.3 through 3.5)

(a) review CT periodic functions
(b) harmonic sums
(c) derive synthesis equation
(d) derive analysis equation
(e) spectrum plots
(f) define mean-square convergence
(g) truncated CT FS
(h) stable LTI system response using CTFS
(i) example of the impulse train (for sampling theory later)
(j) formal Dirichlet conditions
(k) properties of CT FS

TLO-15 DT Fourier Series representation of signals (OW §3.6 and 3.7)

(a) review DT periodic functions
(b) harmonic sums
(c) derive synthesis equation
(d) derive analysis equation
(e) spectrum plots
(f) stable LTI system response using DTFS
(g) properties of DT FS

TLO-16 CT Fourier Transform (OW §4.0 through 4.7)

(a) derive the CTFT pair from the CTFS
(b) Dirichlet existence conditions
(c) CTFT of the CTFS
(d) Properties of the CT Fourier Transform
  i. linearity
  ii. time shift
  iii. conjugacy
  iv. integration and differentiation: application to LCCDE $\mapsto$ CTFR
  v. time scaling
  vi. duality
  vii. convolution: stable LTI system response using CTFT
  viii. multiplication/modulation
  ix. application of the properties in combination

TLO-17 DT Fourier Transform (OW §5.0 though 5.8)

(a) derive the DTFT from DTFS
(b) DTFT of DTFS
(c) Properties of the DT Fourier Transform
  i. periodicity
  ii. linearity
  iii. index-shift: application to LCCDE $\mapsto$ DTFR
  iv. frequency shift
  v. conjugation
  vi. finite difference and accumulation
  vii. interpolation /index expansion
  viii. frequency differentiation
  ix. Parseval's
  x. convolution: stable LTI system response using DTFT
  xi. multiplication/modulation
  xii. application of the properties in combination

TLO-18 CT Frequency Response (OW §6.1, 6.2, 6.5)

(a) review CTFR as CTFT of impulse response
(b) review CTFR to/from LCCDE
(c) review CTFR to/from block diagram
(d) magnitude-phase representation of the frequency response
(e) frequency response acting on sinusoids
(f) Bode plots
  i. why plot it this way: dB units and log time axis
  ii. how to read them (**not** construct them manually)
  iii. Bode plots in software, e.g. Matlab/Python/Julia
(g) CTFR of first and second order systems

TLO-19 DT Frequency Response (OW §6.1, 6.2, 6.6)

(a) review DTFR as DTFT of impulse response
(b) review DTFR to/from LCCDE
(c) review DTFR to/from block diagram
(d) magnitude-phase representation of the frequency response
(e) frequency response acting on sinusoids
(f) DTFR plots
    i. periodicity
    ii. dB units
    iii. DTFR plots in software, e.g. Matlab/Python/Julia
(g) DTFR of first and second order systems

TLO-20 Frequency Selective Filters in CT (OW §3.9, 3.10, 6.3, 6.4)

(a) ideal low-pass
(b) ideal high-pass
(c) ideal bandpass
(d) ideal notch/bandstop
(e) practical filters
(f) transformations
(g) first and second order systems as building blocks
    i. review LCCDE representation
    ii. review block diagram representation
    iii. review CTFR representation
    iv. CT 1st order RC+buffer
    v. CT Sallen-key

TLO-21 Frequency Selective Filters in DT (OW §3.11, 6.3, 6.4)

(a) ideal low-pass
(b) ideal high-pass
(c) ideal bandpass
(d) ideal notch/bandstop
(e) practical filters
(f) transformations
(g) first and second order systems as building blocks
    i. review LCCDE representation
    ii. review block diagram representation
    iii. review DTFR representation
    iv. DT 1st order implementation in code
    v. DT 2nd order implementation in code

TLO-22 The Discrete Fourier Transform

(a) time window the DTFT to get the DFT

(b) interpreting the index axis as DT and CT frequency

(c) zero-padding

(d) offline or batched filtering using the DFT

(e) briefly mention fast algorithms to compute the DFT = FFT

TLO-23 Sampling (OW §7.1, 7.3, 7.4)

(a) sampling using the impulse train

(b) derive the Nyquist rate

(c) effects of aliasing

(d) practical ADC (sample and hold, SAR, bit-width)

(e) designing anti-aliasing filters

TLO-24 Reconstruction (OW §7.2)

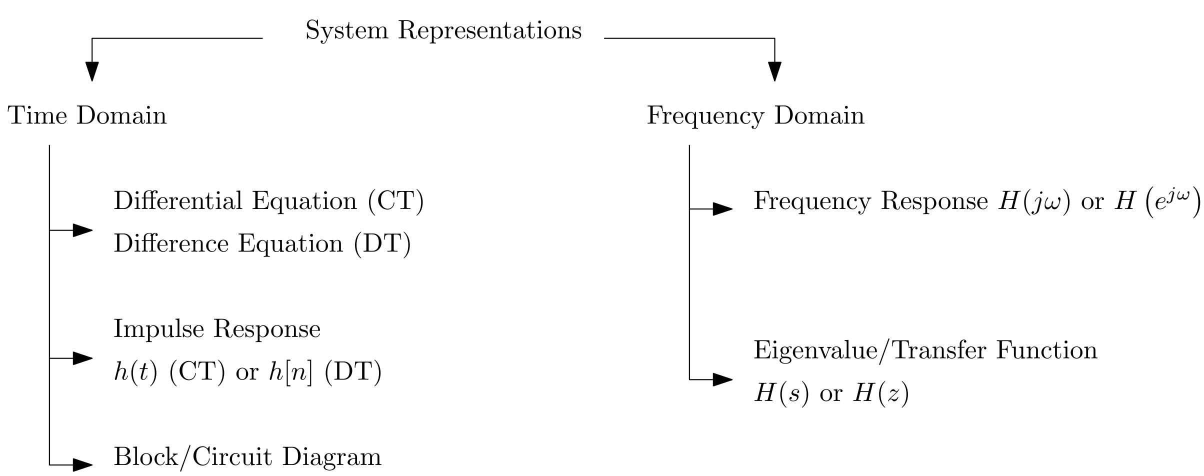
(a) reconstruction as removal of images

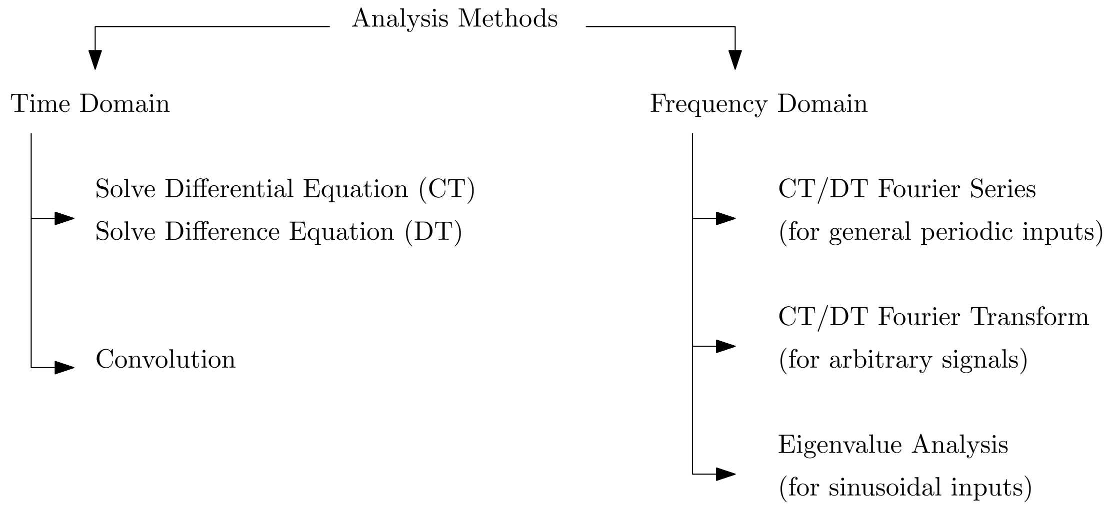
(b) reconstruction as interpolation

(c) practical DAC: R-2R ladder

(d) designing reconstruction filters

1.4 Graphical Outline




Analysis Methods
Time Domain
Frequency Domain
Solve Differential Equation (CT)
Solve Difference Equation (DT)
Convolution
CT/DT Fourier Series
(for general periodic inputs)
CT/DT Fourier Transform
(for arbitrary signals)
Eigenvalue Analysis
(for sinusoidal inputs)

Chapter 2

Continuous-time Signals

A continuous-time (CT) signal is a function of one or more independent variables conveying information about a physical phenomena. This lecture gives an introduction to continuous-time signals as functions. You learn how to characterize such signals in a number of ways and are introduced to two very important signals: the unit impulse and the complex exponential.

2.1 Signals as Functions

In order to reason about signals mathematically we need a representation or *model*. Signals are modeled as functions, mappings between sets

$$f : A \to B$$

where A is a set called the *domain* and B is a set called the *co-domain*.

The most basic classification of signals depends on the sets that makeup the domain and co-domain. We will be interested in two versions of the domain, the reals denoted $\mathbb{R}$ and the integers denoted $\mathbb{Z}$. We will be interested in two versions of the co-domain, the reals $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.

Definition (Analog Signal)**.** If the function $f : \mathbb{R} \to \mathbb{R}$, we call this an analog or real, continuous-time signal, e.g. a voltage at time $t \in \mathbb{R}$, $v(t)$. We will write these as $x(t)$, $y(t)$, etc. The units of t are seconds. Fig. 2.1 shows some graphical representations, i.e. plots.

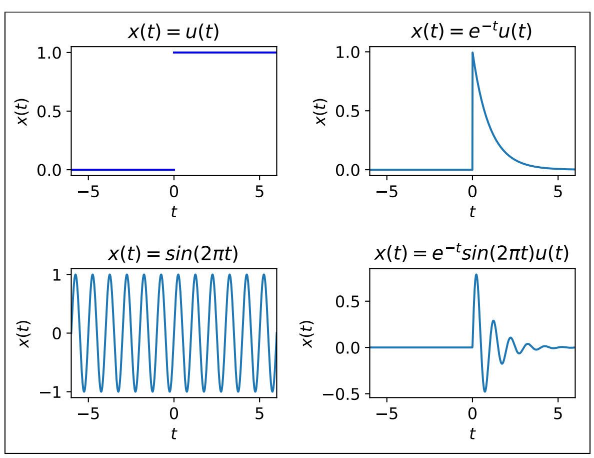


Figure 2.1: Example plots of analog signals.

Definition (Real, Discrete-time Signal)**.** If the function $f : \mathbb{Z} \to \mathbb{R}$, we call this a real, discrete-time signal, e.g. the temperature every day at noon. We will write these as $x[n]$, $y[n]$, etc. Note n is dimensionless.

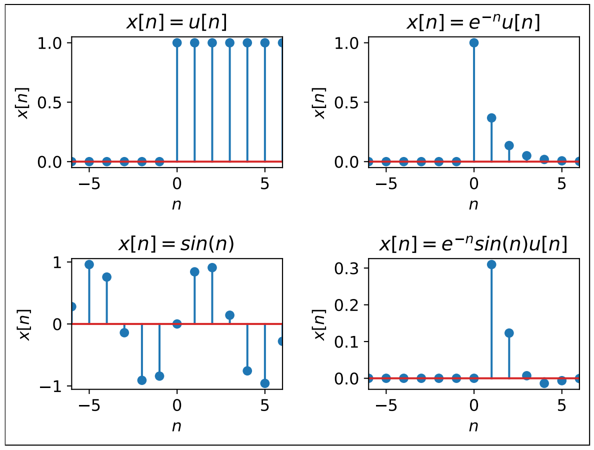


Figure 2.2: Example plots of real discrete-time signals.

Some other possibilities:

- $f : \mathbb{R} \to \mathbb{Z}$, digital, continuous-time signals, e.g. the output of a general purpose pin on a microcontroller
- $f : \mathbb{Z} \to \mathbb{Z}$, digital, discrete-time signals, e.g. the signal on a computer bus

The co-domain can also be complex.

- $f : \mathbb{R} \to \mathbb{C}$, complex-valued, continuous-time signals, e.g.

$$x(t) = e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- $f : \mathbb{Z} \to \mathbb{C}$, complex-valued, discrete-time signals, e.g.

$$x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

Since the domains $\mathbb{R}$ and $\mathbb{Z}$ are usually interpreted as time, we will call these *time-domain* signals. In the time-domain, when the co-domain is $\mathbb{R}$ we call these real signals. All physical signals are real. However complex signals will become important when we discuss the frequency domain.

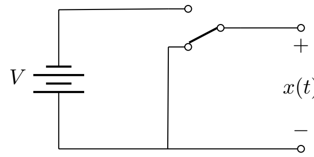
2.2 Primitive Models

We mathematically model signals by combining elementary/primitive functions, for example:

- polynomials: $x(t) = t$, $x(t) = t^2$, etc.
- transendental functions: $x(t) = e^t$, $x(t) = \sin(t)$, $x(t) = \cos(t)$, etc.
- piecewise functions, e.g.

$$x(t) = \begin{cases} f_1(t) & t < 0 \\ f_2(t) & t \geq 0 \end{cases}$$

Example 2.2.1 (Modeling a Switch)**.** Consider a mathematical model of a switch, which moves positions at time $t = 0$.

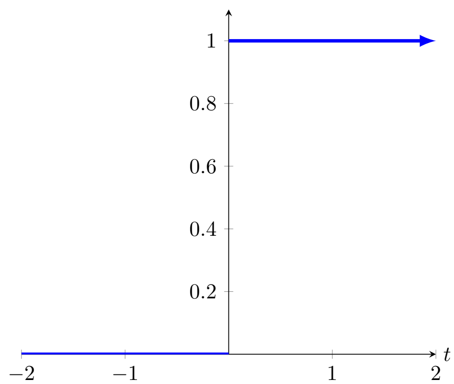


We use this model so much we give it it's own name and symbol: Unit Step, $u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

so a mathematical model of the switch circuit above would be $x(t) = Vu(t)$.

Note: some texts define the step function at $t = 0$ to be 1 or $\frac{1}{2}$. It is typically plotted like so:



Example 2.2.2 (Pure audio tone at "middle C")**.** A signal modeling the air pressure of a specific tone might be

$$x(t) = \sin(2\pi(261.6)t)$$

Example 2.2.3 (Chord)**.** The chord "G", an additive mixture of tones at G, B, and D and might be modeled as

$$x(t) = \sin(2\pi(392)t) + \sin(2\pi(494)t) + \sin(2\pi(293)t)$$

This example shows we can use addition to build-up signals to approximate real signals of interest.

2.3 Basic Transformations

We can also apply transformations to signals to increase their modeling flexibility.

- magnitude scaling

  $$x_2(t) = ax_1(t)$$

  for $a \in \mathbb{R}$.

- derivatives

  $$x_2(t) = x_1'(t) = \frac{dx_1}{dt}(t)$$

- integrals

  $$x_2(t) = \int_{-\infty}^{t} x_1(\tau)\, d\tau$$

- sums
$$y(t) = \sum_i x_i(t)$$
an important example we will see is the CT Fourier series.

- multiplication (modulation)
$$y(t) = x_1(t)x_2(t)$$
For example amplitude modulation $y(t) = x(t)\sin(\omega_0 t)$

- time shift
$$x_2(t) = x_1(t+\tau)$$
  - if $\tau < 0$ it is called a *delay*
  - if $\tau > 0$ it is called an *advance*

- time scaling
$$x_2(t) = x_1\left(\frac{t}{\tau}\right)$$
  - if $\tau > 1$ increasing τ expands in time, slows down the signal
  - if $0 < \tau < 1$ decreasing τ contracts in time, speeds up the signal
  - if $-1 < \tau < 0$ time reverses and increasing τ contracts in time, speeding up the signal
  - if $\tau < -1$ time reverses and decreasing τ expands in time, slows down the signal

  Common uses are time reversal, $x_2(t) = x_1(-t)$, and changing the frequency of of sinusoids.

2.4 Characterization of Signals

There are a few basic ways of characterizing signals.

Definition (Causal CT Signal)**.** A CT signal is *causal* if $x(t) = 0\ \forall t < 0$.

Definition (Anti-Causal CT Signal)**.** A CT signal is *anti-causal* or acausal if $x(t) = 0\ \forall t \geq 0$.

A signal can be written as the sum of a causal and anti-causal signal.

Definition (Periodic Signals)**.** A CT signal is *periodic* if $x(t) = x(t+T)\ \forall t$ for a fixed parameter $T \in \mathbb{R}$ called the *period*.

The simplest periodic signals are those based on the sinusoidal functions.

Definition (Even Signal)**.** A CT signal is *even* if $x(t) = x(-t)\ \forall t$.

Definition (Odd Signal)**.** A CT signal is *odd* if $x(t) = -x(-t)\ \forall t$.

Any CT signal can be written in terms of an even and odd component
$$x(t) = x_e(t) + x_o(t)$$
where
$$\begin{aligned} x_e(t) &= \tfrac{1}{2}\left\{x(t) + x(-t)\right\} \\ x_o(t) &= \tfrac{1}{2}\left\{x(t) - x(-t)\right\} \end{aligned}$$

Definition (Energy of a CT Signal)**.** The *energy* of a CT signal $x(t)$ is defined as a measure of the function

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \ .$$

Definition (Power of a CT Signal)**.** The *power* of a CT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \ .$$

Signals can be characterized based on their energy or power:

- Signals with finite, non-zero energy and zero power are called *energy signals*.
- Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*.

Note, these categories are non-exclusive, some signals are neither energy or power signals.

2.5 Unit Impulse Function

An important CT signal is the unit impulse function, also called the "delta" δ function for the symbol traditionally used to define it. Applying this signal to a system models a "kick" to that system. For example, consider striking a tuning fork. The reason this signal is so important is that it will turn out that the response of the system to this input tells us all we need to know about a linear, time-invariant system!

Definition (CT Impulse Function)**.** The CT impulse function is not really a function at all, but a mathematical object called a "distribution". Some equivalent definitions:

$$\delta(t) = \lim_{\epsilon\to 0} \begin{cases} \frac{1}{2\epsilon} & |t| < \epsilon \\ 0 & \text{else} \end{cases}$$

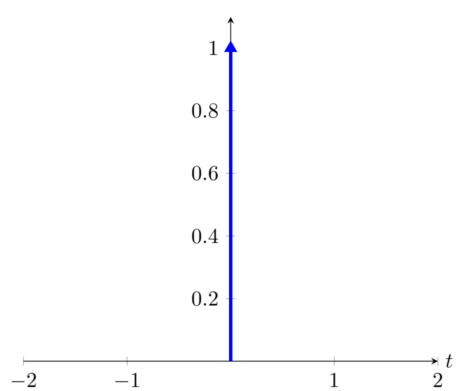
$$\delta(t) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{t^2}{2\epsilon^2}}$$

Note the area under each definition is always one.

In practice we can often use the following definition and some properties, without worrying about the distribution functions.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

which we draw an vertical arrow in plots:



Note the height of the arrow is arbitrary. Often in the case of a non-unit impulse function the area is written in parenthesis near the arrow tip.

The following properties of the impulse function will be used often.

- The area under the unit impulse is unity since by definition

$$\int_{-\infty}^{\infty} \delta(t)\ dt = 1$$

- Sampling property: $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$
- Sifting Property:

$$\int_{a}^{b} x(t)\delta(t - t_0)\ dt = x(t_0)$$

  for any $a < t_0 < b$.

We previously defined the unit step function. The impulse can be defined in terms of the step:

$$\delta(t) = \frac{du}{dt}$$

and vice-versa

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau$$

using the notion of distributions, e.g.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau = \lim_{\epsilon \to 0} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{\tau^2}{2\epsilon^2}}\ d\tau = \lim_{\epsilon \to 0} \frac{1}{2}\left(1 + \text{erf}\left(\frac{t}{\sqrt{2}\epsilon}\right)\right)$$

The step and impulse function are related, but in many cases finding the response of a system to a step input is easier.

We can apply additional transformations to the impulse and step functions to get other useful signals, e.g.

- ramp

$$r(t) = \int_{-\infty}^{t} u(\tau)\ d\tau = tu(t)$$

- causal pulse of width ϵ

$$p(t) = u(t) - u(t - \epsilon)$$

- non-causal pulse of width 2ϵ

$$p(t) = u(t + \epsilon) - u(t - \epsilon)$$

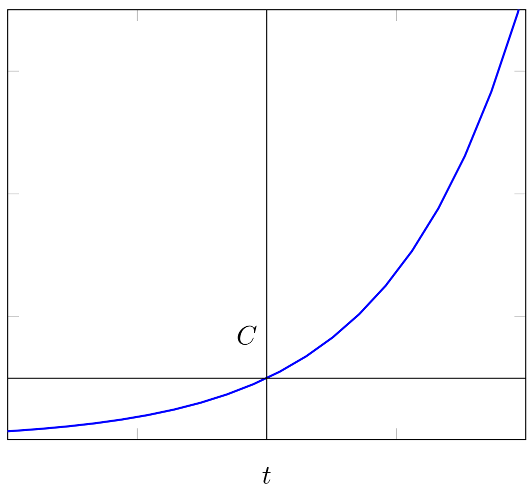
2.6 CT Complex Exponential

One of the most important signals in systems theory is the complex exponential:

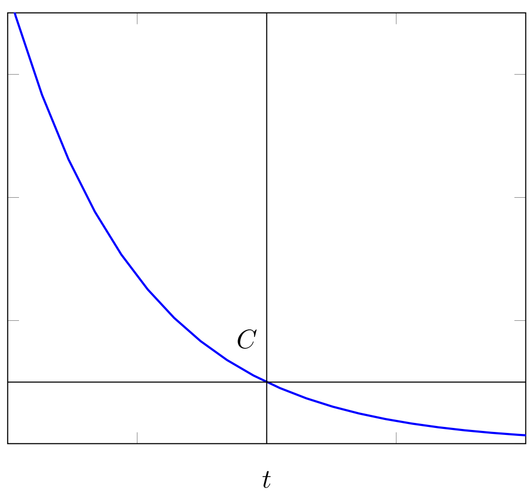
$$x(t) = C\,e^{at}$$

where the parameters $C, a \in \mathbb{C}$ in general.

When C and a are both real ($\text{Im}(C) = \text{Im}(a) = 0$), we have the familiar exponential. When $a > 0$ and $C > 0$, $x(t) = Ce^{at}$ looks like:



When $a < 0$ and $C > 0$, $x(t) = Ce^{at}$ looks like:



If $C < 0$ the signals reflect about the time axis.

To get the pure sinusoidal case, let $C \in \mathbb{R}$ and a be purely imaginary: $a = j\omega_0$:

$$x(t) = Ce^{j\omega_0 t}$$

where ω_0 is the frequency (in radians/sec). This is called the complex sinusoid.

By Euler's identity:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

and

$$\mathrm{Re}(x(t)) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\mathrm{Im}(x(t)) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

are both real sinusoids.

Note that the sinusoids are periodic. Recall a signal $x(t)$ is periodic with period T if

$$x(t) = x(t+T)\ \forall t$$

In the case of the complex sinusoid

$$Ce^{j\omega_0 t} = Ce^{j\omega_0 (t+T)} = Ce^{j\omega_0 t}\ \underbrace{e^{j\omega_0 T}}_{\text{must be 1}}$$

- if $\omega_0 = 0$ this is true for all T
- if $\omega_0 \neq 0$, then to be periodic $\omega_0 T = 2\pi m$ for $m = \pm 1, \pm 2, \cdots$. The smallest T for which this is true is the *fundamental period* T_0

$$T_0 = \frac{2\pi}{|\omega_0|}$$

  or equivalently $\omega_0 = \frac{2\pi}{T_0}$

Some useful properties of sinusoids:

- If $x(t)$ is periodic with period T and g is any function then $g(x(t))$ is periodic with period T.
- If $x_1(t)$ is periodic with period T_1 and $x_2(t)$ is periodic with period T_2, and if there exists positive integers a, b such that

$$aT_1 = bT_2 = P$$

  then $x_1(t) + x_2(t)$ and $x_1(t)x_2(t)$ are periodic with period P

The last property implies that both T_1 and T_2 must both be rational in π or neither should be. For example

- $x(t) = \sin(2\pi t) + \cos(5\pi t)$ is periodic
- $x(t) = \sin(2t) + \cos(5t)$ is periodic
- $x(t) = \sin(2\pi t) + \cos(5t)$ is **not** periodic

When the parameter C is complex we get a phase shift. Again let $a = j\omega_0$. When C is complex we can write it as $C = Ae^{j\phi}$ where $A = |C|$ and $\phi = \angle C$. Then

$$x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)}$$

and

$$\mathrm{Re}(x(t)) = A\cos(\omega_0 t + \phi)$$

$$\mathrm{Im}(x(t)) = A\sin(\omega_0 t + \phi)$$

Since sin is a special case of cos, i.e. $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$, the general real sinusoid is

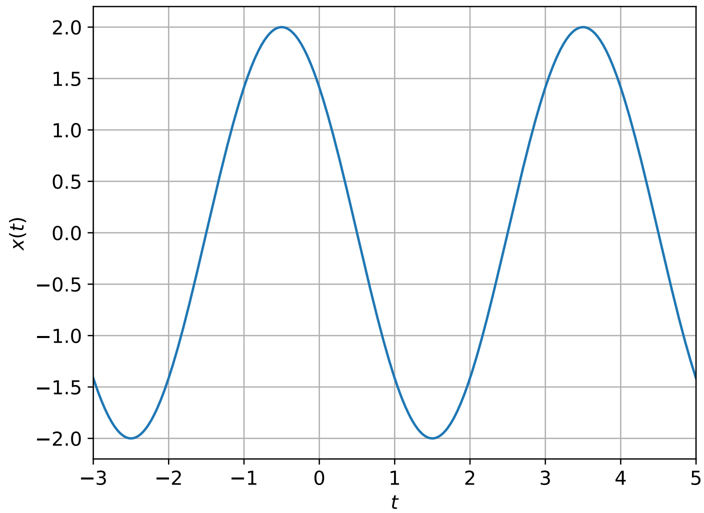
$$A\cos(\omega_0 t + \phi)$$

- A is called the amplitude
- ω_0 is again the frequency in radians/sec.

- ϕ is called the phase shift and is related to a time shift T_s by

$$\phi = \omega_0 T_s$$

For example the signal graphically represented as follows



has the functional representation

$$x(t) = 2\cos\left(\frac{\pi}{2}(t+\tfrac{1}{2})\right) = 2\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$

2.6.1 Energy of CT complex sinusoid

Recall the energy of a CT signal $x(t)$ is

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \; .$$

Substituting $x(t) = e^{j\omega_0 t}$ and letting $T = NT_0$

$$E_x = \lim_{N\to\infty} \int_{-NT_0}^{NT_0} \underbrace{|e^{j\omega_0 t}|^2}_{\text{always } 1} \; dt = \lim_{N\to\infty} 2NT_0 = \infty$$

2.6.2 Power of CT complex sinusoid

Recall the power of a CT signal $x(t)$ is

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \; .$$

Again, substituting $x(t) = e^{j\omega_0 t}$ and letting $T = NT_0$

$$P_x = \lim_{N\to\infty} \frac{1}{2NT_0} \int_{-NT_0}^{NT_0} \underbrace{|e^{j\omega_0 t}|^2}_{\text{always } 1} \; dt = \lim_{N\to\infty} \frac{1}{2NT_0} 2NT_0 = 1$$

2.6.3 Harmonics

Two CT complex sinusoids are *harmonics* of one another is both are periodic in T_0. This occurs when

$$x_k(t) = e^{jk\omega_0 t} \text{ for } k = 0, \pm 1, \pm 2, \cdots$$

The term comes from music where the vibrations of a string instrument are modeled as a weighted combination of harmonic tones.

2.6.4 Geometric interpretation of the Complex Exponential

In the general case we get a sinusoid signal modulated by an exponential. Let $C = Ae^{j\phi}$ and $a = r + j\omega_0$, then

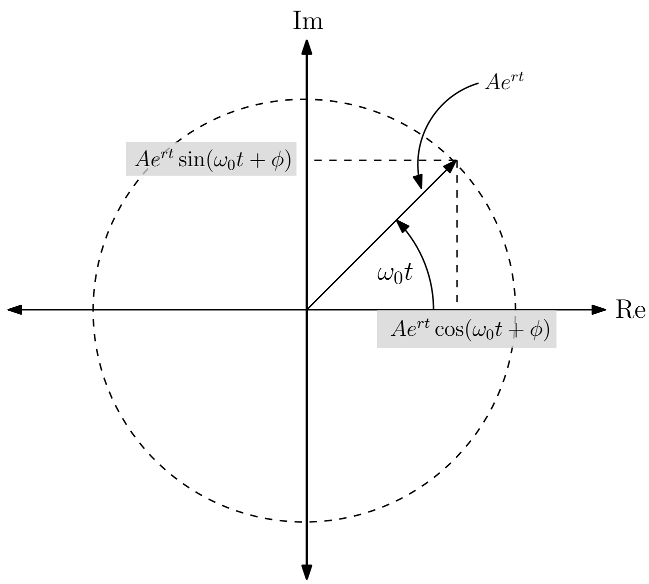
$$x(t) = Ce^{at} = Ae^{j\phi}e^{(r+j\omega_0)t}$$

Expanding the terms and using Euler's identity gives:

$$x(t) = \underbrace{Ae^{rt}\cos(\omega_0 t + \phi)}_{\text{Re part}} + j\underbrace{Ae^{rt}\sin(\omega_0 t + \phi)}_{\text{Im part}}$$

Each part is a real sinusoid whose amplitude is modulated by a real exponential.

An important visualization of the general case is to view the signal $x(t)$ as a vector rotating counter-clockwise in the complex plane for positive t.



For $r < 0$ the tip of the arrow traces out an inward spiral, whereas for $r > 0$ it traces out an outward spiral. For $r = 0$ it traces out the unit circle.

2.7 Solved Problems

1. Consider a signal described by the function

$$x(t) = e^{-3t}\sin(10\pi t)u(t)$$

a) Determine the magnitude and phase of $x\left(\frac{1}{20}\right)$
Solution: Substituting $t = \frac{1}{20}$ gives

$$x\left(\frac{1}{20}\right) = e^{-3\frac{1}{20}} \sin\left(10\pi\frac{1}{20}\right) u\left(\frac{1}{20}\right) = e^{-\frac{3}{20}} \approx 0.86$$

Since the signal is purely real and exponential is always positive, the magnitude is

$$\left|x\left(\frac{1}{20}\right)\right| = \left|e^{-\frac{3}{20}}\right| = e^{-\frac{3}{20}} \approx 0.86$$

and the phase is

$$\angle x\left(\frac{1}{20}\right) = 0$$

b) Using Matlab, plot the signal $|x(t)|$ between $[-2, 2]$. Give your code and embed the plot.

Solution:

```
% Solution to PS01 - Problem 1b
t = -2:0.001:2;
x = exp(-3*t).*sin(10*pi*t).*heaviside(t);
hp = plot(t,abs(x));
grid on;
xh = xlabel('t');
yh = ylabel('|x(t)|');
th = title('Plot for PS01 Problem 1b');

% make the plot more readable
set(gca, 'FontSize', 12, 'Box', 'off', 'LineWidth', 2);
set(hp, 'linewidth', 2);
set([xh, yh, th], 'FontSize', 12);

set(gcf, 'PaperPositionMode', 'auto');
print -dpng ps1p1b.png
```

2. Find a solution to the differential equation

$$\frac{dy}{dt}(t) + 9y(t) = e^{-t}$$

for $t \geq 0$, when $y(0) = 1$.

Solution: The homogeneous equation is

$$\frac{dy_h}{dt}(t) + 9y_h(t) = 0$$

with initial condition $y_h(0) = 1$. Its solution is of the form

$$y_h(t) = C\, e^{-9t}$$

for constant C. Using the initial condition

$$y_h(0) = C\, e^{-0} = C = 1$$

gives

$$y_h(t) = e^{-9t}$$

The particular solution is of the form

$$y_p(t) = C_1 e^{-t} + C_2 e^{-9t}$$

Substitution and equating coefficients gives $C_1 = \frac{1}{8}$ and $C_2 = -\frac{1}{8}$. The total solution is the sum of the two solutions or

$$y(t) = \frac{1}{8}e^{-t} - \frac{1}{8}e^{-9t} + e^{-9t} = \frac{1}{8}e^{-t} + \frac{7}{8}e^{-9t}$$

3. Compute the integral

$$\int_{-\infty}^{\infty} e^{-t^2}\, \delta(t-10)\, dt$$

where $\delta(t)$ is the delta function.
Solution:

Using the sifting property of the delta function

$$\int_a^b f(t)\, \delta(t-t_0)\, dt = f(t_0)$$

for $a < t_0 < b$, we get

$$\int_{-\infty}^{\infty} e^{-t^2}\, \delta(t-10)\, dt = e^{-100} \approx 0$$

Chapter 3

Discrete-time Signals

Recall from the previous meeting that a discrete-time (DT) signal is modeled as a function $f : \mathbb{Z} \to \mathbb{C}$. We will write these as $x[n]$, $y[n]$, etc. Note n is dimensionless. These are graphically plotted as stem or "lollipop" plots, as demonstrated in Fig. 2.2.

Since the domain $\mathbb{Z}$ is usually interpreted as a time index, we will still call these *time-domain* signals. In the time-domain, when the co-domain is $\mathbb{R}$ we call these real DT signals. Unlike with CT signals there are no physical limitations requiring DT signals to be real, since in discrete hardware, a value at a given index can be a complex number, i.e. just a pair of numbers. However it is computationally advantageous to restrict ourselves to real arithmetic and such signals are often converted to or from CT signals, which do have to be real. For this reason, real DT signals dominate in models.

3.1 Primitive Models

As with CT signals, we mathematically model DT signals by combining elementary/primitive functions, for example:

- polynomials: $x[n] = n$, $x[n] = n^2$, etc.
- transendental functions: $x[n] = e^n$, $x[n] = \sin(n)$, $x[n] = \cos(n)$, etc.
- piecewise functions, e.g.

$$x[n] = \begin{cases} f_1[n] & n < 0 \\ f_2[n] & n \geq 0 \end{cases}$$

Example 3.1.1 (Unit Step)**.** The DT counterpart of the CT step function is the *DT Unit Step*, $u[n]$:

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Note, there are not continuity issues at $n = 0$ as DT functions have discrete domains.

Example 3.1.2 (Sampled Pure audio tone at "middle C")**.** A *sampled* signal modeling the air pressure of a specific tone, sampled at 8kHz, might be

$$x[n] = \sin\left(2\pi(261.6)\tfrac{1}{8000}n\right)$$

Such DT signals are commonly used in digital music generation, storage, and playback.

Example 3.1.3 (Sampled Chord)**.** Similarly, the sampled chord "G", an additive mixture of tones at G, B, and D and might be modeled as

$$x[n] = \sin\left(2\pi(392)\tfrac{1}{8000}n\right) + \sin\left(2\pi(494)\tfrac{1}{8000}n\right) + \sin\left(2\pi(293)\tfrac{1}{8000}n\right)$$

again sampled at 8kHz. This example shows we can use addition to build-up signals to approximate real signals of interest.

3.2 Basic Transformations

Similar to CT signals, we can also apply transformations to DT signals to increase their modeling flexibility.

- magnitude scaling

  $$x_2[n] = ax_1[n]$$

  for $a \in \mathbb{R}$.

- time differences

  $$x_2[n] = x_1[n] - x_1[n-1]$$

- running sums

  $$x_2[n] = \sum_{m=-\infty}^{n} x_1[m]$$

- sums

  $$y[n] = \sum_i x_i[n]$$

  an important example we will see is the DT Fourier series.

- multiplication (modulation)

  $$y[n] = x_1[n]x_2[n]$$

- time index shift

  $$x_2[n] = x_1[n+m]$$

  - if $m < 0$ it is called a *delay*
  - if $m > 0$ it is called an *advance*

- time reversal

  $$x_2[n] = x_1[-n]$$

- decimation

  $$y[n] = x[mn]$$

  for $m \in \mathbb{Z}^+$.

  - e.g. for $m = 2$ only keep every other sample
  - e.g. for $m = 3$ only keep every third sample
  - etc.

- interpolation

  $$y[n] = \begin{cases} x\left[\frac{n}{m}\right] & n = 0\ ,\ \pm m,\ ,\ \pm 2m \cdots \\ 0 & \text{else} \end{cases}$$

  When $m = 2$ this inserts a zero sample between every sample of the signal.

3.3 Characterization of Signals

There are a few basic ways of characterizing DT signals.

Definition (Causal DT Signal)**.** A DT signal is *causal* if $x[n] = 0 \; \forall n < 0$.

Definition (Anti-Causal DT Signal)**.** A DT signal is *anti-causal* or acausal if $x[n] = 0 \; \forall n \geq 0$.

A DT signal can be written as the sum of a causal and anti-causal signal.
A DT signal is periodic if $x[n] = x[n + N] \; \forall n$ for a fixed period $N \in \mathbb{Z}$.
A DT signal is even if $x[n] = x[-n] \; \forall n$.
A DT signal is odd if $x[n] = -x[-n] \; \forall n$.
Any DT signal can be written in terms of an even and odd component

$$x[n] = x_e[n] + x_o[n]$$

where

$$\begin{aligned} x_e[n] &= \tfrac{1}{2}\left\{x[n] + x[-n]\right\} \\ x_o[n] &= \tfrac{1}{2}\left\{x[n] - x[-n]\right\} \end{aligned}$$

Analogous to CT signals, the energy of a DT signal is

$$E_x = \lim_{N\to\infty} \sum_{-N}^{N} |x[n]|^2 \; .$$

And the power of a DT signal is the energy averaged over an interval as that interval tends to infinity.

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 \; .$$

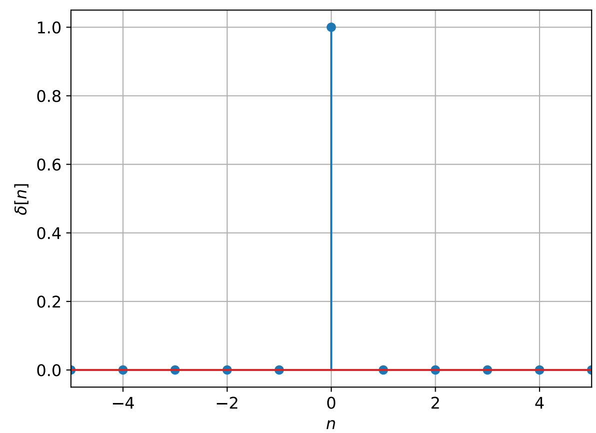
DT Signals with finite, non-zero energy and zero power are called *energy signals*. DT Signals with finite, non-zero power (and by implication infinite energy) are called *power signals*. These categories are non-exclusive, some signals are neither energy or power signals.

3.4 DT Unit Impulse Function

In DT the unit impulse function, denoted $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

Note this definition is straightforward compared to the CT impulse as there are no continuity issues and it is not defined in terms of a distribution. It is typically drawn as



Some useful properties of the DT impulse function are:

- Energy is 1: $\sum_{n=-\infty}^{\infty} \delta[n] = 1$
- Sampling: $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
- Sifting: $\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$

The impulse can be defined in terms of the step:

$$\delta[n] = u[n] - u[n-1]$$

and vice-versa

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

3.5 DT Complex Exponential

The DT Complex Exponential is defined in a similar fashion the the CT version, but with some important differences. The general DT complex exponential is given by the expression:

$$x[n] = Ce^{\beta n}$$

where in general $C \in \mathbb{C}$ and $\beta \in \mathbb{C}$. It is sometimes convenient (for reasons we will see later) to write this as

$$x[n] = C\alpha^n$$

where $\alpha = e^{j\theta}$ is a complex number $\alpha = \cos(\theta) + j\sin(\theta)$.

We now examine several special cases.

3.5.1 DT Complex Exponential: real case

Let C and α be real, then there are four intervals of interest:

- $\alpha > 1$
- $0 < \alpha < 1$
- $-1 < \alpha < 0$
- $\alpha < -1$

Each of these are shown in Fig. 3.1.

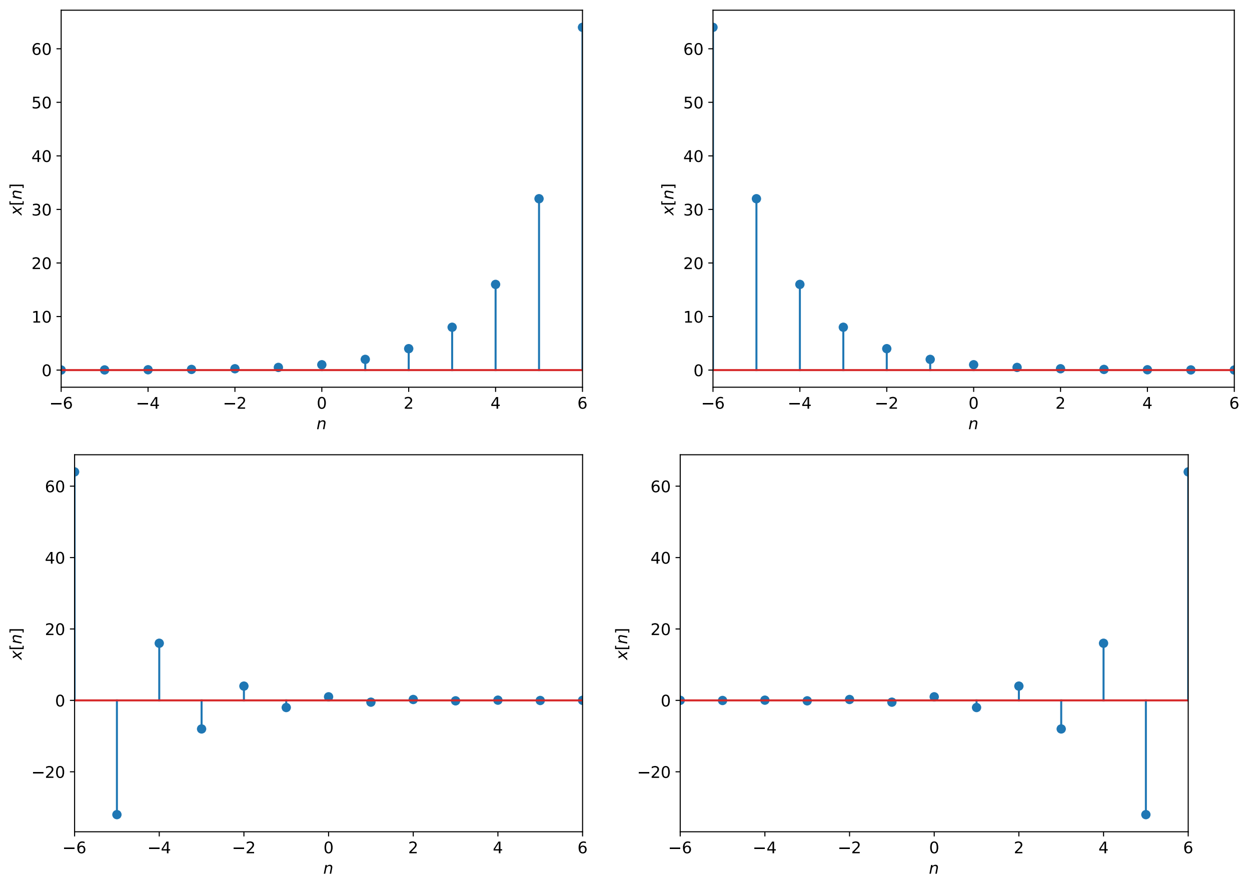


Figure 3.1: DT Complex Exponential: real case, four intervals of interest.

3.5.2 DT Complex Exponential: sinusoidal case

Let $C = 1$. When β is purely imaginary, $\beta = j\omega_0$

$$x[n] = e^{j\omega_0 n}$$

As in CT, by Euler's identity:

$$e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

and

$$\text{Re}(x[n]) = \cos(\omega_0 n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$
$$\text{Im}(x[n]) = \sin(\omega_0 n) = \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)$$

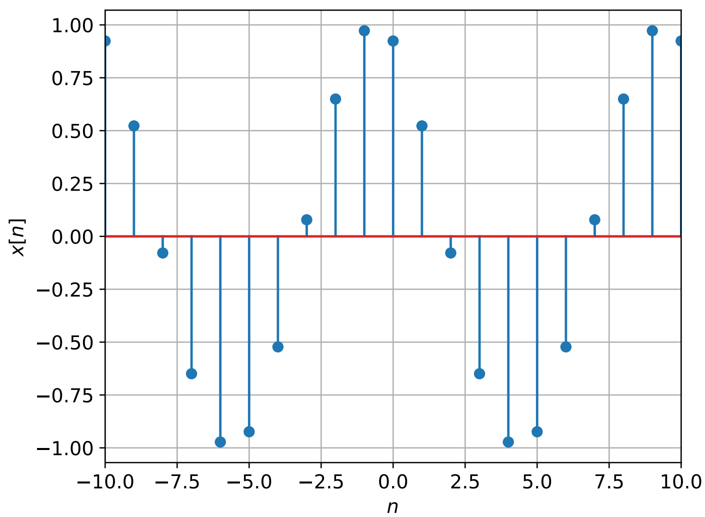
The energy and power are the same as for the CT complex sinusoid: $E_x = \infty$ and $P_x = 1$.

3.5.3 DT Complex Exponential: sinusoidal case with phase shift

The general DT sinusoid is

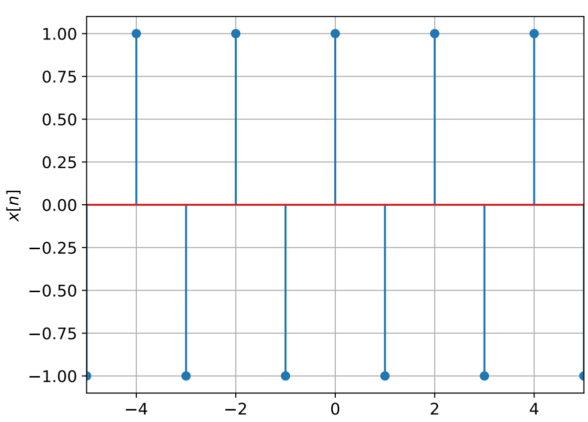
$$x[n] = A\cos(\omega_0 n + \phi)$$

- A is called the amplitude
- ϕ is called the phase shift
- ω_0 is now in radians (assuming n is dimensionless)



For CT sinusoids as ω_0 increases the signal oscillates faster and faster. However for DT sinusoids there is a "fastest" oscillation.

$$e^{j\omega_0 n}\big|_{\omega_0=\pi} = e^{j\pi n} = (-1)^n$$



3.5.4 Properties of DT complex sinusoid

If we consider two frequencies: ω_0 and $\omega_0 + 2\pi$. In the first case:

$$x[n] = e^{j\omega_0 n}$$

In the second case:

$$\begin{aligned} x[n] &= e^{j(\omega_0+2\pi)n} \\ &= \underbrace{e^{j2\pi n}}_{\text{always 1}} \; e^{j\omega_0 n} \\ &= e^{j\omega_0 n} \end{aligned}$$

Thus the two are the same signal. This has important implications later in the course.

Another difference between CT and DT complex sinusoids is periodicity. Recall for a DT signal to be periodic with period N

$$x[n] = x[n+N] \; \forall n$$

Substituting the complex sinusoid

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} = e^{j\omega_0 n} e^{j\omega_0 N}$$

requires $e^{j\omega_0 N} = 1$, which implies $\omega_0 N$ is a multiple of 2π:

$$\omega_0 N = 2\pi m \quad m = \pm 1, \pm 2, \cdots$$

or equivalently

$$\frac{|\omega_0|}{2\pi} = \frac{m}{N}$$

thus ω_0 must be a rational multiple of π.

Two DT complex sinusoids are harmonics of one another is both are periodic in N, i.e when

$$x_k(t) = e^{jk\frac{2\pi}{N}n} \text{ for } k = 0, \pm 1, \pm 2, \cdots$$

This implies there are only N distinct harmonics in DT.

3.5.5 DT Complex Exponential: general case

In the general case we get a sinusoid signal modulated by an exponential. Let $C = Ae^{j\phi}$ and $\beta = r + j\omega_0$, then

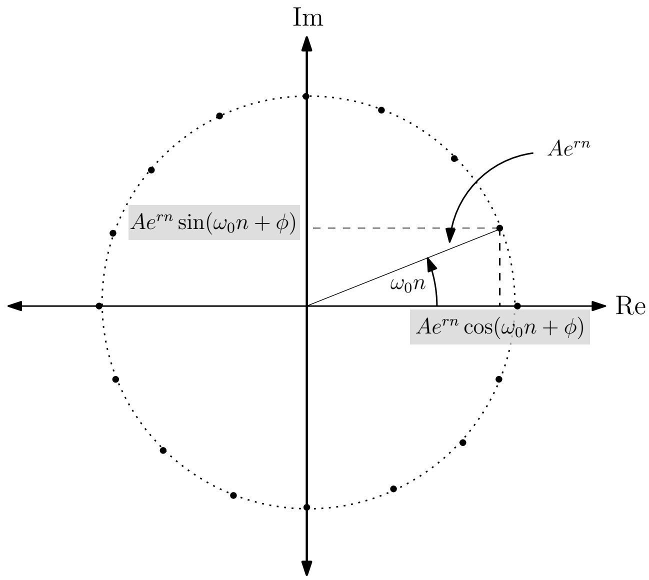
$$x[n] = Ce^{\beta n} = Ae^{j\phi}e^{(r+j\omega_0)n}$$

Expanding the terms and using Euler's identity gives:

$$x[n] = \underbrace{Ae^{rn}\cos(\omega_0 n + \phi)}_{\text{Re part}} + j\underbrace{Ae^{rn}\sin(\omega_0 n + \phi)}_{\text{Im part}}$$

Each part is a real sinusoid whose amplitude is modulated by a real exponential.

The visualization of the general case is to view the signal $x[n]$ as a vector rotating through fixed angles in the complex plane.



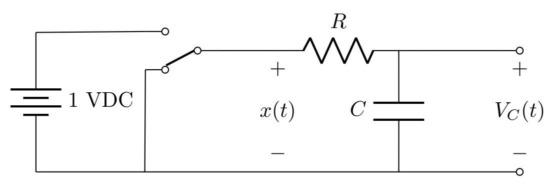
Im
Re
Ae^{rn}
$Ae^{rn}\sin(\omega_0 n+\phi)$
$\omega_0 n$
$Ae^{rn}\cos(\omega_0 n+\phi)$

Chapter 4

CT Systems as Linear Constant Coefficient Differential Equations

Recall a system is a transformation of signals, turning the input signal into the output signal. While this might seem like a new concept to you, you already know something about them from your differential equations course, i.e. MATH 2214 and your circuits course.

For example, consider the following circuit:



where the switch moves position at $t = 0$. The governing equation for the circuit when $t < 0$ is

$$\frac{dV_c}{dt}(t) + \frac{1}{RC}V_c(t) = 0$$

a *homogeneous* differential equation of first-order. From a DC analysis, the initial condition on the capacitor voltage is $V_C(0^-) = 0$, so there is no current flowing prior to $t = 0$ and the solution is $V_C(t) = 0$ for $t < 0$.

After the switch is thrown, the governing equation for the circuit when $t \geq 0$ is

$$\frac{dV_c}{dt}(t) + \frac{1}{RC}V_c(t) = \frac{1}{RC}$$

Since the voltage across the capacitor cannot change instantaneously $V_C(0^-) = V_C(0^+) = 0$, giving the auxillary condition necessary to solve this equation, which has the form

$$V_C(t) = A + Be^{-\frac{1}{RC}t}$$

Using the auxillary condition we find

$$V_C(0) = A + Be^{-\frac{1}{RC}0} = A + B = 0 \text{ which implies } B = -A$$

Subsitution back into the differential equation and equating the coefficients gives $A = 1$. Thus the voltage for $t \geq 0$ is

$$V_C(t) = 1 - e^{-\frac{1}{RC}t}$$

Suppose we consider the voltage after the switch as the input signal $x(t)$ to the system composed of the series RC. As we have seen previously a mathematical model of the switch is the unit step $x(t) = u(t)$.

Suppose we consider the capacitor voltage at the output of the system, so that $y(t) = V_C(t)$. Then we can consider the system to be represented by the *linear, constant-coefficient differential equation*

$$\frac{dy}{dt}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

where $x(t) = u(t)$ and the solution $y(t)$ is the *step response*

$$y(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

As we will see later this representation of systems is central to the course, so we take some time here to review the solution of such equations.

4.1 Solving Linear, Constant Coefficient Differential Equations

A linear, constant coefficient (LCC) differential equation is of the form

$$a_0\, y + a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} + \cdots + a_N \frac{d^Ny}{dt^N} = b_0\, x + b_1 \frac{dx}{dt} + b_2 \frac{d^2x}{dt^2} + \cdots + b_M \frac{d^Mx}{dt^M}$$

which can be written compactly as

$$\sum_{k=0}^{N} a_k \frac{d^ky}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^kx}{dt^k}$$

It is helpful to clean up this notation using the derivative operator $D^n = \frac{d^n}{dt^n}$. For example $D^2y = \frac{d^2y}{dt^2}$ and $D^0y = y$. To give for form as

$$\sum_{k=0}^{N} a_k\, D^k y = \sum_{k=0}^{M} b_k\, D^k x$$

We can factor out the derivative operators

$$a_0y + a_1Dy + a_2D^2y + \cdots + a_ND^Ny = b_0\, x + b_1\, Dx + b_2\, D^2x + \cdots + b_M\, D^Mx$$

$$\underbrace{\left(a_0 + a_1D + a_2D^2 + \cdots + a_ND^N\right)}_{\text{Polynomial in } D, Q(D)} y = \underbrace{\left(b_0 + b_1D + b_2D^2 + \cdots + b_MD^M\right)}_{\text{Polynomial in } D, P(D)} x$$

to give:

$$Q(D)y = P(D)x$$

You learned how to solve these in differential equations (Math 2214) as

$$y(t) = y_\text{h}(t) + y_\text{p}(t)$$

The term $y_\text{h}(t)$ is the solution of the homogeneous equation

$$Q(D)y = 0$$

Given the $N-1$ auxillary conditions $y(t_0) = y_0$, $Dy(t_0) = y_1$, $D^2y(t_0) = y_2$, up to $D^{N-1}y(t_0) = y_{N-1}$.

The term $y_\text{p}(t)$ is the solution of the particular equation

$$Q(D)y = P(D)x$$

for a given $x(t)$.

Rather than recapitulate the solution to $y_\mathrm{h}(t)$ and $y_\mathrm{p}(t)$ in the general case we focus on the homogeneous solution $y_\mathrm{h}(t)$ only. The reason is that we will use the homogeneous solution to find the impulse response below and take a different approach to solving the general case for an arbitrary input using the impulse response and convolution (next week).

To solve the homogenous system:

Step 1: Find the *characteristic equation* by replacing the derivative operators by powers of an arbitrary complex variable s.

$$Q(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_N D^N$$

becomes

$$Q(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_N s^N$$

a polynomial in s with N roots s_i for $i = 1, 2, \cdots, N$ such that

$$(s - s_1)(s - s_2)\cdots(s - s_N) = 0$$

Step 2: Select the form of the solution, a sum of terms corresponding to the roots of the characteristic equation.

- For a real root $s_1 \in \mathbb{R}$ the term is of the form
$$C_1 e^{s_1 t}.$$
- For a pair of complex roots (they will always be in pairs) $s_{1,2} = a \pm jb$ the term is of the form
$$C_1 e^{s_1 t} + C_2 e^{s_2 t} = e^{at}\left(C_3 \cos(bt) + C_4 \sin(bt)\right) = C_5 e^{at}\cos(bt + C_6).$$
- For a repeated root s_1, repeated r times, the term is of the form
$$e^{s_1 t}(C_0 + C_1 t + \cdots + C_{r-1} t^{r-1}).$$

Step 3: Solve for the unknown constants in the solution using the auxillary conditions.

We now examine two common special cases, when $N = 1$ (first-order) and when $N = 2$ (second-order).

4.1.1 First-Order Homogeneous LCCDE

Consider the first order homogeneous differential equation

$$\frac{dy}{dt}(t) + ay(t) = 0 \text{ for } a \in \mathbb{R}$$

The characteristic equation is given by

$$s + a = 0$$

which has a single root $s_1 = -a$. The solution is of the form

$$y(t) = Ce^{s_1 t} = Ce^{-at}$$

where the constant C is found using the auxillary condition $y(t_0) = y_0$.

Example 4.1.1. Consider the homogeneous equation

$$\frac{dy}{dt}(t) + 3y(t) = 0 \text{ where } y(0) = 10$$

The solution is

$$y(t) = Ce^{-3t}$$

To find C we use the auxillary condition

$$y(0) = Ce^{-3\cdot 0} = C = 10$$

and the final solution is

$$y(t) = 10e^{-3t}$$

4.1.2 Second-Order Homogeneous LCCDE

Consider the second-order homogeneous differential equation

$$\frac{d^2y}{dt^2}(t) + a\frac{dy}{dt}(t) + by(t) = 0 \text{ for } a, b \in \mathbb{R}$$

The characteristic equation is given by

$$s^2 + as + b = 0$$

Let's look at several examples to illustrate the functional forms.

Example 4.1.2.

$$\frac{d^2y}{dt^2}(t) + 7\frac{dy}{dt}(t) + 10y(t) = 0$$

The characteristic equation is given by

$$s^2 + 7s + 10 = 0$$

which has roots $s_1 = -2$ and $s_2 = -5$. Thus the form of the solution is

$$y(t) = C_1 e^{-2t} + C_2 e^{-5t}$$

Example 4.1.3.

$$\frac{d^2y}{dt^2}(t) + 2\frac{dy}{dt}(t) + 5y(t) = 0$$

The characteristic equation is given by

$$s^2 + 2s + 5 = 0$$

which has complex roots $s_1 = -1 + j2$ and $s_1 = -1 - j2$. Thus the form of the solution is

$$y(t) = e^{-t}\left(C_1 \cos(2t) + C_2 \sin(2t)\right)$$

Example 4.1.4.

$$\frac{d^2y}{dt^2}(t) + 2\frac{dy}{dt}(t) + y(t) = 0$$

The characteristic equation is given by

$$s^2 + 2s + 1 = 0$$

which has a root $s_1 = -1$ repeated $r = 2$ times. Thus the form of the solution is

$$y(t) = e^{-t}\left(C_1 + C_2 t\right)$$

In each of the above cases the constants, C_1 and C_2, are found using the auxillary conditions $y(t_0)$ and $y\prime(t_0)$.

4.2 Finding the impulse response of a system described by a LCCDE

As we will see next week an important response of a system is the one that corresponds to an impulse input, i.e. the *impulse response* $y(t) = h(t)$ when $x(t) = \delta(t)$. Thus we focus here on a recipe for solving LCCDEs for this special case when $M \leq N$. We will skip the derivation of why this works.

Our goal is to find the solution to $Q(D)y = P(D)x$ when $x(t) = \delta(t)$.

Step 1: Rearrange the LCCDE so that $a_N = 1$, i.e. divide through by a_N to put it into a standard form.

Step 2: Let $y_h(t)$ be the homogeneous solution to $Q(D)y_h = 0$ for auxillary conditions

$$D^{N-1}y_h(0^+) = 1\ ,\ D^{N-2}y_h(0^+) = 0\ ,\ \text{etc. } y_h(0^+) = 0$$

Step 3: Assume a form for $h(t)$ given by:

$$h(t) = \underbrace{b_N\delta(t)}_{=0 \text{ unless } N=M} + \underbrace{[P(D)y_h]}_{\text{apply } P(D) \text{ to } y_n(t)} u(t)$$

Recall from above the homogeneous solution depends on the roots of the characteristic equation $Q(D) = 0$.

- roots are either real, or
- roots occur in complex conjugate pairs, or
- repeated roots.

Example 4.2.1. Find the impulse response of the LCCDE

$$2\frac{dy}{dt}(t) + 2y(t) = 2x(t)$$

In the standard for the LCCDE is

$$\frac{dy}{dt}(t) + y(t) = x(t)$$

The characteristic equation is given by

$$s + 1 = 0$$

which has a single root $s_1 = -1$. The solution is of the form

$$y_h(t) = Ce^{-t}$$

with the special auxillary condition $y(0) = 1$, so that

$$y_h(t) = e^{-t}$$

Since $P(D) = 1$ and $N = 1 \neq M = 0$ the impulse response is

$$h(t) = \underbrace{b_N\delta(t)}_{=0} + \left[\underbrace{P(D)}_{1}\, y_h(t)\right] u(t) = e^{-t}u(t)$$

Example 4.2.2. Find the impulse response of the LCCDE

$$\frac{dy}{dt}(t) + y(t) = \frac{dx}{dt}(t) + x(t)$$

It is already in the standard form. The homogeneous solution is the same as in Example 1,

$$y_h(t) = e^{-t}$$

however now $M = N = 1$ with $b_1 = 1$ and $P(D) = D + 1$. Thus, the impulse response is

$$h(t) = \underbrace{b_N}_{=1}\delta(t) + \left[\underbrace{P(D)}_{D+1}\, y_h(t)\right] u(t) = \delta(t) + \left\{[D+1]e^{-t}\right\} u(t) = \delta(t) + [-e^{-t} + e^{-t}]u(t) = \delta(t)$$

Example 4.2.3. Find the impulse response of the LCCDE

$$\frac{d^2y}{dt^2}(t) + 7\frac{dy}{dt}y(t) + 10y(t) = x(t)$$

It is already in the standard form. The characteristic equation is given by

$$s^2 + 7s + 10 = 0$$

which has roots $s_1 = -2$ and $s_2 = -5$. Thus the form of the solution is

$$y_h(t) = C_1 e^{-2t} + C_2 e^{-5t}$$

The special auxillary conditions are $y_h(0) = 0$ and $y_h'(0) = 1$. Using these conditions

$$y_h(0) = C_1 e^{-2t} + C_2 e^{-5t}|_{t=0} = C_1 + C_2 = 0$$

$$y_h'(0) = -2C_1 e^{-2t} - 5C_2 e^{-5t}|_{t=0} = -2C_1 - 5C_2 = 1$$

Solving for the constants gives $C_1 = \frac{1}{3}$ and $C_2 = -\frac{1}{3}$. Since $P(D) = 1$ and $N = 2 \neq M = 0$ the impulse response is

$$h(t) = \underbrace{b_N \delta(t)}_{=0} + \left[\underbrace{P(D)}_{1} y_h(t) \right] u(t) = \frac{1}{3} e^{-2t} u(t) - \frac{1}{3} e^{-5t} u(t)$$

Chapter 5

DT systems as linear constant coefficient difference equations

A *difference equation* is a relation among combinations of two DT functions and shifted versions of them. Similar to differential equations where the solution is a CT function, the solution to a difference equation is a DT function. For example:

$$y[n+1] + \frac{1}{2}y[n] = x[n]$$

is a first order, linear, constant-coefficient difference equation. Given $x[n]$ the solution is a function $y[n]$. We can view this as a representation of a DT system, where $x[n]$ is the input signal and $y[n]$ is the output.

There is a parallel theory to differential equations for solving difference equations. However in this lecture we will focus specifically on the iterative solution of linear, constant-coefficient difference equations and the case when the input is a delta function, as this is all we need for this course.

5.1 Definition of linear constant coefficient difference equation

A *linear*, *constant-coefficient*, difference equation (LCCDE) comes in one of two forms.

- Delay form.

  $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  or

  $$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N] = b_0 x[n] + \cdots b_M x[n-M]$$

- Advance form. Let $n \to n+N$, then the delay form becomes

  $$\sum_{k=0}^{N} a_k y[n+N-k] = \sum_{k=0}^{M} b_k x[n+N-k]$$

  or

  $$a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = b_0 x[n+N] + \cdots b_M x[n+N-M]$$

The *order* of the system is given by N. The delay and advance forms are equivalent because the equation holds for any n, and we can move back and forth between them as needed by a constant index-shift.

Example 5.1.1 ($N = 2$, $M = 1$)**.** The delay form is

$$a_0 y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1]$$

Replacing $n \to n+2$, the advance form is

$$a_0 y[n+2] + a_1 y[n+1] + a_2 y[n] = b + 0x[n+2] + b_1 x[n+1]$$

■

It will be convenient to define the operator E^m as shifting a DT function by positive m, i.e. $E^m x[n] = x[n+m]$, and the operator D^m as shifting a DT function by negative m, i.e. $D^m x[n] = x[n-m]$. These are called the advance and delay operators respectively. Then, the advance form of the difference equation using this operator notation is

$$a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = b_0 x[n+N] + \cdots b_M x[n+N-M]$$

$$a_0 E^N y + a_1 E^{N-1} y + \cdots a_N y = b_0 E^N x + \cdots b_M E^{N-M} x$$

Factoring out the advance operators gives

$$\underbrace{\left(a_0 E^N + a_1 E^{N-1} + \cdots a_N\right)}_{Q(E)} y = \underbrace{\left(b_M E^N + \cdots b_M E^{N-M}\right)}_{P(E)} x$$

or

$$Q(E) y[n] = P(E) x[n]$$

Similarly, the delay form of the difference equation using this operator notation is

$$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N] = b_0 x[n] + \cdots b_M x[n-M]$$

$$a_0 y[n] + a_1 D y + \cdots a_N D^N y = b_0 x + \cdots b_M D^M x$$

Note: The DT delay operator D is similar, but *not* identical to the derivative operator D in CT.

Example 5.1.2. Consider the difference equation

$$3y[n+1] + 4y[n] + 5y[n-1] = 2x[n+1]$$

The advance form would be:

$$3y[n+2] + 4y[n+1] + 5y[n] = 2x[n+2]$$

or using the advance operator

$$\left(3E^2 + 4E + 5\right) y = 2E^2 x$$

with $Q(E) = 3E^2 + 4E + 5$ and $P(E) = 2E^2$.

The delay form would be:

$$3y[n] + 4y[n-1] + 5y[n-2] = 2x[n]$$

or using the delay operator

$$\left(5D^2 + 4D + 3\right) y = 2x$$

with $Q(D) = 5D^2 + 4D + 3$ and $P(D) = 2$. ■

5.2 Iterative solution of LCCDEs

Difference equations are different (pun!) from differential equations in that they can be solved by manually running the equation forward using previous values of the output and current and previous values of the input, given some initial conditions. This is called an *iterative* solution for this reason.

To perform an iterative solution we need the difference equation in delay form

$$a_0 y[n] + a_1 y[n-1] + \cdots a_N y[n-N] = b_0 x[n] + \cdots b_M x[n-M]$$

We then solve for the current output $y[n]$

$$y[n] = -\left(\frac{a_1}{a_0} y[n-1] + \cdots \frac{a_N}{a_0} y[n-N]\right) + \frac{b_0}{a_0} x[n] + \cdots \frac{b_M}{a_0} x[n-M]$$

Now lets examine what this expression says in words. To compute the current output $y[n]$ we need the value of the *previous* $N-1$ outputs, the value of the *current* input $x[n]$ and $M-1$ *previous* inputs (and the coefficients). Then we can compute the next output $y[n+1]$ by adding the previous computation result for $y[n]$ to our list of things to remember, and forgetting one previous value of y. This can continue as long as we like.

Example 5.2.1. Consider the first-order difference equation

$$y[n+1] + y[n] = x[n+1]$$

where $y[-1] = 1$ and $x[n] = u[n]$. We first convert this to delay form

$$y[n] = -y[n-1] + x[n] \; .$$

Then we can compute $y[0]$ as

$$y[0] = -y[-1] + x[0] = -1 + 1 = 0$$

and continuing

$$\begin{aligned} &y[1] = -y[0] + x[1] = 0 + 1 = 1 \\ &y[2] = -y[1] + x[2] = -1 + 1 = 0 \\ &y[3] = -y[2] + x[3] = 0 + 1 = 1 \\ &\text{etc.} \end{aligned}$$

We can see that this will continue to give the alternating sequence $1, 0, 1, 0, 1, \cdots$. ■

5.3 Solution of the homogeneous LCCDE

Note the iterative solution does not give us (directly) and analytical expression for the output at arbitrary n. We have to start at the initial conditions and compute our way up to n. We now consider an analytical solution when the input is zero, the solution to the *homogeneous* difference equation

$$Q(E)\, y = a_0 y[n+N] + a_1 y[n+N-1] + \cdots a_N y[n] = 0 \; .$$

given N sequential auxiliary conditions on y.

Similar to differential equations, the homogeneous solution depends on the roots of the characteristic equation $Q(E) = 0$ whose roots are either real or occur in complex conjugate pairs. Let λ_i be the i-th root of $Q(E) = 0$, then the solution is of the form

$$y[n] = \sum_{i=1}^{N} C_i \lambda_i^n$$

where the parameters C_i are determined from the auxiliary conditions.

For a real system (when the coefficients of the difference equation are real) and when the roots are complex $\lambda_{1,2} = |\lambda|e^{\pm j\beta}$, it is cleaner to assume a form for those terms as

$$y[n] = C|\lambda|^n \cos(\beta n + \theta)$$

for constants C and θ.

Example 5.3.1 (First-Order)**.** Find the solution to the first-order homogeneous LCCDE

$$y[n+1] + \frac{1}{2}y[n] = 0 \text{ with } y[0] = 5 \; .$$

Note $Q(E) = E + \frac{1}{2}$ has a single root $\lambda_1 = -\frac{1}{2}$. Thus the solution is of the form

$$y[n] = C\left(-\frac{1}{2}\right)^n$$

where the parameter C is found using

$$y[0] = C = 5$$

to give the final solution

$$y[n] = 5\left(-\frac{1}{2}\right)^n$$

■

Example 5.3.2 (Second-Order, Complex Roots)**.** Find the solution to the second-order homogeneous LCCDE

$$y[n+2] + y[n+1] + \frac{1}{2}y[n] = 0 \text{ with } y[0] = 1 \text{ and } y[1] = 0 \; .$$

Note $Q(E) = E^2 + E + \frac{1}{2}$ has a pair of complex roots $\lambda_{1,2} = -\frac{1}{2} \pm j\frac{1}{2}$. Thus the solution is of the form

$$y[n] = C\left|\frac{1}{\sqrt{2}}\right|^n \cos\left(\frac{\pi}{4}n + \theta\right)$$

where the parameters are found using

$$y[0] = C\cos(\theta) = 1$$

$$y[1] = C\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4} + \theta\right) = 0$$

This is true when

$$C = \sqrt{2} \text{ and } \theta = \frac{\pi}{4} + 2\pi m$$

or

$$C = -\sqrt{2} \text{ and } \theta = -\frac{3\pi}{4} + 2\pi m$$

for any $m \in \mathbb{Z}$ since cos is periodic in 2π. A final solution is then

$$y[n] = \sqrt{2}\left|\frac{1}{\sqrt{2}}\right|^n \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$$

■

See the appendix for a general tehnique to solve for these constants.

5.4 Impulse response from LCCDE

Today our goal is to find the solution to $Q(E)y = P(E)x$ when $x[n] = \delta[n]$ assuming $y[n] = 0$ for $n < 0$, giving the *impulse response* $y[n] = h[n]$. We skip the derivation here and just give a procedure.

Step 1: Let y_h be the homogeneous solution to $Q(E)y_h = 0$ for $n > N$.

Step 2: Assume a form for $h[n]$ given by

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_h[n]u[n]$$

Step 3: Using the iterative procedure above find the N auxiliary conditions we need by,

- first, rewrite the equation in delay form and solve for $y[n]$,
- then let $x[n] = \delta[n]$ and manually compute $h[0]$ assuming $h[n] = 0$ for $n < 0$,
- repeating the previous step for $h[1]$, continuing up to $h[N-1]$.

Step 4: Using the auxillary conditions in step 3, solve for the constants in the solution $h[n]$ from step 2.

Example 5.4.1. Find the impulse response of the system given by

$$y[n+2] - \frac{1}{4}y[n+1] - \frac{1}{8}y[n] = 2x[n+1]$$

For step 1 we solve the equation

$$y_h[n+2] - \frac{1}{4}y_h[n+1] - \frac{1}{8}y_h[n] = 0$$

which is of the form

$$y_h[n] = C_1\left(-\frac{1}{4}\right)^n + C_2\left(\frac{1}{2}\right)^n$$

since the roots of $Q(E) = E^2 - \frac{1}{4}E - \frac{1}{8}$ are $-\frac{1}{4}$ and $\frac{1}{2}$.

For step 3, we find the auxiliary conditions needed to find C_1 and C_2 by rewriting the original equation in delay form and solving for $y[0]$ and $y[1]$ when $x[n] = \delta[n]$.

$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + 2x[n-1]$$

Let $x[n] = \delta[n]$ and manually compute $y[0]$ assuming $y[n] = 0$ for $n < 0$

$$y[0] = \frac{1}{4}\underbrace{y[0-1]}_{0} + \frac{1}{8}\underbrace{y[0-2]}_{0} + 2\underbrace{\delta[0-1]}_{0} = 0$$

Repeat for $y[1]$

$$y[1] = \frac{1}{4}\underbrace{y[1-1]}_{0} + \frac{1}{8}\underbrace{y[1-2]}_{0} + 2\underbrace{\delta[1-1]}_{1} = 2$$

Now we find the constants using step 4

$$h[0] = C_1 + C_2 = 0$$

$$h[1] = C_1\left(-\frac{1}{4}\right) + C_2\left(\frac{1}{2}\right) = 2$$

which gives $C_1 = -\frac{8}{3}$ and $C_2 = \frac{8}{3}$. Thus the final impulse response is

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_h[n]u[n] = -\frac{8}{3}\left(-\frac{1}{4}\right)^n u[n] + \frac{8}{3}\left(\frac{1}{2}\right)^n u[n]$$

since $b_N = 0$. ■

Note we can confirm our closed-form result in the previous example, for a few values of n, by iteratively solving the difference equation

$$h[0] = \frac{1}{4}\underbrace{h[0-1]}_{0} + \frac{1}{8}\underbrace{h[0-2]}_{0} + 2\underbrace{\delta[0-1]}_{0} = 0$$

$$h[1] = \frac{1}{4}\underbrace{h[1-1]}_{0} + \frac{1}{8}\underbrace{h[1-2]}_{0} + 2\underbrace{\delta[1-1]}_{1} = 2$$

$$h[2] = \frac{1}{4}\underbrace{h[2-1]}_{2} + \frac{1}{8}\underbrace{h[2-2]}_{0} + 2\underbrace{\delta[2-1]}_{0} = \frac{1}{2}$$

$$h[3] = \frac{1}{4}\underbrace{h[3-1]}_{\frac{1}{2}} + \frac{1}{8}\underbrace{h[3-2]}_{2} + 2\underbrace{\delta[2-1]}_{0} = \frac{3}{8}$$

and comparing to our closed-form solution a the same values of n

$$h[0] = -\frac{8}{3} + \frac{8}{3} = 0$$

$$h[1] = -\frac{8}{3}\left(-\frac{1}{4}\right) + \frac{8}{3}\left(\frac{1}{2}\right) = 2$$

$$h[2] = -\frac{8}{3}\left(-\frac{1}{4}\right)^2 + \frac{8}{3}\left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$h[3] = -\frac{8}{3}\left(-\frac{1}{4}\right)^3 + \frac{8}{3}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

Example 5.4.2. Find the impulse response of the system given by

$$y[n+1] - \frac{1}{2}y[n] = x[n+1] + x[n]$$

In step 1 we note the solution to $Q(E)y[n] = 0$ is of the form

$$y_h[n] = C\left(\frac{1}{2}\right)^n$$

From step 2 we note $b_N = 1$ and $a_N = -\frac{1}{2}$, so that

$$h[n] = -2\delta[n] + C\left(\frac{1}{2}\right)^n u[n]$$

In step 3 we manually find $h[0]$

$$y[n] = \frac{1}{2}y[n-1] + x[n] + x[n-1]$$

$$h[n] = \frac{1}{2}y[n-1] + \delta[n] + \delta[n-1]$$

$$h[0] = 0 + 1 + 0 = 1$$

And in step 4 we solve for C

$$h[0] = -2 + C = 1 \text{ implies } C = 3$$

to give

$$h[n] = -2\delta[n] + 3\left(\frac{1}{2}\right)^n u[n]$$

■

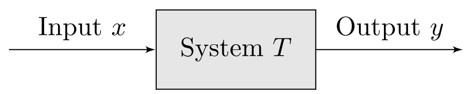
Chapter 6

Linear time invariant CT systems

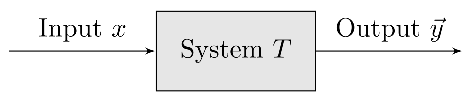
6.1 System types

A system is an interconncted set of components or sub-systems. Mathematically a system is a transformation between one or more signals, a rule that maps functions to functions.

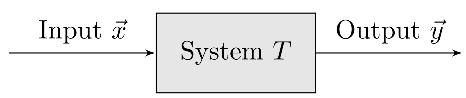
- single input - single output (SISO) system.



- single input - multiple output (SIMO) system

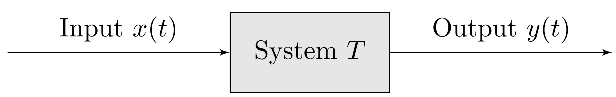


- general case, multiple input - multiple output (MIMO)

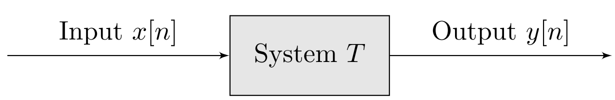


We will focus on single input - single output, CT and DT systems.

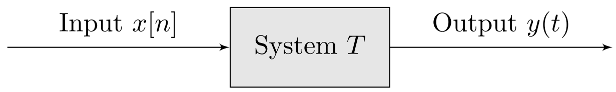
- If both input and output are CT signals, it is a CT system.

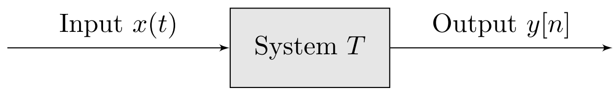


- If both input and output are DT signals, it is a DT system.



- If input and output are not both CT or DT signals, it is a hybrid CT-DT system.

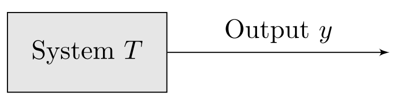




As a shorthand notation for the graphical description above we can use $x \mapsto y$. A system maps a function x to a function y:

- CT system
$$x(t) \mapsto y(t)$$
- DT system
$$x[n] \mapsto y[n]$$
- Hybrid CT-DT system
$$x[n] \mapsto y(t)$$
or
$$x(t) \mapsto y[n]$$

When a system has no input, the system is *autonomous*. An autonomous system just produces output: $\mapsto y$.



We can think of an autonomous system as a function generator, producing signals for use.

6.2 CT system representations

We can mathematically represent, or model, systems multiple ways.

- purely mathematically - in time domain we will use
  - for CT systems: linear, constant coefficient differential equations. e.g.
  $$y'' + ay' + by = x$$
  - for DT systems: linear, constant coefficient difference equation, e.g.
  $$y[n] = ay[n-1] + by[n-2] + x[n]$$

  or
  - for CT systems: CT impulse response
  - for DT systems: DT impulse response
- purely mathematically - in frequency domain we will use
  - frequency response

– transfer function (complex frequency, covered in ECE 3704)

- graphically, using a mixture of math and block diagrams

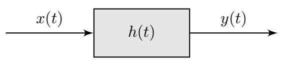
Mathematical models:

- provide abstraction, removing (often) irrelevant detail.
- can be more or less detailed, an *internal* v.s. *external* (block box) description
- are not unique with respect to instantiation (implementation)
- are limited to the regime they were designed for

Example 6.2.1 (RC Circuit)**.** Consider the RC circuit. It is a single input - single output system. We will be able to represent it mathematically or graphically and internally or externally.

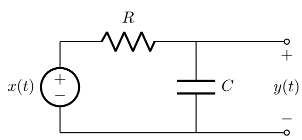
Graphical — Symbolic

External:



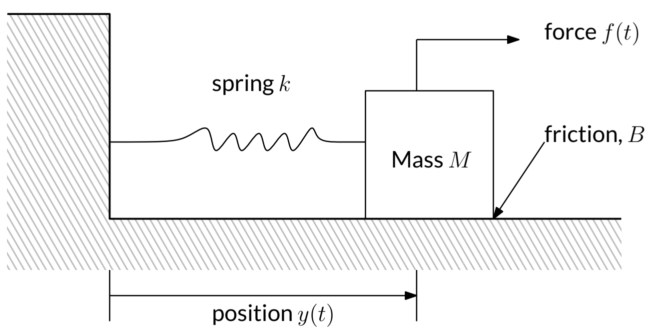
$$y(t) = h(t) * x(t)$$

Internal:



$$y' + \frac{1}{RC}y = \frac{1}{RC}x(t)$$

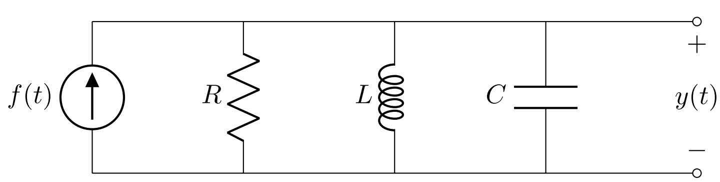
It does not matter what the underlying system implementation is. For example, consider a mechanical system, described by a second-order ODE:



y = position	M = mass
y' = velocity	K = spring constant
y'' = acceleration	B = coefficient of friction

$$y'' + \frac{B}{M}y' + \frac{K}{M}y = \frac{1}{M}f(t)$$

Compare this to the parallel RLC circuit, described by the second-order ODE:



y = voltage	R = resistance
Cy' = capacitor current	L = inductance
	C = capacitance

$$y'' + \frac{1}{RC}y' + \frac{1}{LC}y = \frac{1}{LC}f(t)$$

Comparing these systems, if $R = \frac{1}{B}$, $L = \frac{1}{K}$, and $C = M$, they are mathematically identical.

6.3 System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- Memory
- Invertability
- Causality
- Stability
- Time-invariance
- Linearity

Let's define each it turn.

6.3.1 Memory

The output of a system with memory depends on previous or future inputs and is said to be *dynamic*. Otherwise the system is memoryless or *instantaneous*, and the output $y(t)$ at time t depends only on $x(t)$. For example in CT:

$$y(t) = 2x(t)$$

is a memoryless system, while

$$y(t) = \int_{-\infty}^{t} x(\tau)\, dt$$

has memory.

6.3.2 Invertability

A system is invertable if there exists a system that when placed in series with the original recovers the input.

$$x(t) \mapsto Ty(t) \mapsto T^{-1}x(t)$$

where T^{-1} is the inverse system of T. For example, consider a system

$$x(t) \mapsto y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

and a system

$$y(t) \mapsto z(t) = \frac{dy}{dt}$$

The combination in series $x(t) \mapsto y(t) \mapsto z(t) = x(t)$, i.e. the derivative undoes the integral.

6.3.3 Causality

A CT system is causal if the output at time t depends on the input for time values at or before t:

$$y(t) \text{ depends on } x(\tau) \text{ for } \tau \leq t$$

All physical CT systems are causal, even if all continuous systems are not (e.g. continuous 2D images $f(u, v)$, have no "before" and "after").

For example, consider a CT system whose impulse response is $h(t) = e^{-t^2}$. This implies the system produces output *before* (i.e. for $t < 0$) the impulse is applied at $t = 0$, somehow anticipating the arrival of the impulse. Barring time-travel, this is physically impossible.

6.3.4 Stability

A CT system is (BIBO) stable if applying a bounded-input (BI)

$$|x(t)| < \infty \ \forall \ t$$

results in a bounded-output (BO) $x(t) \mapsto y(t)$ and

$$|y(t)| < \infty \ \forall \ t$$

Note, bounded in practice is limited by the physical situation, e.g. positive and negative rails in a physical circuit.

For example, a CT system described by the LCCDE

$$\frac{dy}{dt}(t) - 2y(t) = x(t)$$

is unstable because the solution $y(t)$ will have one term of the form Ce^{2t}, for most non-zero inputs $x(t)$ or any non-zero initial condition, that grows unbounded as time increases.

6.3.5 Time-invariance

A CT system is time-invariant if, given

$$x(t) \mapsto y(t)$$

then a time-shift of the input leads to the same time-shift in the output

$$x(t - \tau) \mapsto y(t - \tau)$$

An important counterexample is a CT system described by a LCCDE, e.g.

$$\frac{dy}{dt}(t) + y(t) = x(t)$$

but non-zero auxillary conditions at some t_0, $y(t_0) = y_0 \neq 0$. Such systems will have a term in its solution that depends on y_0. However if I time shift the input, the term that depends on y_0 does not shift (since it is anchored to t_0) and the total output does not shift identically with the input. Thus the system cannot be time-invariant.

6.3.6 Linearity

A CT system is linear if the output due to a sum of scaled individual inputs is the same as the scaled sum of the individual outputs with respect to those inputs. In other words given

$$x_1(t) \mapsto y_1(t) \text{ and } x_2(t) \mapsto y_2(t)$$

then

$$ax_1(t) + bx_2(t) \mapsto ay_1(t) + by_2(t)$$

for constants a and b. Note this property extends to sums of arbitrary signals, e.g. if

$$x_i(t) \mapsto y_i(t) \ \forall \ i \in [1 \cdots N]$$

then given N constants a_i, if the system is linear

$$\sum_{i=1}^{N} a_i x_i(t) \mapsto \sum_{i=1}^{N} a_i y_i(t)$$

This is a very important property, called *superposition*, and it simplifies the analysis of systems greatly.

Similar to time-invariance an important non-linear system is that is described by a LCCDE with non-zero auxillary conditions at some t_0, $y(t_0) = y_0$. Again such systems will have a term in it's solution that depends on y_0. Given two inputs, each individual response will have that term in it, so thier sum has double that term. However the response due to the sum of the inputs would again only have one and the sum of the responses would not be the same as the response of the sum. Such a system cannot be linear.

6.4 Stable LTI Systems

The remainder of this course is about stable, linear, time-invariant (LTI) systems. As we have seen in CT such systems can be described by a LCCDE with zero auxillary (initial) conditions (the system is *at rest*).

We have seen previously how to find the impulse response, $h(t)$, of such systems. We now note some relationships between the impulse response and the system properties described above.

- If a system is memoryless then $h(t) = C\delta(t)$ for some constant C.
- If a system is causal then $h(t) = 0$ for $t < 0$.
- If a system is BIBO stable then

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$$

Chapter 7

Linear time invariant DT systems

7.1 DT system representations

We can mathematically represent, or model, DT systems multiple ways.

- purely mathematically - in time domain we will use
  - linear, constant coefficient difference equations, e.g.

$$y[n] = ay[n-1] + by[n-2] + x[n]$$

  - DT impulse response $h[n]$
- purely mathematically - in frequency domain we will use
  - frequency response
  - transfer function (complex frequency, covered in ECE 3704)
- graphically, using a mixture of math and block diagrams

7.2 System properties and classification

Choosing the right kind of system model is important. Here are some important properties that allow us to broadly classify systems.

- Memory
- Invertability
- Causality
- Stability
- Time-invariance
- Linearity

Let's define each it turn.

7.2.1 Memory

The output of a DT system with memory depends on previous or future inputs and is said to be *dynamic*. Otherwise the system is memoryless or *instantaneous*, and the output $y[n]$ at index n depends only on $x[n]$. For example:

$$y[n] = 2x[n]$$

is a memoryless system, while

$$y[n+1] + y[n] = x[n]$$

has memory. To see this, write the difference equation in recursive form

$$y[n] = -y[n-1] + x[n-1]$$

and we see explicitly the current output $y[n]$ depends on past values of output and input.

7.2.2 Invertability

A system is invertible if there exists a system that when placed in series with the original recovers the input.

$$x[n] \mapsto Ty[n] \mapsto T^{-1}x[n]$$

where T^{-1} is the inverse system of T. For example, consider a system

$$x[n] \mapsto y[n] = \sum_{m=-\infty}^{n} x[m]$$

and a system

$$y[n] \mapsto z[n] = y[n] - y[n-1]$$

The combination in series $x[n] \mapsto y[n] \mapsto z[n] = x[n]$, since

$$z[n] = y[n] - y[n-1] = \sum_{m=-\infty}^{n} x[m] - \sum_{m=-\infty}^{n-1} x[m] = x[n]$$

i.e. the difference undoes the accumulation.

7.2.3 Causality

A DT system is causal if the output at index n depends on the input for index values at or before n:

$$y[n] \text{ depends on } x[m] \text{ for } m \leq n$$

While all physical CT systems are causal, practical DT systems may not be since we can use memory to "shift time". For CT systems we cannot store the infinite number of values between two time points t_1 and t_2, but we can store the $n_2 - n_1$ values of a DT system between between two indices n_1 and n_2 (assuming infinite precision).

Example 7.2.1. Consider a DT system whose difference equation is

$$y[n] = -x[n-1] + 2x[n] - x[n+1]$$

We see the current output $y[n]$ depends on a "future" value of the input $x[n+1]$. Thus the system **is not** causal. In practice we can shift the difference equation to

$$y[n-1] = -x[n-2] + 2x[n-1] - x[n]$$

and then delay the output by one sample to get $y[n]$.

Example 7.2.2. Consider a DT system whose difference equation is

$$y[n] = -y[n-1] + 2x[n]$$

We see the current output $y[n]$ depends on a "past" value of the output $y[n-1]$ and the current input $x[n]$. Thus the system **is** causal. In practice we can immediately compute $y[n]$ with no delay.

7.2.4 Stability

A DT system is (BIBO) stable if applying a bounded-input (BI)

$$|x[n]| < \infty \; \forall \; n$$

results in a bounded-output (BO) $x[n] \mapsto y[n]$ and

$$|y[n]| < \infty \; \forall \; n$$

Note, bounded in practice is limited by the physical situation, e.g. the number of bits used to store values.

For example, a DT system described by the LCCDE

$$y[n+1] - 2y[n] = x[n+1]$$

is unstable because the solution $y[n]$ will have one term of the form $(2)^n$, for most non-zero inputs $x[n]$ or any non-zero initial condition, that grows unbounded as n increases.

7.2.5 Time-invariance

A DT system is time(index)-invariant if, given

$$x[n] \mapsto y[n]$$

then an index-shift of the input leads to the same index-shift in the output

$$x[n-m] \mapsto y[n-m]$$

An important example is a DT system described by a LCCDE, e.g.

$$y[n+1] - \frac{1}{2}y[n] = x[n+1]$$

or in recursive form

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

If we index shift the input $x[n-m]$ we replace n by $n-m$ and the difference equation becomes

$$y[n-m+1] - \frac{1}{2}y[n-m] = x[n-m+1]$$

which has the same solution shifted by m

$$y[n-m] = \frac{1}{2}y[n-m-1] + x[n-m]$$

If a coefficient depends on n however, e.g

$$y[n+1] - \frac{n}{2}y[n] = x[n+1]$$

so that it is no longer LCC then the solution depends on m and the system is no longer time-invariant.

7.2.6 Linearity

A DT system is linear if the output due to a sum of scaled individual inputs is the same as the scaled sum of the individual outputs with respect to those inputs. In other words given

$$x_1[n] \mapsto y_1[n] \text{ and } x_2[n] \mapsto y_2[n]$$

then

$$ax_1[n] + bx_2[n] \mapsto ay_1[n] + by_2[n]$$

for constants a and b. Note this property extends to sums of arbitrary signals, e.g. if

$$x_i[n] \mapsto y_i[n] \; \forall \; i \in [1 \cdots N]$$

then given N constants a_i, if the system is linear

$$\sum_{i=1}^{N} a_i x_i[n] \mapsto \sum_{i=1}^{N} a_i y_i[n]$$

This is a very important property, called *superposition*, and it simplifies the analysis of systems greatly.

An important non-linear system is that is described by a LCCDE with non-zero auxiliary conditions at some n_0, $y[n_0] = y_0$. As in CT, such systems will have a term in it's solution that depends on y_0. Given two inputs, each individual response will have that term in it, so their sum has double that term. However the response due to the sum of the inputs would again only have one and the sum of the responses would not be the same as the response of the sum. Such a system cannot be linear. Thus the system must be "at rest" before applying the input in order to be a linear system.

7.3 Stable LTI Systems

The remainder of this course is about stable, linear, time-invariant (LTI) systems. As we have seen in DT such systems can be described by a LCCDE with zero auxiliary (initial) conditions (the system is *at rest*).

We have seen previously how to find the impulse response, $h[n]$, of such systems. We now note some relationships between the impulse response and the system properties described above.

- If a system is memoryless then $h[n] = C\delta[n]$ for some constant C.
- If a system is causal then $h[n] = 0$ for $n < 0$.
- If a system is BIBO stable then

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

Chapter 8

CT Convolution

8.1 Review CT LTI systems and superposition property

Recall the superposition property of LTI systems. If a CT system is LTI then the superposition property holds. Given a system where

$$x_i(t) \mapsto y_i(t) \; \forall \; i$$

then

$$\sum_i a_i x_i(t) \mapsto \sum_i a_i y_i(t)$$

Superposition enables a powerful problem reduction strategy. The overall idea for is that if:

- we can write an arbitrary signal as a sum of simple signals, and
- we can determine the response to the simple signals, then
- we can easily express the output due to the input using superposition

This will be a recurring pattern in this course. In this lecture, the simple signals are weighted, time shifts of one signal, the delta function, $\delta(t)$.

8.2 Convolution Integral

To derive this we start with the sifting property of the CT impulse function (from chapter 2)

$$\int_a^b x(t)\delta(t - t_0) \; dt = x(t_0)$$

for any $a < t_0 < b$. A slight change of variables ($t_0 \to \tau$) and limits ($a \to -\infty$ and $b \to \infty$) gives:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) \; d\tau$$

showing that we can write any CT signal as an infinite sum (integral) of weighted and time-shifted impluse functions.

Let $h(t)$ be the CT *impulse response*, the output due to the input $\delta(t)$, i.e. $\delta(t) \mapsto h(t)$. Then if the system is time-invariant: $\delta(t-\tau) \mapsto h(t-\tau)$ and by superposition if the input is writen as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

then the output is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = x(t) * h(t)$$

This is called the *convolution integral* .

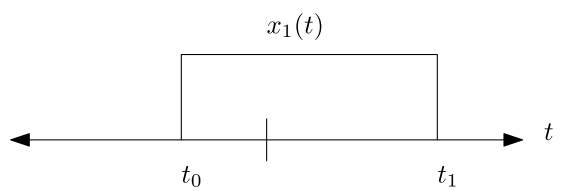
It is worth pausing here to see the signifigance. For a LTI CT system, if I know its impulse response $h(t)$, I can find the response due to **any** input using convolution. For this reason the impulse response is another way to represent an LTI system.

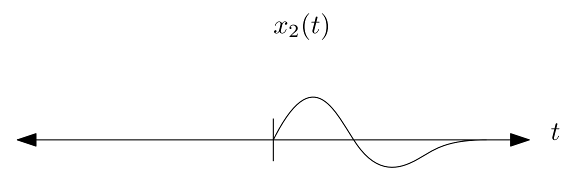
8.3 Graphical View of the Convolution Integral.

Let us break the convolution expression down into pieces. In its general form the convolution of two signals $x_1(t)$ and $x_2(t)$ is

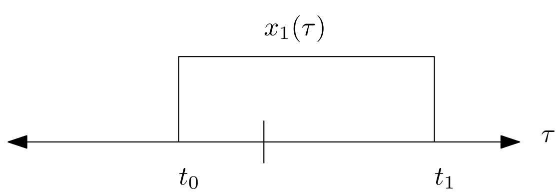
$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$$

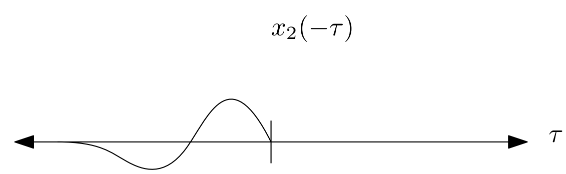
Suppose $x_1(t)$ and $x_2(t)$ are signals that look like



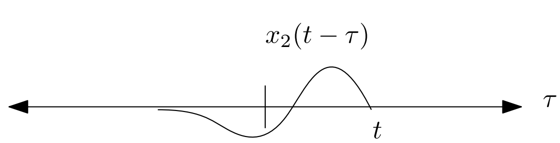


Then $x_1(\tau)$ and $x_2(-\tau)$ look like

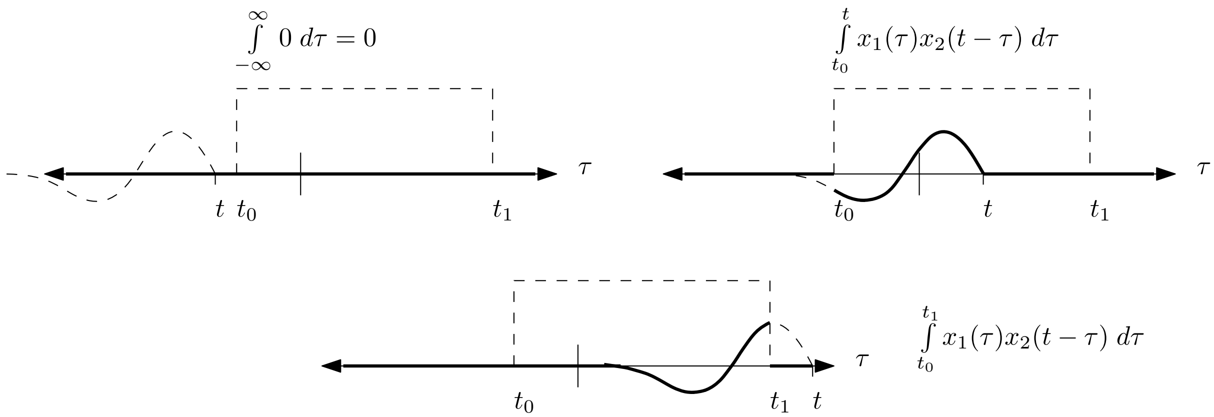




The signal $x_2(t-\tau)$ is $x_2(-\tau)$ shifted by t (since $x_2(-\tau+t) = x_2(t-\tau)$) and then looks like



Then the integrand of convolution is the product $x_1(\tau)x_2(t-\tau)$ whose plot depends of the value of t. Some examples, where the individual signals are dashed and their product is in bold:



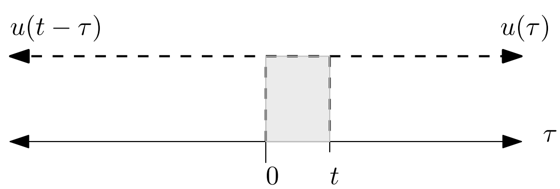
Then convolution is the total integral of the product (bold curves above) for that value of t. For the example above we see the integral will be zero for t less than t_0 since the two signals do not overlap and their product is zero. For $t_0 < t < t_1$ the signals overap and the product is non-zero, and the effective bounds of integration are $[t_0, t]$. For $t > t_1$ the signals again overap and the product is non-zero, but the effective bounds of integration are $[t_0, t_1]$.

8.4 Examples of CT Convolution

Example 8.4.1 $(u(t) * u(t))$**.** Consider the convolution of two unit step functions.

$$u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\, d\tau$$

The product $u(\tau)u(t-\tau)$ is non-zero only when $t \geq 0$ as illustrated here



The convolution integral is then the shaded area

$$u(t) * u(t) = \begin{cases} 0 & t < 0 \\ \int_0^t d\tau = t & t \geq 0 \end{cases}$$

Combining this back into a single expression gives:

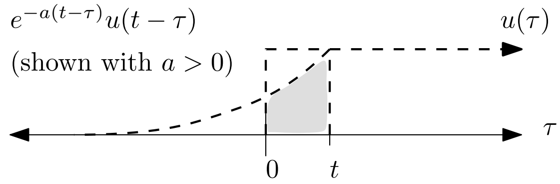
$$u(t) * u(t) = tu(t)$$

Thus the convolution of two step signals is a ramp signal.
■

Example 8.4.2 $(u(t) * e^{-at}u(t))$**.** Let $x_1(t) = u(t)$ and $x_2(t) = e^{-at}u(t)$ for constant $a \in \mathbb{C}$, then

$$u(t) * e^{-at}u(t) = \int_{-\infty}^{\infty} u(\tau)e^{-a(t-\tau)}u(t-\tau)\, d\tau$$

Similar to the previous example, the product $u(\tau)e^{-a(t-\tau)}u(t-\tau)$ is non-zero only when $t \geq 0$



The convolution integral is then the shaded area

$$u(t) * e^{-at}u(t) = \begin{cases} 0 & t < 0 \\ \int_0^t e^{-a(t-\tau)} d\tau = \frac{1-e^{-at}}{a} & t \geq 0 \end{cases}$$

Combining this back into a single expression gives:

$$u(t) * e^{-at}u(t) = \frac{1 - e^{-at}}{a} u(t)$$

■

Example 8.4.3 (Convolution with a delta function)**.** Let $x_1(t) = \delta(t)$ and $x_2(t)$ be an arbitrary signal. Then

$$\delta(t) * x_2(t) = \int_{-\infty}^{\infty} \delta(\tau) x_2(t - \tau) \, d\tau$$

By the sifting property of the delta function this evaluates to

$$\delta(t) * x_2(t) = x_2(t)$$

or in other words convolution with a delta function just results in the signal it was convolved with. That is it acts like the identity function, with respect to convolution.
■

The following table lists several convolution results.

Short Table of Representative Convolution Integrals

$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t)$
$e^{at}u(t)$	$u(t)$	$\frac{1-e^{at}}{-a}u(t)$
$u(t)$	$u(t)$	$tu(t)$
$e^{a_1 t}u(t)$	$e^{a_2 t}u(t)$	$\frac{e^{a_1 t}-e^{a_2 t}}{a_1-a_2}u(t)$ for $a_1 \neq a_2$
$e^{at}u(t)$	$e^{at}u(t)$	$te^{at}u(t)$
$te^{a_1 t}u(t)$	$e^{a_2 t}u(t)$	$\frac{e^{a_2 t}-e^{a_1 t}+(a_1-a_2)te^{a_1 t}}{(a_1-a_2)^2}u(t)$ for $a_1 \neq a_2$
$e^{a_1 t}\cos(\beta t + \theta)u(t)$	$e^{a_2 t}u(t)$	$\frac{\cos(\theta-\phi)e^{a_2 t}-e^{a_1 t}\cos(\beta t+\theta-\phi)}{\sqrt{(a_1+a_2)^2+\beta^2}}u(t)$ $\phi = \arctan\left(\frac{-\beta}{a_1+a_2}\right)$

8.5 Properties of CT Convolution

There are several useful properties of convolution. We do not prove these here, but it is not terribly difficult to do so. Given signals $x_1(t)$, $x_2(t)$, and $x_3(t)$:

Communative Property The ordering of the signals does not matter.

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive Propery Convolution is distributed over addition.

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

Associative Property The order of convolution does not matter.

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Time Shift Given $x_3(t) = x_1(t) * x_2(t)$ then for time shifts $\tau_1, \tau_2 \in \mathbb{R}$

$$x_1(t - \tau_1) * x_2(t - \tau_2) = x_3(t - \tau_1 - \tau_2)$$

Multiplicative Scaling Given $x_3(t) = x_1(t) * x_2(t)$ then for constants $a, b \in \mathbb{C}$

$$[a\, x_1(t)] * [b\, x_2(t)] = a\, b\, x_3(t)$$

These properties can be used in combination with a table like that above to compute the convolution of a wide variety of signals without evaluating the integrals.

Example 8.5.1. Here is a simple example. Let $x_1(t) = e^t u(t)$ and $x_2(t) = 2\delta(t) + 5e^{-3t}u(t)$.

$$x_1(t) * x_2(t) = e^t u(t) * \left[2\delta(t) + 5e^{-3t}u(t)\right]$$

Using the distributive property

$$x_1(t) * x_2(t) = 2\left[\delta(t) * e^t u(t)\right] + 5\left[e^t u(t) * e^{-3t}u(t)\right]$$

Using previously derived results involving the delta function and the table row 3

$$x_1(t) * x_2(t) = 2e^t\, u(t) + 5\left[\frac{e^t - e^{-3t}}{4}\right] u(t)$$

Doing some simplification gives the result

$$x_1(t) * x_2(t) = \left[\frac{13}{4}e^t - \frac{5}{4}e^{-3t}\right] u(t)$$

■

Example 8.5.2. Here is a more complicated example. Let $x_1(t) = 2e^{-5t}u(t-1)$ and $x_2(t) = \left(1 - e^{-t}\right) u(t)$.

$$x_1(t) * x_2(t) = \left[2e^{-5t}u(t-1)\right] * \left[\left(1 - e^{-t}\right) u(t)\right]$$

We first rewrite $e^{-5t}u(t-1) = e^{-5}e^{-5(t-1)}u(t-1) = e^{-5}e^{-5t}u(t)\Big|_{t=t-1}$ so that we can remove the time shift

$$x_1(t) * x_2(t) = 2e^{-5}\left[e^{-5t}u(t)\right] * \left[\left(1 - e^{-t}\right) u(t)\right]\Big|_{t=t-1}$$

We now apply the distributive property

$$x_1(t) * x_2(t) = 2e^{-5}\left[\left(e^{-5t}u(t) * u(t)\right) - \left(e^{-5t}u(t) * e^{-t}u(t)\right)\right]\Big|_{t=t-1}$$

Using the table rows 1 and 3 we get

$$x_1(t) * x_2(t) = 2e^{-5}\left[\frac{1}{5}\left(1 - e^{-5t}\right)u(t) + \frac{1}{4}\left(e^{-5t} - e^{-t}\right)u(t)\right]\Big|_{t=t-1}$$

Combining terms we simplify to

$$x_1(t) * x_2(t) = 2e^{-5}\left[\frac{1}{5} - \frac{1}{4}e^{-t} + \frac{1}{20}e^{-5t}\right]u(t)\Big|_{t=t-1}$$

Replacing the time shift gives the final result

$$x_1(t) * x_2(t) = 2e^{-5}\left[\frac{1}{5} - \frac{1}{4}e^{-(t-1)} + \frac{1}{20}e^{-5(t-1)}\right]u(t-1)$$

which can be cleaned up a bit more by distributing the leading term

$$x_1(t) * x_2(t) = \left[\frac{2}{5}e^{-5} - \frac{1}{2}e^{-(t+4)} + \frac{1}{10}e^{-5t}\right]u(t-1)$$

■

Chapter 9

DT Convolution

9.1 Review DT LTI systems and superposition property

Recall the superposition property of LTI systems. If a DT system is LTI then the superposition property holds. Given a system where

$$x_i[n] \mapsto y_i[n] \; \forall \; i$$

then

$$\sum_i a_i x_i[n] \mapsto \sum_i a_i y_i[n]$$

As in CT we can use superposition to enable a problem reduction strategy in DT systems, where we write the input as a weighted sum of simple signals. In this lecture, the simple signals are weighted, time shifts of one signal, the DT delta function, $\delta[n]$.

9.2 Convolution Sum

To derive this we start with the sifting property of the DT impulse function (from lecture 3)

$$\sum_a^b x[n]\delta[n - n_0] = x[n_0]$$

for any $a < n_0 < b$. A slight change of variables ($n_0 \to m$) and limits ($a \to -\infty$ and $b \to \infty$) gives:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n - m]$$

showing that we can write any DT signal as an infinite sum of weighted and time-shifted impluse functions.

Let $h[n]$ be the DT *impulse response*, the output due to the input $\delta[n]$, i.e. $\delta[n] \mapsto h[n]$. Then if the system is time-invariant: $\delta[n - m] \mapsto h[n - m]$ and by superposition, if the input is writen as

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n - m]$$

then the output is given by

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] = x[n] * h[n]$$

This is called the *convolution sum* .

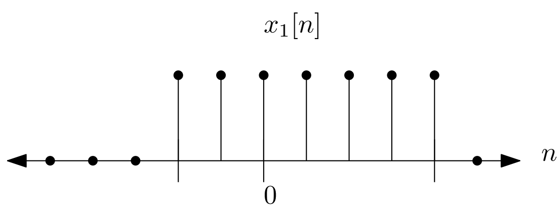
The significance is similar to that in CT convolution. For a LTI DT system, if I know its impulse response $h[n]$, I can find the response due to **any** input using convolution. For this reason the impulse response is another way to represent an LTI system.

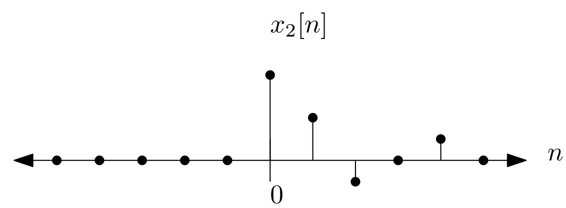
9.3 Graphical View of the Convolution Sum.

As in CT, let us break the convolution expression down into pieces. In its general form the convolution of two signals $x_1[n]$ and $x_2[n]$ is

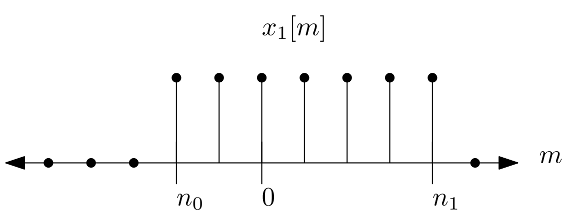
$$x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m] x_2[n-m]$$

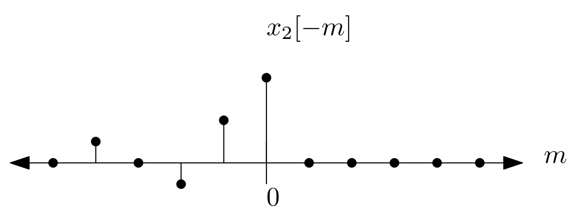
Suppose $x_1[n]$ and $x_2[n]$ are signals that look like



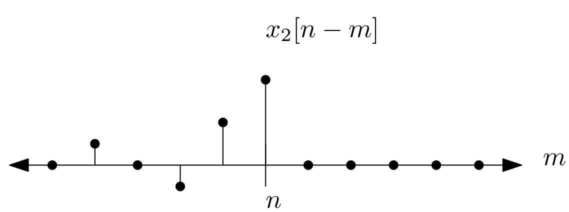


Then $x_1[m]$ and $x_2[-m]$ look like



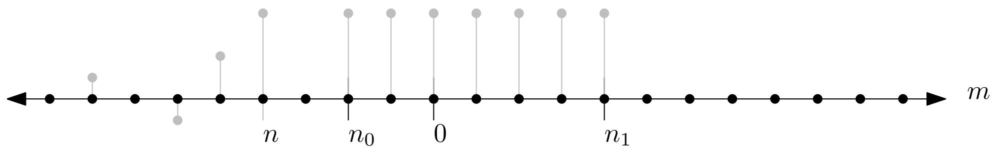


The signal $x_2[n-m]$ is $x_2[-m]$ shifted by n (since $x_2[-m+n] = x_s[n-m]$) and looks like

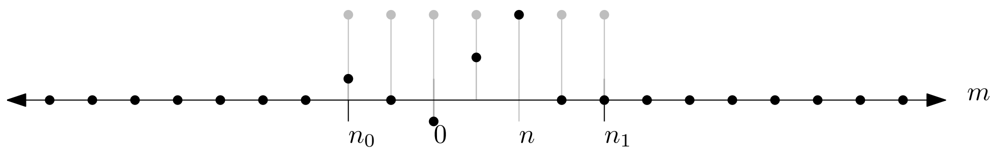


Then the terms of the convolution sum is the product $x_1[m]x_2[n-m]$ whose plot depends of the value of n. Some examples, where the individual signals are in grey and their product is in bold:

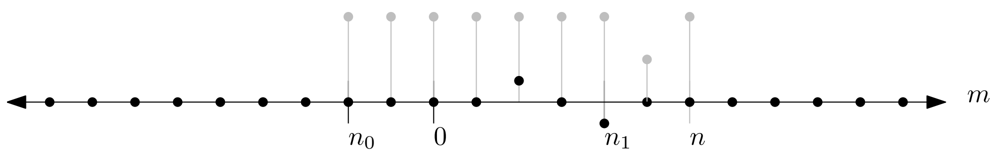
$x_1[m] \cdot x_2[n-m]$ when $n < n_0$



$x_1[m] \cdot x_2[n-m]$ when $n_0 < n < n_1$



$x_1[m] \cdot x_2[n-m]$ when $n > n_1$



Then convolution is the total sum of the product (bold plots above) for that value of n. For the example above we see the sum will be zero for n less than n_0 since the two signals do not overlap and their product is zero. For $n_0 \leq n \leq n_1$ the signals overap and the product is non-zero, and the effective bounds of summation are $[n_0, n]$. For $n > n_1$ the signals again overap and the product is non-zero, but the effective bounds of summation are $[n_0, n_1]$.

9.4 DT Convolution of Finite-Length Signals

For finite-length signals, DT convolution gives us an algorithm to determine their convolution. Suppose the signal x_1 is non-zero only over the interval $[N_1, M_1]$, and the signal x_2 is non-zero only over the interval $[N_2, M_2]$. The *length* of the signals are $L_1 = M_1 - N_1 + 1$ and $L_2 = M_2 - N_2 + 1$ respectively. The non-zero terms of the convolution sum (when the signals overlap) is then the range $[N_1 + N_2, M_1 + M_2]$ and the sum can be truncated as:

$$x_1[n] * x_2[n] = \sum_{m=N_1+N_2}^{M_1+M_2} x_1[m]x_2[n-m]$$

It is common to shift both signals so that they both start at index 0 (in order to be represented as arrays in a zero-based index programming language like C or C++), zero-padding them both to have length $L = L_1 + L_2 - 1$ (zero-pad means to just add zero values to the end of the sequence). Then the convolution becomes

$$y = x_1 * x_2 = \sum_{m=0}^{L} x_1[m]x_2[n-m]$$

where the indexing of x_2 is modulo the signal length, i.e. $x_2[(n-m) \bmod L]$. The resulting signal after convolution, y, is also of length L, and can then be shifted back to start at $N_1 + N_2$.

Example 9.4.1. The following C++ code computes the convolution of the DT signals $\{1, -1, 1\}$ and $\{1, 1, 1, 1\}$.

```
const unsigned int L = 6;
double x1[L] = {1., -1., 1., 0, 0, 0};
double x2[L] = {1., 1., 1., 1., 0, 0};
double y[L];

for(int n = 0; n < L; n++){
  double sum = 0.;
  for(int m = 0; m < L; m++){
    int idx = (L+n-m) % L;
    sum += x1[m]*x2[idx];
  }
  y[n] = sum;
}
```

Note that $L_1 = 3$, $L_2 = 4$, so that $L = 6$. ■

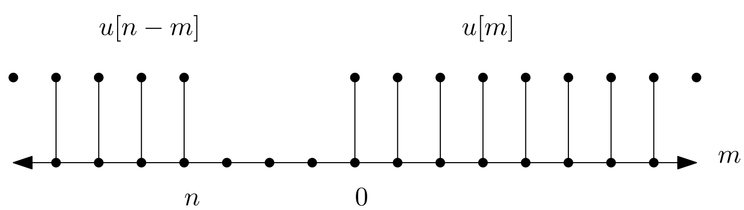
An interesting aside, convolution of finite length signals is equivalent to multiplication of two polynomials, where the signal values are the coefficients.

9.5 Examples of DT Convolution

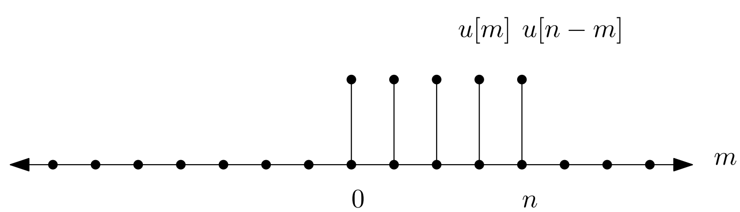
Example 9.5.1. Consider the convolution of two unit step functions:

$$u[n] * u[n] = \sum_{m=-\infty}^{\infty} u[m]u[n-m]$$

Note for $n < 0$ the product of the signals $u[m]$ and $u[n-m]$ is zero as shown in the following figure



so that the resulting sum is zero for any $n < 0$. For $n \geq 0$ the signals $u[m]$ and $u[n-m]$ overlap from 0 to n as shown below



and the convolution sum is

$$\sum_{m=0}^{n} 1 = (n+1)$$

so that

$$u[n] * u[n] = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0 \end{cases}$$

Putting the piecewise result into a single expression gives

$$u[n] * u[n] = (n+1)u[n]$$

■

Example 9.5.2. Consider the convolution of a unit step and the function $\gamma^n\, u[n]$ for some constant $\gamma \neq 1$:

$$\gamma^n\, u[n] * u[n] = \sum_{m=-\infty}^{\infty} \gamma^m u[m]u[n-m]$$

Since both signals are multiplied by a step, the product of $\gamma^m u[m]u[n-m]$ is non-zero only for $0 \leq m \leq n$ (for the same reason as in the previous example). Thus for $n \geq 0$ the convolution sum is:

$$\sum_{m=0}^{n} \gamma^m = \frac{\gamma^{n+1}-1}{\gamma-1} = \frac{1-\gamma^{n+1}}{1-\gamma}$$

Putting the two piecewise results together gives

$$\gamma^n\, u[n] * u[n] = \frac{1-\gamma^{n+1}}{1-\gamma}\, u[n]$$

■

Example 9.5.3. Consider the convolution of an arbitrary signal $x[n]$ with the impulse function

$$x[n] * \delta[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]$$

By the sifting property we get

$$\sum_{m=-\infty}^{\infty} x[m]\delta[n-m] = x[n]$$

Thus the convolution with the impulse gives back the same signal (the δ is the *identity* signal). ■

The following table lists several DT convolution results.

Table of Representative Convolution Sums

$x_1[n]$	$x_2[n]$	$x_1[n] * x_2[n]$
$u[n]$	$u[n]$	$(n+1)u[n]$
$\gamma^n u[n]$	$u[n]$	$\frac{1-\gamma^{n+1}}{1-\gamma} u[n]$ for $\gamma \neq 1$
$\gamma_1^n u[n]$	$\gamma_2^n u[n]$	$\frac{\gamma_1^{n+1}-\gamma_2^{n+1}}{\gamma_1-\gamma_2} u[n]$ for $\gamma_1 \neq \gamma_2$
$\gamma^n u[n]$	$\gamma^n u[n]$	$(n+1)\gamma^n u[n]$
$\lvert\gamma_1\rvert^n \cos(\beta n + \theta)\, u[n]$	$\lvert\gamma_2\rvert^n u[n]$	$\frac{1}{R}\left[\lvert\gamma_1\rvert^{n+1}\cos(\beta(n+1)+\theta-\phi) - \lvert\gamma_2\rvert^{n+1}\cos(\theta-\phi)\right] u[n]$
		$R = \left[\lvert\gamma_1\rvert^2 + \lvert\gamma_2\rvert^2 - 2\lvert\gamma_1\rvert\lvert\gamma_2\rvert\cos(\beta)\right]^{\frac{1}{2}}$
		$\phi = \arctan\left(\frac{\lvert\gamma_1\rvert\sin(\beta)}{\lvert\gamma_1\rvert\cos(\beta)-\lvert\gamma_2\rvert}\right)$

9.6 Properties of DT Convolution

There are several useful properties of convolution. We do not prove these here, but it is not terribly difficult to do so. Given signals $x_1[n]$, $x_2[n]$, and $x_3[n]$:

Commutative Property The ordering of the signals does not matter.

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

Distributive Propery Convolution is distributed over addition.

$$x_1[n] * (x_2[n] + x_3[n]) = (x_1[n] * x_2[n]) + (x_1[n] * x_3[n])$$

Associative Property The order of convolution does not matter.

$$x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$$

Index Shift Given $x_3[n] = x_1[n] * x_2[n]$ then for index shifts $m_1, m_2 \in \mathbb{R}$

$$x_1[n - m_1] * x_2[n - m_2] = x_3[n - m_1 - m_2]$$

Multiplicative Scaling Given $x_3[n] = x_1[n] * x_2[n]$ then for constants $a, b \in \mathbb{C}$

$$(a\, x_1[n]) * (b\, x_2[n]) = a\, b\, x_3[n]$$

These properties can be used in combination with a table like that above to compute the convolution of a wide variety of signals without evaluating the summations.

Example 9.6.1. Consider the convolution of the causal DT pulse of length N, $x_1[n] = u[n] - u[n-N]$, and the signal $x_2[n] = \left(\frac{1}{2}\right)^n u[n]$.

$$\begin{aligned}
x_1[n] * x_2[n] &= (u[n] - u[n-N]) * \left(\left(\frac{1}{2}\right)^n u[n]\right) \\
&= (u[n]) * \left(\left(\frac{1}{2}\right)^n u[n]\right) - (u[n-N]) * \left(\left(\frac{1}{2}\right)^n u[n]\right) \quad \text{using distributive property} \\
&= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \left(\frac{1}{2}\right)} u[n] - \left.\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \left(\frac{1}{2}\right)} u[n]\right|_{n \to n-N} \quad \text{from Table row 2 and index shift property} \\
&= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)} u[n] - \frac{1 - \left(\frac{1}{2}\right)^{n-N+1}}{\left(\frac{1}{2}\right)} u[n-N] \\
&= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)} u[n] - \frac{1 - \left(\frac{1}{2}\right)^{-N} \left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)} u[n-N] \\
&= 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) u[n] - 2\left(1 - \left(\frac{1}{2}\right)^{-N} \left(\frac{1}{2}\right)^{n+1}\right) u[n-N] \\
&= \left(2 - \left(\frac{1}{2}\right)^{n}\right) u[n] - \left(2 - \left(\frac{1}{2}\right)^{-N} \left(\frac{1}{2}\right)^{n}\right) u[n-N]
\end{aligned}$$

■

Chapter 10

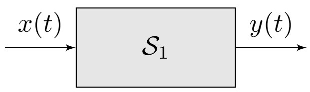
CT Block Diagrams

10.1 The Four Basic Motifs

Understanding complex systems, with many interconnections, is aided by graphical representations, generally called block diagrams [1]. They are a hybrid graphical-analytical approach.

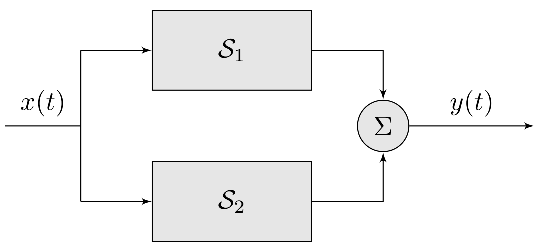
There are just four basic motifs needed to build any block diagram. Let $\mathcal{S}_i$ denote a (sub) system. Then the four motifs are:

- A single block.



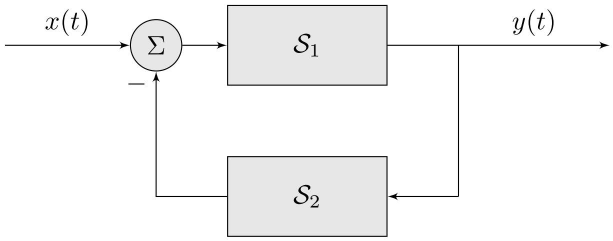
- A *series* connection of two blocks

- A *parallel* connection of two blocks



- A *feedback* connection

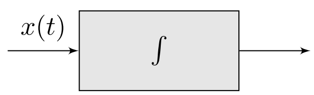
[1] There is a closely related graphical approach called *signal flow graphs* that you may learn about in upper-level courses. They are equivalent to block diagrams, but are more amenable to computer representation and manipulation.



Note the feedback is negative (the minus sign on the feedback summation input). These can be use in various combinations, as we shall see shortly.

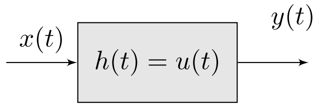
10.2 Connections to Convolution

Each subsystem, $\mathcal{S}_i$, can be represented by a basic time-domain operation (e.g. derivatives, integrals, addition, and scaling) or more generally by its impulse response $h_i(t)$. For example a block representing an system acting as integrator is typically drawn as



$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

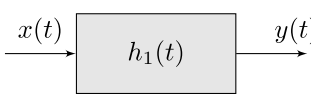
This is equivalent to an impulse response $h(t) = u(t)$ so that it might also be drawn as



$$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

We can use the concept of convolution to connect block diagrams to the properties of convolution

- A single block is equivalent to convolution with the impulse response for that subsystem



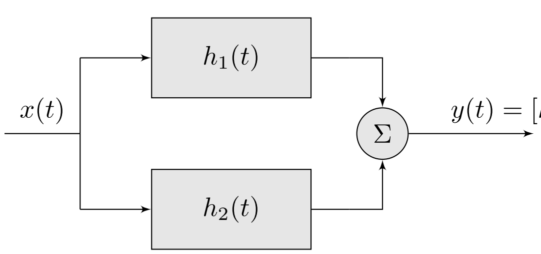
$$y(t) = h_1(t) * x(t)$$

- Using the associative property, a series connection of two blocks becomes

$$y(t) = [h_1(t) * h_2(t)] * x(t)$$

  which can be reduced to a single convolution $y(t) = h_3(t) * x(t)$ where $h_3(t) = h_1(t) * h_2(t)$.

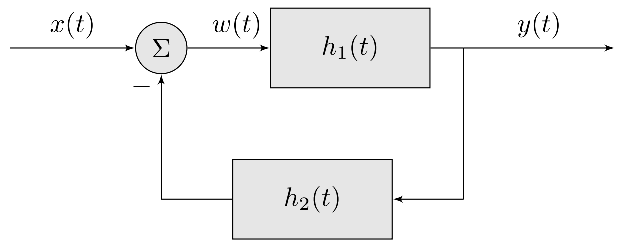
- Using the distributive property, a parallel connection of two blocks becomes



$$y(t) = [h_1(t) * x(t)] + [h_2(t) * x(t)] = [h_1(t) + h_2(t)] * x(t)$$

which is equivalent to a single convolution $y(t) = h_3(t) * x(t)$ where $h_3(t) = h_1(t) + h_2(t)$.

- In the feedback connection let $w(t)$ be the output of the summation



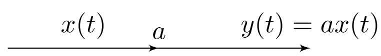
Then $y(t) = h_1(t) * w(t)$ and $w(t) = x(t) - h_2(t) * y(t)$. Substituting the later into the former gives $y(t) = h_1 * (x - h_2(t) * y(t))$. Using the distributive property we get $y(t) = h_1(t) * x(t) - h_1(t) * h_2(t) * y(t)$. Isolating the input on the right-hand side and using $y(t) = \delta(t) * y(t)$ we get

$$y(t) + h_1(t) * h_2(t) * y(t) = [\delta(t) + h_1(t) * h_2(t)] * y(t) = h_1(t) * x(t)$$

We can solve this for $y(t)$ using the concept of inverse systems. Let $h_3(t) * [\delta(t) + h_1(t) * h_2(t)] = \delta(t)$, i.e. h_3 is the inverse system of $\delta(t) + h_1(t) * h_2(t)$. Then

$$y(t) = h_3(t) * h_1(t) * x(t)$$

Recall, when the system is instantaneous (memoryless) the impulse response is $a\delta(t)$ for some constant a. This is the same as scaling the signal by a. We typically drop the block in such cases and draw the input-output operation as

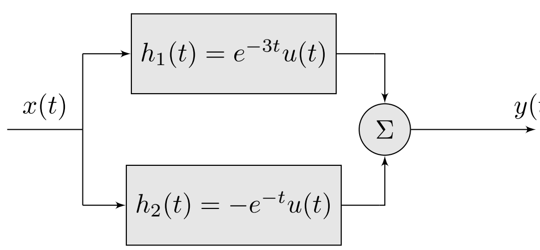


These properties allow us to perform transformations, either breaking up a system into subsystems, or reducing a system to a single block.

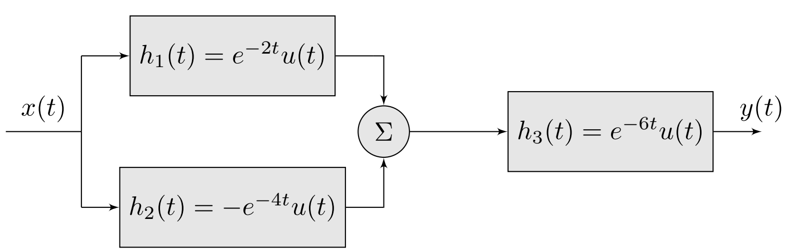
Example 10.2.1. Consider a second-order system system with impulse response

$$h(t) = \left(e^{-3t} - e^{-t}\right) u(t)$$

We can express this as a block diagram consisting of two parallel blocks



Example 10.2.2. Consider a system with block diagram



We can determine the overall impulse response of this system using the distributive and associative properties

$$\begin{aligned} h(t) &= [h_1(t) + h_2(t)] * h_3(t) \\ &= h_1(t) * h_3(t) + h_2(t) * h_3(t) \\ &= \left[e^{-2t}u(t)\right] * \left[e^{-6t}u(t)\right] + \left[-e^{-4t}u(t)\right] * \left[e^{-6t}u(t)\right] \end{aligned}$$

Using the convolution table from Lecture 8 we get the overall impulse response

$$h(t) = \frac{e^{-2t} - e^{-6t}}{4}u(t) - \frac{e^{-4t} - e^{-6t}}{2}u(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t) + \frac{1}{4}e^{-6t}u(t)$$

10.3 Connections to LCCDE

The other system representation we have seen are linear, constant-coefficient differential equations. These can be expressed as combinations of derivative and/or integration blocks.

First-Order System

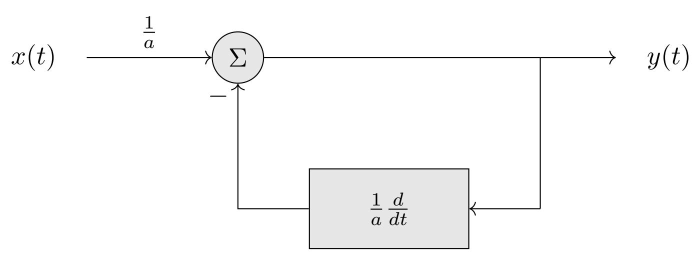
To illustrate this consider the first-order LCCDE

$$\frac{dy}{dt}(t) + ay(t) = x(t)$$

We can solve this for $y(t)$

$$y(t) = -\frac{1}{a}\frac{dy}{dt}(t) + \frac{1}{a}x(t)$$

and can express this as a feedback motif



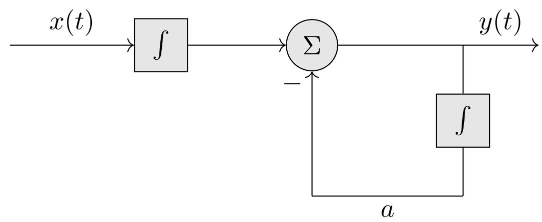
Alternatively we could integrate the differential equation

$$\begin{aligned} \frac{dy}{dt}(t) + ay(t) &= x(t) \\ \int_{-\infty}^{t} \frac{dy}{dt}(\tau)\, d\tau + a\int_{-\infty}^{t} y(\tau)\, d\tau &= \int_{-\infty}^{t} x(\tau)\, d\tau \\ y(\tau)\Big|_{-\infty}^{t} + a\int_{-\infty}^{t} y(\tau)\, d\tau &= \int_{-\infty}^{t} x(\tau)\, d\tau \end{aligned}$$

Under the assumption $y(-\infty) = 0$ we can solve this for $y(t)$ to get

$$y(t) = -a\int_{-\infty}^{t} y(\tau)\, d\tau + \int_{-\infty}^{t} x(\tau)\, d\tau$$

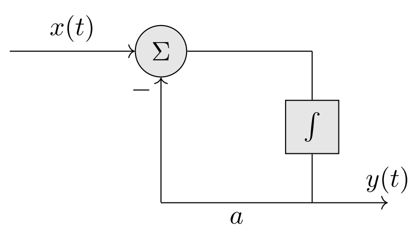
which can be expressed as the block diagram



We can simplify this block diagram, by noting

$$
\begin{aligned}
y(t) &= -a \int_{-\infty}^{t} y(\tau)\, d\tau + \int_{-\infty}^{t} x(\tau)\, d\tau \\
&= \int_{-\infty}^{t} \left(-ay(\tau) + x(\tau)\right)\, d\tau
\end{aligned}
$$

which requires only a single integrator



The choice of using derivative or integrator blocks is not arbitrary in practice. Derivatives are sensitive to noise at high frequencies (for reasons we will see later in the semester) and so integrators perform much better when implemented in hardware.

Second-Order System

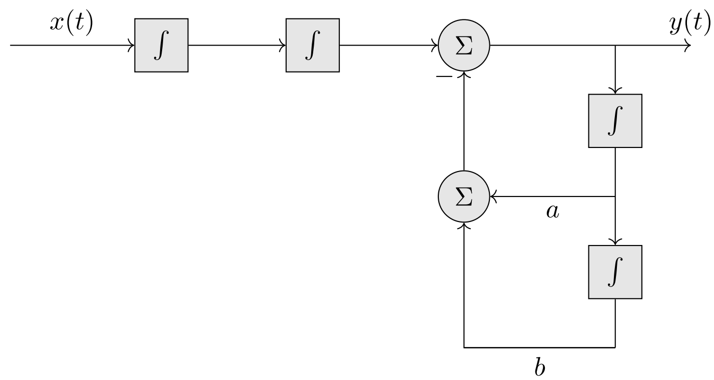
Now consider the second-order system

$$
\frac{d^2y}{dt^2}(t) + a\frac{dy}{dt}(t) + by(t) = x(t)
$$

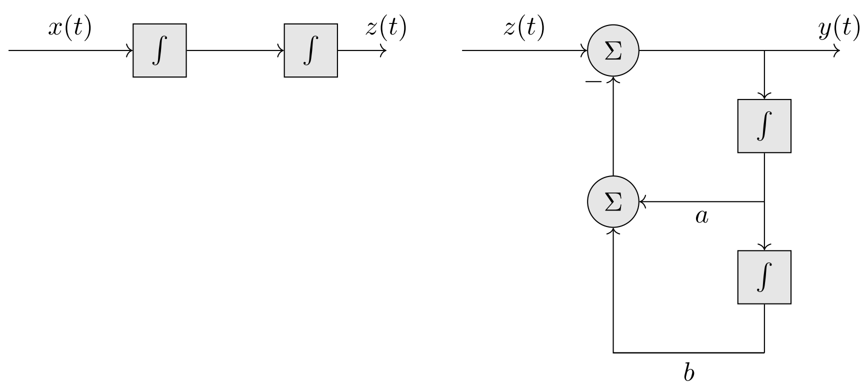
Using a similar process to the first-order system, we can express this as (dropping the limits of integration for clarity):

$$
y(t) = -a \int y(\tau)\, d\tau + \int \int \left(-by(\tau) + x(\tau)\right)\, d\tau^2
$$

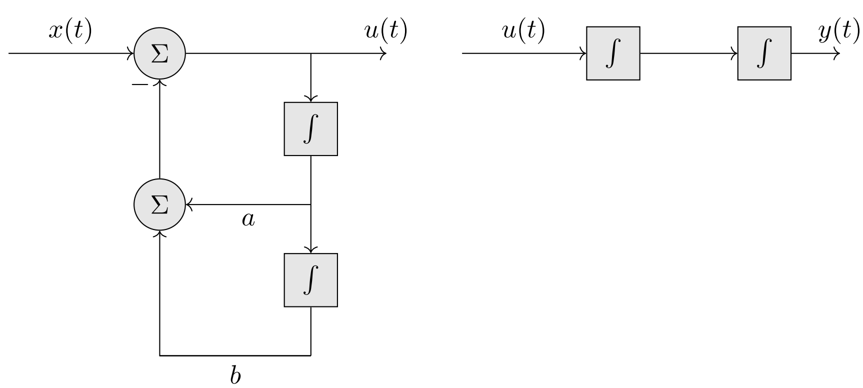
which has the block diagram



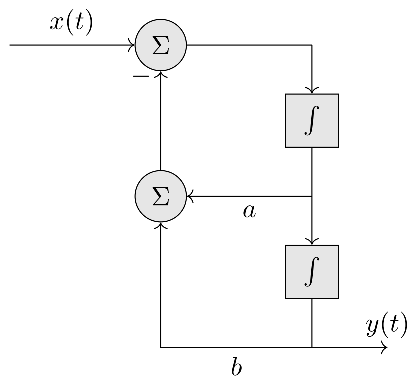
This is equivalent to two systems in series



Recall that, from the commutative property of convolution, the order of systems in series can be swapped

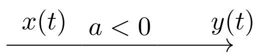
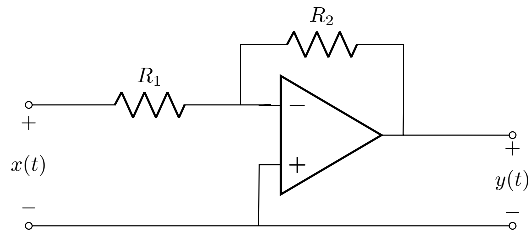
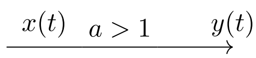
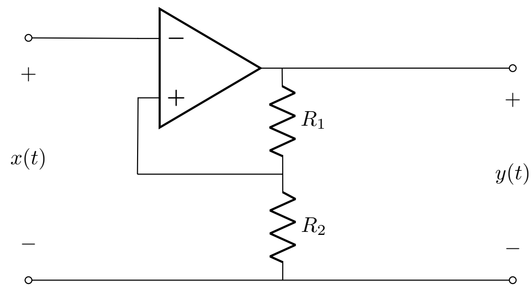
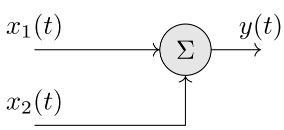
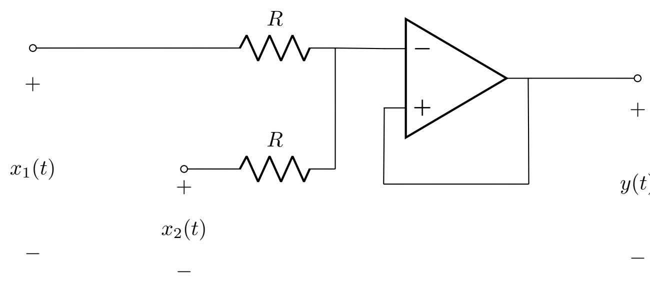
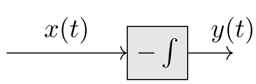
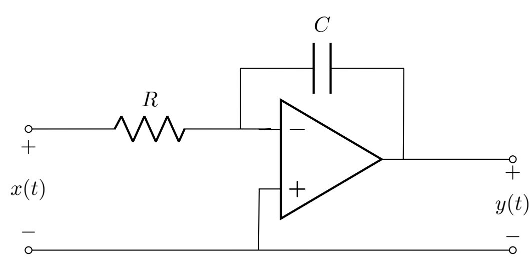


We then note that the signal z and the output of the integrator blocks are the same in both systems so that they can be combined into a single block diagram as follows, reducing the number of integrators by two



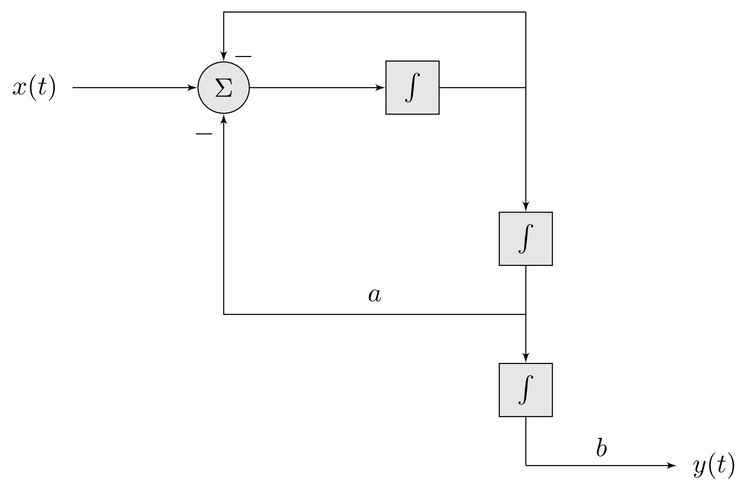
10.4 Implementing a System in Hardware

One of the most powerful uses of block diagrams is the implementation of a CT system in hardware. As we shall see later in the semester, designing CT systems for a particular purpose leads to a mathematical description that is equivalent to either an impulse response, or a LCCDE. We have seen how these can be represented as block diagrams. Once we have reduced a system to blocks consisting of simple operations, we can then convert the block diagram to a circuit.

Block	Typical Circuit
	
	
	
	

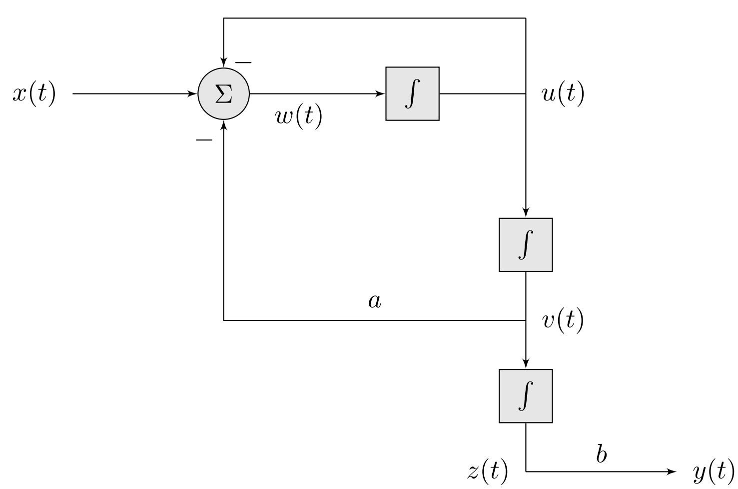
Solved Problems

1. Consider a system with the following block diagram:



Determine the differential equation representation of this system.

Solution: We can convert this back to a differential equation representation as follows. First label the output of each block as a signal (called the internal states of the system), which we denote as $u(t)$, $v(t)$, $w(t)$, and $z(t)$ below.



Now we can read off the input-output relationships moving from input to output. Starting with the output of the summation

$$w(t) = x(t) - u(t) - a\,v(t) \ .$$

The outputs of each integrator are:

$$u(t) = \int_{-\infty}^{t} w(\tau)\ d\tau\ ,\ v(t) = \int_{-\infty}^{t} u(\tau)\ d\tau\ ,\ \text{and}\ \ z(t) = \int_{-\infty}^{t} v(\tau)\ d\tau$$

or equivalently

$$\frac{du}{dt}(t) = w(t)\ ,\ \frac{dv}{dt}(t) = u(t)\ ,\ \text{ and }\ \frac{dz}{dt}(t) = v(t)$$

Finally, the output is:

$$y(t) = b\, z(t)\ .$$

We now do a series of derivatives and substitutions

$$\begin{aligned}
y(t) &= b\, z(t) \\
\frac{dy}{dt}(t) &= b\, \frac{dz}{dt}(t) \\
&= b\, v(t) \\
\frac{d^2y}{dt^2}(t) &= b\, \frac{dv}{dt}(t) \\
&= b\, u(t) \\
\frac{d^3y}{dt^3}(t) &= b\, \frac{du}{dt}(t) \\
&= b\, w(t) \\
&= b\,(x(t) - u(t) - a\, v(t))
\end{aligned}$$

Rearranging the last equation to isolate the input on the right hand side gives

$$\frac{d^3y}{dt^3}(t) + b\, u(t) + ab\, v(t) = b\, x(t) \quad \text{(Eqn. 1)}$$

We can now note from above

$$u(t) = \frac{dv}{dt}(t) = \frac{d^2z}{dt^2}(t) = \frac{1}{b}\frac{d^2y}{dt^2}(t) \text{ and}$$

$$v(t) = \frac{dz}{dt}(t) = \frac{1}{b}\frac{dy}{dt}(t)\ .$$

Substituting these back into Eqn. 1 gives

$$\frac{d^3y}{dt^3}(t) + \frac{d^2y}{dt^2}(t) + a\,\frac{dy}{dt}(t) = b\, x(t)$$

Which is a LCCDE.

■

Chapter 11

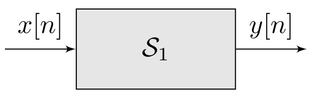
DT Block Diagrams

11.1 The Four Basic Motifs

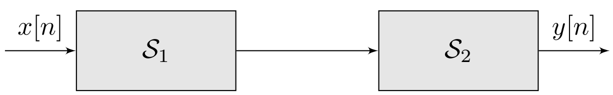
Block diagrams of DT systems are similar to CT systems.

The four motifs are:

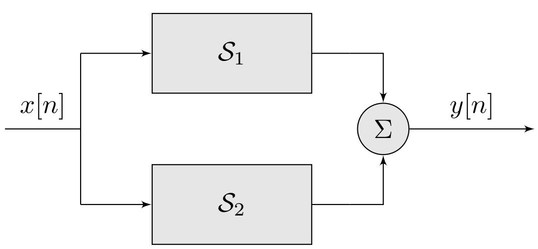
- A single block.



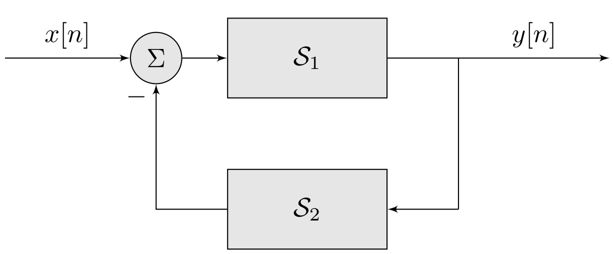
- A *series* connection of two blocks



- A *parallel* connection of two blocks



- A *feedback* connection

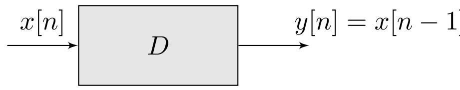


Note the feedback is negative (the minus sign on the feedback summation input). As in CT, these can be used in various combinations.

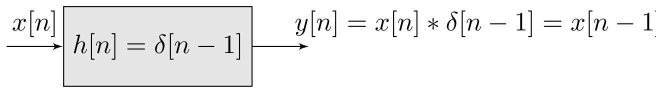
11.2 Connections to Convolution

Each subsystem, $\mathcal{S}_i$, can be represented by a basic discrete time-domain operation (e.g. differences, running sums, addition, and scaling) or more generally by its impulse response $h_i[n]$.

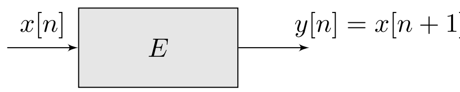
For example a block representing an system acting as a delay of one sample is typically drawn as



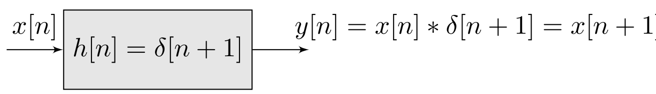
This is equivalent to an impulse response $h[n] = \delta[n-1]$ so that it might also be drawn as



Similarly, a block representing an system acting as an advance of one sample is typically drawn as

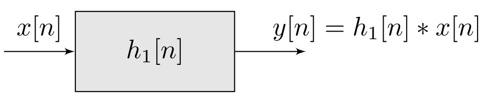


This is equivalent to an impulse response $h[n] = \delta[n+1]$ so that it might also be drawn as

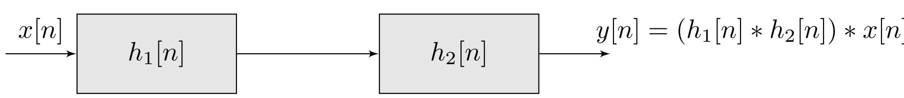


We can use the concept of convolution to connect block diagrams to the properties of convolution

- A single block is equivalent to convolution with the impulse response for that subsystem

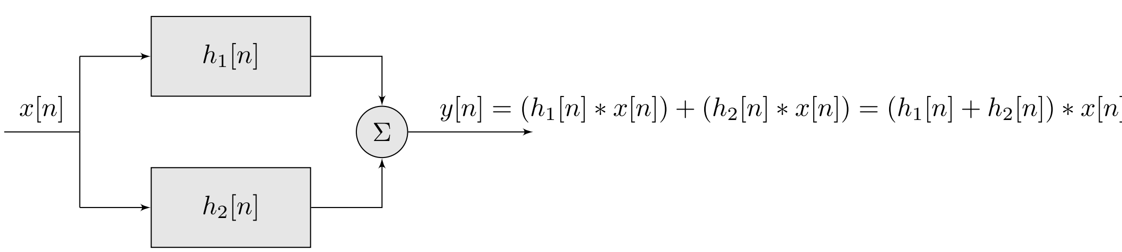


- Using the associative property, a series connection of two blocks becomes



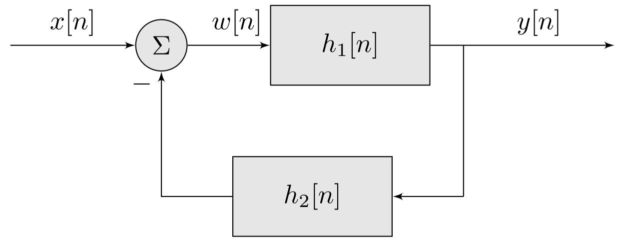
  which can be reduced to a single convolution $y[n] = h_3[n] * x[n]$ where $h_3[n] = h_1[n] * h_2[n]$.

- Using the distributive property, a parallel connection of two blocks becomes



  which is equivalent to a single convolution $y[n] = h_3[n] * x[n]$ where $h_3[n] = h_1[n] + h_2[n]$.

- In the feedback connection let $w[n]$ be the output of the summation



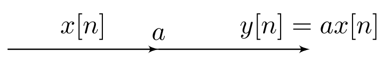
Then $y[n] = h_1[n] * w[n]$ and $w[n] = x[n] - h_2[n] * y[n]$. Substituting the later into the former gives $y[n] = h_1 * (x - h_2[n] * y[n])$. Using the distributive property we get $y[n] = h_1[n] * x[n] - h_1[n] * h_2[n] * y[n]$. Isolating the input on the right-hand side and using $y[n] = \delta[n] * y[n]$ we get

$$y[n] + h_1[n] * h_2[n] * y[n] = (\delta[n] + h_1[n] * h_2[n]) * y[n] = h_1[n] * x[n]$$

We can solve this for $y[n]$ using the concept of inverse systems. Let $h_3[n] * (\delta[n] + h_1[n] * h_2[n]) = \delta[n]$, i.e. h_3 is the inverse system of $\delta[n] + h_1[n] * h_2[n]$. Then

$$y[n] = h_3[n] * h_1[n] * x[n]$$

Recall, when the system is instantaneous (memoryless) the impulse response is $a\delta[n]$ for some constant a. This is the same as scaling the signal by a. We typically drop the block in such cases and draw the input-output operation as

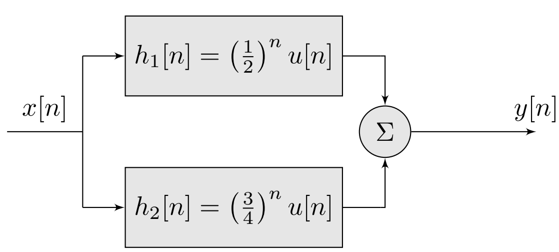


These properties allow us to perform transformations, either breaking up a system into subsystems, or reducing a system to a single block.

Example 11.2.1. Consider a second-order system system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{3}{4}\right)^n u[n]$$

We can express this as a block diagram consisting of two parallel blocks



11.3 Connections to LCCDE

The other DT system representation we have seen are linear, constant-coefficient difference equations. These can be expressed as combinations of advance or delay blocks. This is straightforward compared to the CT system case.

First-Order System

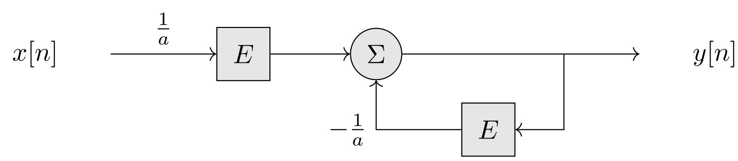
To illustrate this consider the first-order LCCDE

$$y[n+1] + ay[n] = x[n+1]$$

We can solve this for $y[n]$

$$y[n] = -\frac{1}{a}y[n+1] + \frac{1}{a}x[n+1]$$

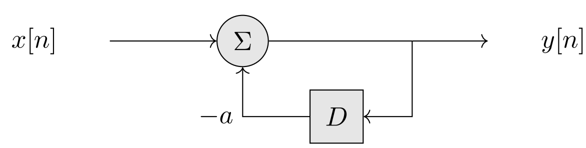
and can express this as a feedback motif using the advance operator E



Alternatively we could rewrite the difference equation in recursive delay form

$$y[n] = -ay[n-1] + x[n]$$

which can be expressed as a block diagram using the delay operator, D



The choice of using advance or delay blocks results in a non-causal or causal (respectively) system. Thus, delay blocks are required for real-time DT system implementations.

Second-Order System

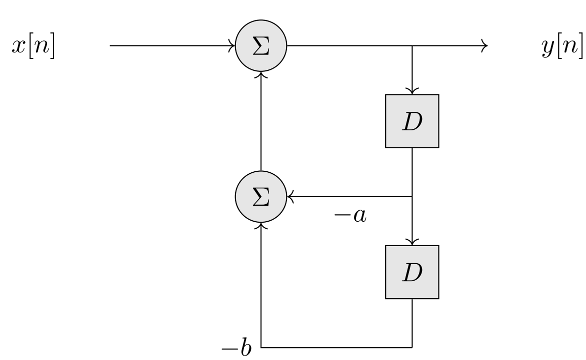
Now consider the second-order system

$$y[n+2] + ay[n+1] + by[n] = x[n+2]$$

Again, writing in recursive delay form

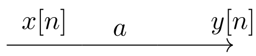
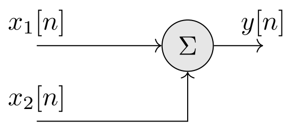
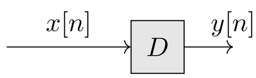
$$y[n] = -ay[n-1] - by[n-2] + x[n]$$

we obtain the block diagram

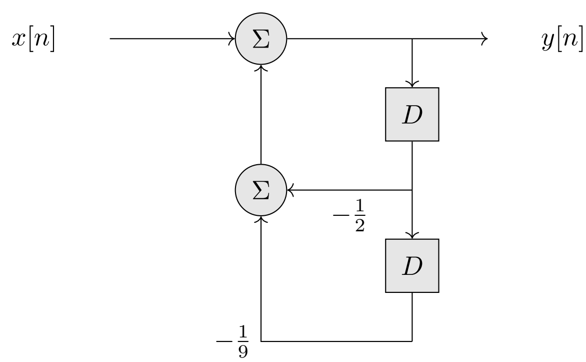


11.4 Implementing a DT System

As in the CT case, one of the most powerful uses of block diagrams is the implementation of a DT system in hardware. As we shall see later in the semester, designing a DT system for a particular purpose leads to a mathematical description that is equivalent to either an impulse response or a LCCDE. We have seen how these can be represented as block diagrams. Once we have reduced a system to blocks consisting of simple operations, we can then convert the block diagram to a digital circuit, implement using a digital signal processor, or write an equivalent program to run on an embedded or general purpose computer.

Block	Typical Digital Circuit
	Multiplier (ALU)
	Adder (ALU)
	Shift Register

Example 11.4.1. The following C++ code implements the second order system given by



using floating point calculations. It assumes the current input is obtained via the function `read`, and the output written using the function `write`. The delayed values of the output are stored in the array `buffer` and are initialized to zero ("at rest" prior to application of the input).

```
double buffer[2] = {0.0,0.0};
while(true){
  double x = read();
  double y = -0.5*buffer[1] - buffer[0]/9.0 + x;
  write(y);
  buffer[0] = buffer[1];
  buffer[1] = y;
}
```

Note in real applications it is common to replace the floating point calculations with fixed-width (scaled integer) ones. ■

Chapter 12

Eigenfunctions of CT systems

To summarize the course so far given an input signal $x(t)$ and a LTI system described (equivalently) by a linear, constant coefficient differential equation, impulse response, or a block diagram, we can determine the output using convolution. This is referred to as *time-domain* analysis.

The advantages of this approach are that the analysis is straightforward (if cumbersome) and it applies to all LTI systems, stable or otherwise. Time-domain representations of signals are also intuitive given their direct application in physical systems.

There are also some disadvantages. First, time-domain analysis does not scale well to larger systems since analysis with block diagram decompositions requires convolution, and in the case of the feedback motif dealing with inverse systems or de-convolution. Second, it is difficult to design an impulse responses for a given purpose. Finally implementing a system directly from an impulse response is not intuitive.

We can borrow a technique from mathematics to overcome these disadvantages by transforming the *domain* of the representations to one in which the operation of convolution becomes one of multiplication. This approach, called generally *frequency domain* analysis has a number of advantages and will be our focus for the remainder of the course.

12.1 The Response of LTI Systems to Complex Exponentials

Recall convolution can be viewed as a decomposition of a signal into an infinite sum of δ functions plus the linearity property.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau \;\longrightarrow\; y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

We now consider a different decomposition based on the complex exponential, e^{st} for $s \in \mathbb{C}$, rather than δ functions. As we will see this decomposition simplifies convolution, turning it into multiplication.

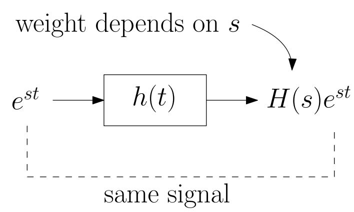
12.1.1 Eigenfunction e^{st} and Transfer Function $H(s)$

Let $x(t) = e^{st}$ for $s \in \mathbb{C}$, then $y(t) = h(t) * x(t) = x(t) * h(t)$ and by the definition of convolution

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\, d\tau = \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}\, d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau \\ &= e^{st}H(s) \end{aligned}$$

where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau$ is the *Laplace Transform* of the impulse response, $h(t)$. $H(s)$ is called the *transfer function* or *Eigenvalue* of the system and e^{st} is the *Eigenfunction* for CT LTI systems.

Similar to the impulse function, the complex exponential is a special signal because it's response is easy to determine. It is just the same signal scaled by a multiplicative factor as illustrated below:



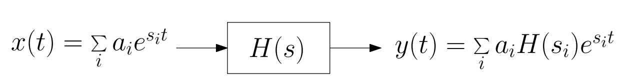
Example 12.1.1. Suppose $H(s) = \frac{1}{s+1}$ and $x(t) = e^{(-4+j2\pi)t}$. Then the output is

$$\begin{aligned} y(t) &= H(-4+j2\pi)e^{(-4+j2\pi)t} \\ &= \frac{1}{-4+j2\pi+1}e^{(-4+j2\pi)t} \\ &= \frac{1}{-3+j2\pi}e^{(-4+j2\pi)t}\ , \end{aligned}$$

another complex exponential.

■

Given $H(s)$ and inputs that are sums of complex exponentials, the output is easy to determine.



In some cases the sums are countably infinite while in others the uncountably infinite so that the sums become integrals.

Example 12.1.2. Consider the CT system with impulse response response

$$h(t) = e^{-5t}u(t)$$

Determine the Eigenvalues that corresponds to the input $x(t) = \cos(t)$ and the output $y(t)$.

Solution: We note the cosine can be decomposed into two complex exponentials as

$$\cos(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$$

Thus in terms of the general decomposition there are two terms with complex constants $s_1 = 0 + j$ and $s_2 = 0 - j$ and real constants $a_1 = a_2 = \frac{1}{2}$.

$$x(t) = \sum_i a_i e^{s_i t} = a_1 e^{s_1 t} + a_2 e^{s_2 t} = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} = \cos(t)$$

Then the output is given by

$$y(t) = \sum_i H(s_i) a_i e^{s_i t} = H(s_1) a_1 e^{s_1 t} + H(s_2) a_2 e^{s_2 t} = H(j)\frac{1}{2}e^{jt} + H(-j)\frac{1}{2}e^{-jt}$$

which requires we find the Eigenvalues $H(j)$ and $H(-j)$. To do so we use the Laplace integral

$$H(j) = \int_{-\infty}^{\infty} h(\tau) e^{-j\tau}\, d\tau = \int_{0}^{\infty} e^{-5\tau}\, e^{-j\tau}\, d\tau = \int_{0}^{\infty} e^{-(j+5)\tau}\, d\tau = \frac{-1}{j+5} e^{-(j+5)\tau}\Big|_0^{\infty} = \frac{1}{j+5}$$

Similarly

$$H(-j) = \int_{-\infty}^{\infty} h(\tau) e^{j\tau}\, d\tau = \int_{0}^{\infty} e^{-5\tau}\, e^{j\tau}\, d\tau = \int_{0}^{\infty} e^{-(-j+5)\tau}\, d\tau = \frac{-1}{-j+5} e^{-(j+5)\tau}\Big|_0^{\infty} = \frac{1}{-j+5}$$

Substituting back into the output equation gives

$$\begin{aligned} y(t) &= H(j)\frac{1}{2}e^{jt} + H(-j)\frac{1}{2}e^{-jt} \\ &= \frac{1}{j+5}\frac{1}{2}e^{jt} + \frac{1}{-j+5}\frac{1}{2}e^{-jt} \end{aligned}$$

We can simplify this expression using the polar form of the Eigenvalues

$$\begin{aligned} y(t) &= \frac{1}{j+5}\frac{1}{2}e^{jt} + \frac{1}{-j+5}\frac{1}{2}e^{-jt} \\ &= Re^{j\theta}\frac{1}{2}e^{jt} + Re^{-j\theta}\frac{1}{2}e^{-jt} \\ &= R\frac{1}{2}e^{jt+j\theta} + R\frac{1}{2}e^{-jt-j\theta} \\ &= R\cos(t+\theta) \end{aligned}$$

where

$$R = \left|\frac{1}{j+5}\right| = \frac{1}{\sqrt{26}} \text{ and } \theta = \angle\frac{1}{j+5} = -\arctan\frac{1}{5}$$

Note for this system, given a sinusoidal input, the output is a scaled and phase shifted sinusoid at the same frequency, where the scaling factor and phase shift is system dependent. It is illustrative to compare this analysis to the time-domain analysis of the same impulse response and input using convolution. ■

12.2 Decomposition of signals using complex exponentials

In this course we consider the cases of stable CT systems. Recall a stable system is one in which a bounded input leads to a bounded output, or equivalently the impulse response is absolutely integrable. We will consider two decompositions of the input:

- *Fourier Series*: When $x(t)$ is periodic with fundamental frequency ω_0, $\mathrm{Re}\,(s) = 0$ so that $s = jk\omega_0$, and the decomposition is a countably infinite sum. This gives the input-output relationship

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \;\longrightarrow\; y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) \, a_k \, e^{jk\omega_0 t}$$

  where $H(jk\omega_0)$ are the Eigenvalues, also called the *frequency response*.

- *Inverse Fourier Transform*: When $x(t)$ is a-periodic, $\mathrm{Re}\,(s) = 0$ so that $s = j\omega$, and the decomposition is an uncountably infinite sum (real integral over ω). This gives the input-output relationship

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, e^{j\omega t} \, d\omega \;\longrightarrow\; y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) X(\omega) \, e^{j\omega t} \, d\omega$$

  where $H(j\omega)$ are the Eigenvalues, again called the *frequency response*.

Other courses (e.g. ECE 3704) look at the general case of unstable systems and $s \in \mathbb{C}$ with decompositions:

- *One-Sided Laplace Transform*: $x(t)$ is causal and the decomposition is an uncountably infinite sum (complex integral)
- *Two-Sided (Bilateral) Laplace Transform*: $x(t)$ is non-causal and the decomposition is an uncountably infinite sum (complex integral). This is the most general case for CT LTI systems.

While the Laplace decompositions require complex integration, they can be understood and computed using algebra and a table of forward transforms, which only require integration of a complex function of a real variable t (this is the general approach taken in upper level courses). However, this is outside the scope of this course because of time limitations.

Instead, we will be spending the next few weeks going through the CT Fourier decompositions in some detail. You will also learn how to find the CT frequency response for a stable system, and see how to use both for analysis.

Chapter 13

Eigenfunctions of DT systems

To summarize the course so far for DT analysis, given an input signal $x[n]$ and a LTI system described (equivalently) by a linear, constant coefficient difference equation, impulse response, or a block diagram, we can determine the output using convolution. This is referred to as *discrete time-domain* analysis since the index n usually refers to a time index.

Like in CT, the advantages of this approach are that the analysis is straightforward and applies to all LTI systems, stable or otherwise. Discrete time-domain representations of signals are also intuitive when viewed as equally-spaced samples of physical signals.

As in CT, there are disadvantages. It does not scale well to larger systems since analysis with block diagram decompositions requires convolution, and in the case of the feedback motif dealing with inverse systems or de-convolution. It is difficult to design an impulse responses for a given purpose. Finally implementing a DT system directly from an impulse response is not intuitive.

Similar to CT we can transform the domain of the signal representations to one in which the operation of DT convolution becomes one of multiplication.

13.1 The Response of DT LTI Systems to Complex Exponentials

Recall convolution can be viewed as a decomposition of a signal into an infinite sum of δ functions plus the linearity property.

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

We now consider a different decomposition based on the complex exponential, z^n for $z \in \mathbb{C}$, rather than δ functions. As we will see this decomposition simplifies convolution, turning it into multiplication.

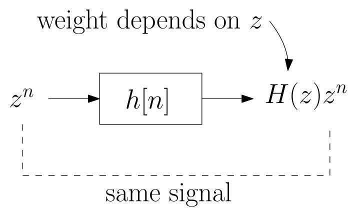
13.1.1 Eigenfunction z^n and Transfer Function $H(z)$

Let $x[n] = z^n$ for $z \in \mathbb{C}$, then $y[n] = h[n] * x[n] = x[n] * h[n]$ and by the definition of DT convolution

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \sum_{m=-\infty}^{\infty} h[m]z^{n-m} = \sum_{m=-\infty}^{\infty} h[m]z^n z^{-m} \\ &= z^n \sum_{m=-\infty}^{\infty} h[m]z^{-m} \\ &= z^n H(z) \end{aligned}$$

where $H(z) = \sum_{m=-\infty}^{\infty} h[m]z^{-m}$ is the *Z Transform* of the impulse response, $h[n]$. $H(z)$ is called the *transfer function* or *Eigenvalue* of the system and z^n is the *Eigenfunction* for DT LTI systems.

Similar to the impulse function, the complex exponential is a special signal because its response is easy to determine. It is just the same signal scaled by a multiplicative factor as illustrated below:



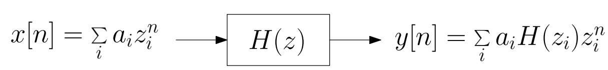
Example 13.1.1. For example, suppose $H(z) = \frac{z}{z-\frac{1}{2}}$ and $x[n] = \left(-\frac{1}{4}\right)^n$. Then the output is

$$\begin{aligned} y[n] &= H\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^n \\ &= \frac{-\frac{1}{4}}{-\frac{1}{4}-\frac{1}{2}}\left(-\frac{1}{4}\right)^n \\ &= \frac{1}{3}\left(-\frac{1}{4}\right)^n , \end{aligned}$$

another complex exponential.

■

Given $H(z)$ and inputs that are sums of complex exponentials, the output is easy to determine.



In some cases the sums are countably infinite while in others the uncountably infinite so that the sums become integrals.

Example 13.1.2. Consider the DT system with impulse response response

$$h[n] = \left(\frac{3}{4}\right)^n u[n]$$

Determine the Eigenvalues that corresponds to the input $x[n] = \cos(n)$ and the output $y[n]$.

Solution: We note the cosine can be decomposed into two complex exponentials as

$$\cos(n) = \frac{1}{2}e^{jn} + \frac{1}{2}e^{-jn} = \frac{1}{2}\left(e^j\right)^n + \frac{1}{2}\left(e^{-j}\right)^n$$

Thus in terms of the general decomposition there are two terms with complex constants $z_1 = e^j$ and $z_2 = e^{-j}$ and real constants $a_1 = a_2 = \frac{1}{2}$.

$$x[n] = \sum_i a_i z_i^n = a_1 z_1^n + a_2 z_2^n = \frac{1}{2}\left(e^j\right)^n + \frac{1}{2}\left(e^{-j}\right)^n = \cos(n)$$

Then the output is given by

$$y[n] = \sum_i H(z_i) a_i z_i^n = H(z_1)a_1 z_1^n + H(z_2)a_2 z_2^n = H\left(e^j\right)\frac{1}{2}\left(e^j\right)^n + H\left(e^{-j}\right)\frac{1}{2}\left(e^{-j}\right)^n$$

which requires we find the Eigenvalues $H\left(e^j\right)$ and $H\left(e^{-j}\right)$. To do so we use the Z transform summation

$$H\left(e^j\right) = \sum_{m=-\infty}^{\infty} h[m]\left(e^j\right)^{-m} = \sum_{m=0}^{\infty}\left(\frac{3}{4}\right)^m\left(e^j\right)^{-m} = \sum_{m=0}^{\infty}\left(\frac{3}{4\left(e^j\right)}\right)^m = \frac{-1}{\left(\frac{3}{4e^j}\right)-1} = \frac{e^j}{e^j-\left(\frac{3}{4}\right)}$$

Similarly

$$H\left(e^{-j}\right) = \sum_{m=-\infty}^{\infty} h[m]\left(e^{-j}\right)^{-m} = \sum_{m=0}^{\infty}\left(\frac{3}{4}\right)^m\left(e^{-j}\right)^{-m} = \sum_{m=0}^{\infty}\left(\frac{3}{4\left(e^{-j}\right)}\right)^m = \frac{-1}{\left(\frac{3}{4e^{-j}}\right)-1} = \frac{e^{-j}}{e^{-j}-\left(\frac{3}{4}\right)}$$

Substituting back into the output equation gives

$$\begin{aligned} y[n] &= H\left(e^j\right)\frac{1}{2}\left(e^j\right)^n + H\left(e^{-j}\right)\frac{1}{2}\left(e^{-j}\right)^n \\ &= \frac{e^j}{e^j-\left(\frac{3}{4}\right)}\frac{1}{2}\left(e^j\right)^n + \frac{e^{-j}}{e^{-j}-\left(\frac{3}{4}\right)}\frac{1}{2}\left(e^{-j}\right)^n \end{aligned}$$

We can simplify this expression using the polar form of the Eigenvalues

$$\begin{aligned} y[n] &= \frac{e^j}{e^j-\left(\frac{3}{4}\right)}\frac{1}{2}\left(e^j\right)^n + \frac{e^{-j}}{e^{-j}-\left(\frac{3}{4}\right)}\frac{1}{2}\left(e^{-j}\right)^n \\ &= Re^{j\theta}\frac{1}{2}e^{jn} + Re^{-j\theta}\frac{1}{2}e^{-jn} \\ &= R\frac{1}{2}e^{jn+j\theta} + R\frac{1}{2}e^{-jn-j\theta} \\ &= R\cos(n+\theta) \end{aligned}$$

where

$$R = \left|\frac{e^j}{e^j-\left(\frac{3}{4}\right)}\right| \approx 1.153 \text{ and } \theta = \angle\frac{e^j}{e^j-\left(\frac{3}{4}\right)} \approx -0.815$$

Note for this system, given a sinusoidal input, the output is a scaled and phase shifted sinusoid at the same frequency, where the scaling factor and phase shift is system dependent. It is illustrative to compare this analysis to the time-domain analysis of the same impulse response and input using convolution. ■

13.2 Decomposition of signals using DT complex exponentials

Similar to CT, in this course we consider the cases of stable DT systems. Recall a stable system is one in which a bounded input leads to a bounded output, or equivalently the impulse response is absolutely summable. We will consider two decompositions of the input:

- *Fourier Series*: When $x[n]$ is periodic with fundamental frequency $\omega_0 = \frac{2\pi}{N}$, $|z| = 1$ so that $z = e^{jk\omega_0}$, and the decomposition is a finite sum. This gives the input-output relationship

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \longrightarrow y[n] = \sum_{k=N_0}^{N_0+N-1} H\left(e^{jk\omega_0}\right) a_k e^{jk\omega_0 n}$$

  where $H\left(e^{jk\omega_0}\right)$ are the Eigenvalues, also called the DT *frequency response*.

- *Inverse Fourier Transform*: When $x[n]$ is a-periodic, $|z| = 1$ so that $z = e^{j\omega}$, and the decomposition is an integral over a finite length set. This gives the input-output relationship

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \longrightarrow y[n] = \frac{1}{2\pi} \int_{2\pi} H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

  where $H\left(e^{j\omega}\right)$ are the Eigenvalues, again called the DT *frequency response*.

Other courses such as ECE 3704 look at the general case of unstable systems and $z \in \mathbb{C}$ with decompositions:

- *One-Sided Z Transform*: $x[n]$ is causal and the decomposition is an uncountably infinite sum (complex integral)
- *Two-Sided (Bilateral) Z Transform*: $x[n]$ is non-causal and the decomposition is an uncountably infinite sum (complex integral). This is the most general case for DT LTI systems.

While the Z decompositions require complex integration, like for the Laplace transform in CT, they can be understood and computed using algebra and a table of forward transforms, which only require summations of a complex function over a real variable n (this is the general approach taken in upper level courses). However, this is outside the scope of this course because of time limitations.

Instead, we will be spending the next few weeks going through the DT Fourier decompositions in some detail. You will also learn how to find the DT frequency response for a stable system, and see how to use both for analysis.

Chapter 14

CT Fourier Series

Recall the complex exponential e^{st} for $s \in \mathbb{C}$ is the Eigenfunction of CT LTI systems. If we can decompose an input into a (possibly infinite) sum of such signals, we can easily determine the output using the superposition principle. In this section we consider the decomposition when the input is periodic, called the CT *Fourier Series* (CTFS).

Recall a signal $x(t)$ is periodic, with fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$ rad/sec or $f_0 = \frac{1}{T_0}$ Hertz, if $x(t) = x(t + kT_0)$ for integer multiple k and fundamental period $T_0 \in \mathbb{R}$. As we shall see, in this case the complex exponent of the Eigenfunction becomes $s_k = jk\omega_0$, and the decomposition is a countably infinite sum. This gives the input-output relationship for a stable LTI system as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)\, a_k\, e^{jk\omega_0 t}$$

where $H(jk\omega_0)$ are the Eigenvalues or frequency response. We now turn to determining under what circumstances the decomposition exists and how to find the coefficients a_k.

14.1 Synthesis and Analysis Equation

Suppose we can approximate (we will revisit shortly when this approximation is exact) the periodic function $x(t)$ by the sum

$$\boxed{x(t) \approx \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\ .}$$

This is called the *synthesis equation* of the CT Fourier series.

Assuming equivalence, let us multiply both sides by the function $e^{-jn\omega_0 t}$,

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

and integrate over one period

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t}\, dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt$$

Exchanging the order of integration and summation in the right-hand expression gives

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^{T_0} e^{jk\omega_0 t} e^{-jn\omega_0 t}\, dt \right]$$

The bracketed term can be rewritten as

$$\int_0^{T_0} e^{jk\omega_0 t}e^{-jn\omega_0 t}\,dt = \int_0^{T_0} e^{j(k-n)\omega_0 t}\,dt = \int_0^{T_0} \cos((k-n)\omega_0 t)\,dt + j\int_0^{T_0} \sin((k-n)\omega_0 t)\,dt$$

We now note that for $k \neq n$ the integrals of the real and imaginary parts are zero

$$\int_0^{T_0} \cos((k-n)\omega_0 t)\,dt = \frac{1}{(k-n)\omega_0}\sin((k-n)\omega_0 t)\Big|_0^{T_0} = \frac{1}{(k-n)\omega_0}\sin((k-n)2\pi) - \frac{1}{(k-n)\omega_0}\sin(0) = 0$$

$$\int_0^{T_0} \sin((k-n)\omega_0 t)\,dt = -\frac{1}{(k-n)\omega_0}\cos((k-n)\omega_0 t)\Big|_0^{T_0} = -\frac{1}{(k-n)\omega_0}\cos((k-n)2\pi) + \frac{1}{(k-n)\omega_0}\cos(0) = 0$$

When $k = n$

$$\int_0^{T_0} e^{j(k-n)\omega_0 t}\,dt = \int_0^{T_0} dt = T_0$$

Thus the bracketed term above is

$$\int_0^{T_0} e^{jk\omega_0 t}e^{-jn\omega_0 t}\,dt = T_0\,\delta[k-n]$$

and the right-hand side is

$$\sum_{k=-\infty}^{\infty} a_k \left[\int_0^{T_0} e^{jk\omega_0 t}e^{-jn\omega_0 t}\,dt\right] = \sum_{k=-\infty}^{\infty} a_k T_0\,\delta[k-n] = T_0\,a_n$$

Thus we obtain the *analysis equation* of the CT Fourier series:

$$\boxed{a_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}\,dt}$$

where the integration can be over any interval of length T_0 and the symbol for the subscript (integer n) is arbitrary. The CT Fourier Series coefficients are also called the *spectrum* of the signal. In general the a_k are complex. The function of k, $|a_k|$ is called the *amplitude spectrum*. The function of k, $\angle a_k$ is called the *phase spectrum*. When plotting the coefficients it is common to plot the amplitude and phase spectrum together.

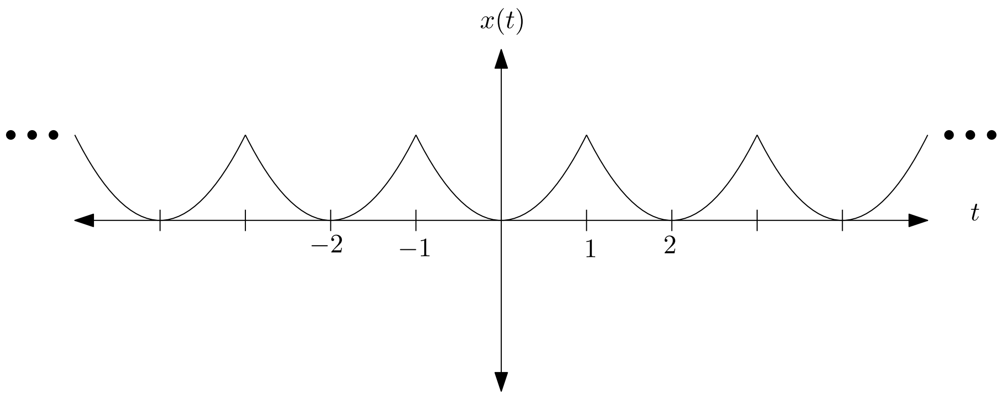
Example 14.1.1. Consider the signal

$$x_p(t) = \begin{cases} t^2 & -1 < t < 1 \\ 0 & \text{else} \end{cases}$$

periodically extended with period $T_0 = 2$

$$x(t) = \sum_{i=-\infty}^{\infty} x_p(t-2i)$$

as shown below:



To find the Fourier Series approximation of $x(t)$,

$$x(t) \approx \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \, ,$$

we need to find the coefficients

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} \, dt$$

Since the integration can be over any period, we can use the limits $[-1, 1]$ and note that $T_0 = 2$ so that $\omega_0 = \pi$, giving the sequence of expressions

$$\begin{aligned}
a_k &= \frac{1}{2} \int_{-1}^{1} t^2 \, e^{-jk\pi t} \, dt \\
&= \frac{1}{2} \left[\int_{-1}^{1} t^2 \, \cos(-k\pi t) \, dt + j \int_{-1}^{1} t^2 \, \sin(-k\pi t) \, dt \right] \\
&= \frac{1}{2} \left[\int_{-1}^{1} t^2 \, \cos(k\pi t) \, dt + j \int_{-1}^{1} -t^2 \, \underbrace{\sin(k\pi t)}_{\text{always} \, = \, 0} \, dt \right] \\
&= \frac{1}{2} \int_{-1}^{1} t^2 \, \cos(k\pi t) \, dt \;\; \text{using an integration table} \\
&= \frac{1}{2} \frac{4k\pi \overbrace{\cos(k\pi)}^{(-1)^k} + 2(k^2\pi^2 - 2) \overbrace{\sin(k\pi)}^{\text{always} \, = \, 0}}{k^3\pi^3} \\
a_k &= \frac{2}{k^2\pi^2} \, (-1)^k
\end{aligned}$$

This result is undefined for when $k = 0$. In that case note the original integral is

$$a_0 = \frac{1}{2} \int_{-1}^{1} t^2 \, dt = \frac{1}{6} t^3 \Big|_{-1}^{1} = \frac{1}{3}$$

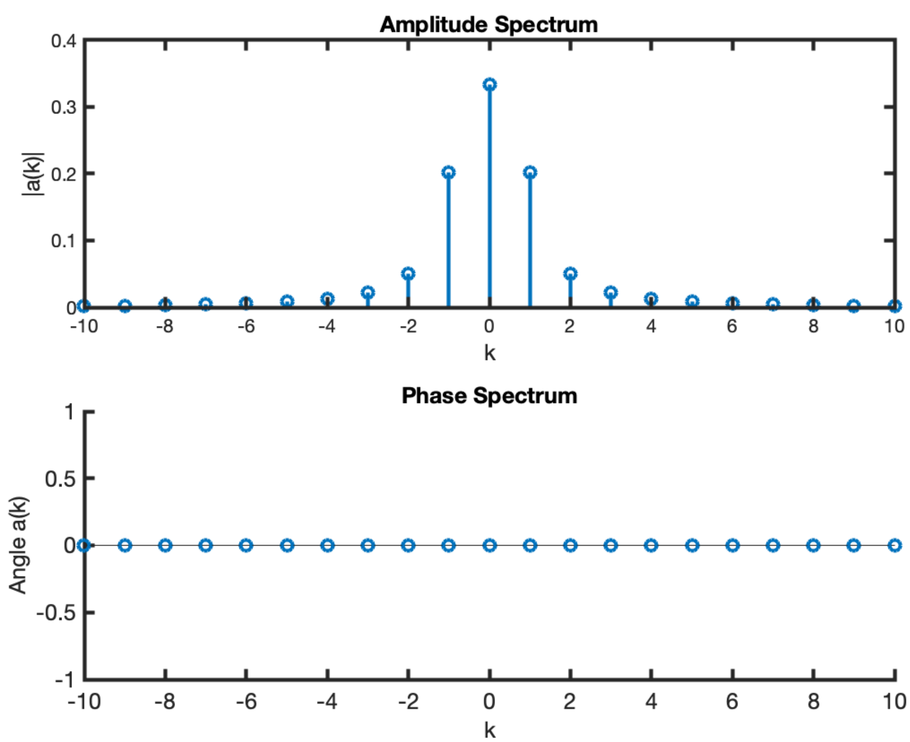
Thus the final approximation is

$$x(t) \approx \sum_{k=-\infty}^{\infty} \underbrace{\frac{2}{k^2\pi^2} \, (-1)^k}_{a_k} \, e^{jk\pi t} \, .$$

We can plot the spectrum of this signal (using for example Matlab)

```
k = -10:10;
a = 2./(pi^2*k.^2);
a(11) = 1/3;

subplot(2,1,1);
stem(k, abs(a));
xlabel('k');
ylabel('|a(k)|');
title('Amplitude Spectrum');

subplot(2,1,2);
stem(k, angle(a));
xlabel('k');
ylabel('Angle a(k)');
title('Phase Spectrum');
```

Giving the amplitude and phase spectrum plot



■

14.2 Variations on the Synthesis and Analysis Equations

There are two commonly used, equivalent, expressions for computing the CTFS coefficients. They can be derived using Euler's formula and related trig identities.

- **Exponential Form**. This is the form derived above

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

where

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \, dt$$

- **Trig Form**

$$x(t) = b_0 + \sum_{k=1}^{\infty} b_k \, \cos(k\omega_0 t) + c_k \, \sin(k\omega_0 t)$$

where

$$b_0 = \frac{1}{T_0} \int_{T_0} x(t) \, dt$$

is the average value of the signal, and

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t) \, dt$$

$$c_k = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t) \, dt$$

- **Compact Trig Form**

$$x(t) = d_0 + \sum_{k=1}^{\infty} d_k \, \cos(k\omega_0 t + \theta_k)$$

where

$$d_0 = \frac{1}{T_0} \int_{T_0} x(t) \, dt$$

is the average value of the signal, and

$$d_k = \sqrt{b_k^2 + c_k^2}$$

$$\theta_k = \arctan\left(\frac{-c_k}{b_k}\right)$$

Note that $2a_k = b_k - jc_k$ for $k \geq 1$ and $a_0 = b_0$.

14.3 Convergence of the CT Fourier Series

As mentioned above the Fourier Series is strictly speaking an approximation

$$x(t) \approx \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \text{ where } a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \, dt$$

to determine when this approximation is an equivalence (and in what sense) we need to establish the existence and convergence of the integral and summation respectively.

The coefficients a_k will exist when the integral converges, or equivalently when

$$\int_{T_0} |x(t)| \; dt < \infty$$

i.e. the signal is absolutely integrable over any period.

To determine when the summation converges, first consider the *truncated* CT Fourier Series

$$x_N(t) \approx \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t}$$

where the infinite sum has been truncated to the finite range $[-N, N]$. Define the error between the original signal $x(t)$ and the truncated approximation $x_N(t)$ at each time point as

$$E(N,t) = x(t) - x_N(t)$$

There are two relevant notions of convergence. If

$$\lim_{N\to\infty} \int_{T_0} |E(N,t]| \; dt = 0$$

we say the CT Fourier Series converges *exactly* to the signal. If

$$\lim_{N\to\infty} \int_{T_0} |E(N,t)|^2 \; dt = 0$$

we say the CT Fourier Series converges in the *mean-square* sense to the signal.

More formally the CTFS exists if the *Dirichlet Conditions* hold for the signal:

- The signal has a finite number of discontinuities per period.
- The signal has a finite number of maxima and minima per period.
- The signal is bounded, i.e.

$$\int_{T_0} |x(t)| \, dt < \infty$$

These conditions rule out pathological functions. For most practical signals of interest, the conditions hold.

Example 14.3.1. Consider the *impulse train* signal defined as

$$x(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

which we be important later when we discuss sampling CT signals. Do the Dirichlet conditions hold? Yes. It has one discontinuity, one maximum, and one minimum per period. It is also bounded since

$$\int_{T_0} |\delta(t)| \, dt = 1 \text{ by definition.}$$

The spectrum for the impulse train is given by

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} \; dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-jk\omega_0 t} \; dt = \frac{1}{T_0}$$

■

Example 14.3.2. Consider the signal $x(t) = \cos(\omega t)$. We can write this as the sum of two complex exponentials using Euler's formula

$$x(t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

Comparing this to the synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} = \cdots + a_{-2}\, e^{j(-2)\omega_0 t} + a_{-1}\, e^{j(-1)\omega_0 t} + a_0 + a_1\, e^{j(1)\omega_0 t} + a_2\, e^{j(2)\omega_0 t} + \cdots$$

we note that if $\omega_0 = \omega$ and

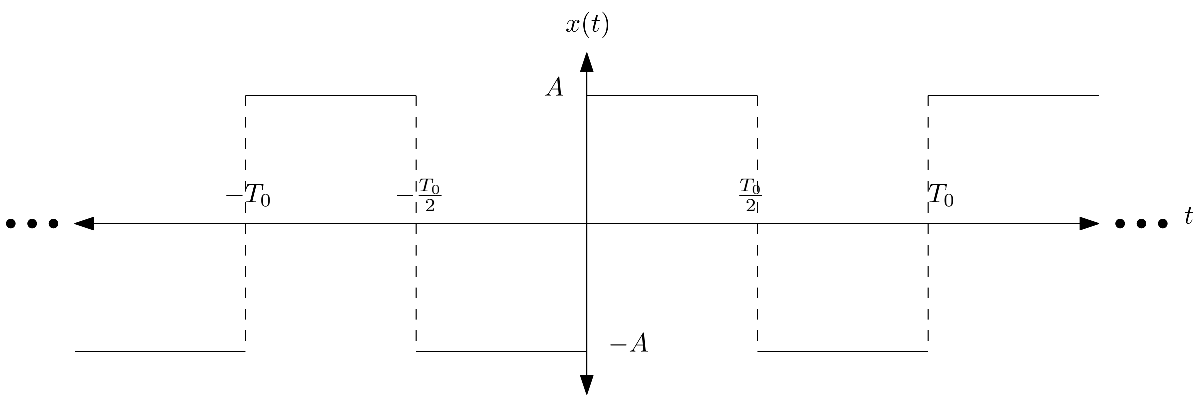
$$a_k = \begin{cases} \frac{1}{2} & k = -1 \\ \frac{1}{2} & k = 1 \\ 0 & \text{else} \end{cases}$$

then the two expressions are identical and the CT Fourier Series is an exact representation.
■

Example 14.3.3. Consider the square wave signal of amplitude $A > 0$

$$x(t) = \sum_{m=-\infty}^{\infty} \begin{cases} -A & \frac{T_0}{2} < t - mT_0 < 0 \\ A & 0 < t - mT_0 < \frac{T_0}{2} \end{cases}$$

shown below



The coefficients are given by

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_0}\left[\int_0^{\frac{T_0}{2}} Ae^{-jk\omega_0 t}\, dt + \int_{\frac{T_0}{2}}^{T_0} -Ae^{-jk\omega_0 t}\, dt\right] \\ &= \frac{1}{T_0}\left[\frac{A}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_0^{\frac{T_0}{2}} + \frac{-A}{-jk\omega_0}e^{-jk\omega_0 t}\Big|_{\frac{T_0}{2}}^{T_0}\right] \\ &= \frac{1}{T_0}\frac{A}{jk\omega_0}\left[-\left(e^{-jk\omega_0\frac{T_0}{2}} - e^0\right) + \left(e^{-jk\omega_0 T_0} - e^{-jk\omega_0\frac{T_0}{2}}\right)\right] \end{aligned}$$

Note that $\omega_0 \frac{T_0}{2} = \frac{2\pi}{T_0}\frac{T_0}{2} = \pi$ and $\omega_0 T_0 = \frac{2\pi}{T_0} T_0 = 2\pi$. Thus

$$\begin{aligned} a_k &= \frac{1}{T_0}\frac{A}{jk\frac{2\pi}{T_0}}\left[-\left(e^{-jk\pi} - e^{0}\right) + \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right] \\ &= \frac{A}{jk\pi}\left(1 - e^{-jk\pi}\right) \\ &= \begin{cases} 0 & k \text{ even} \\ \frac{2A}{jk\pi} & k \text{ odd} \end{cases} \end{aligned}$$

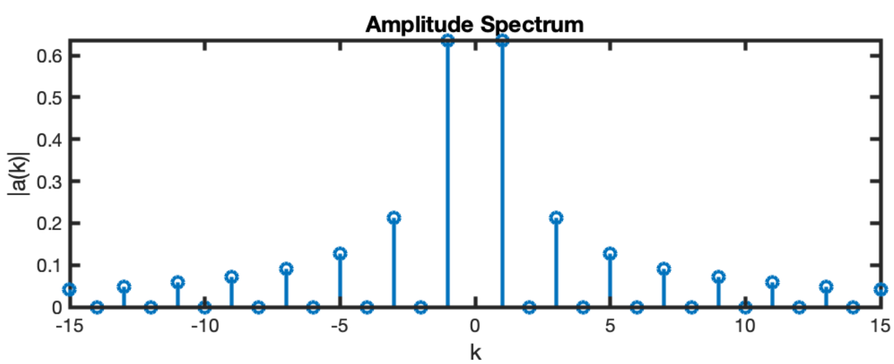
The amplitude spectrum is given by

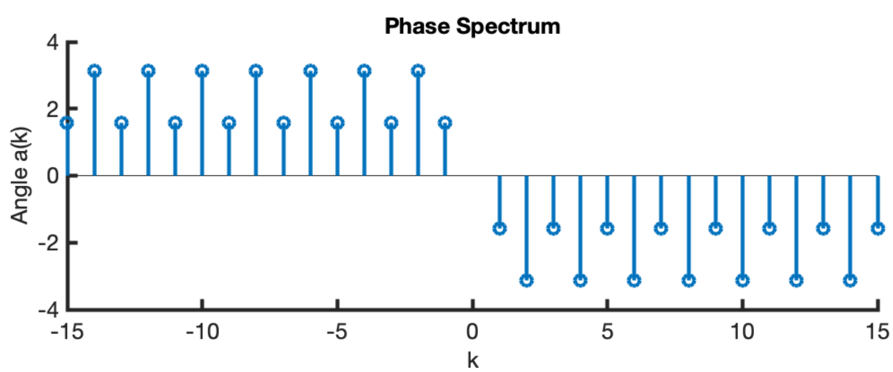
$$|a_k| = \begin{cases} 0 & k \text{ even} \\ \left|\frac{2A}{k\pi}\right| & k \text{ odd} \end{cases}$$

The phase spectrum is given by

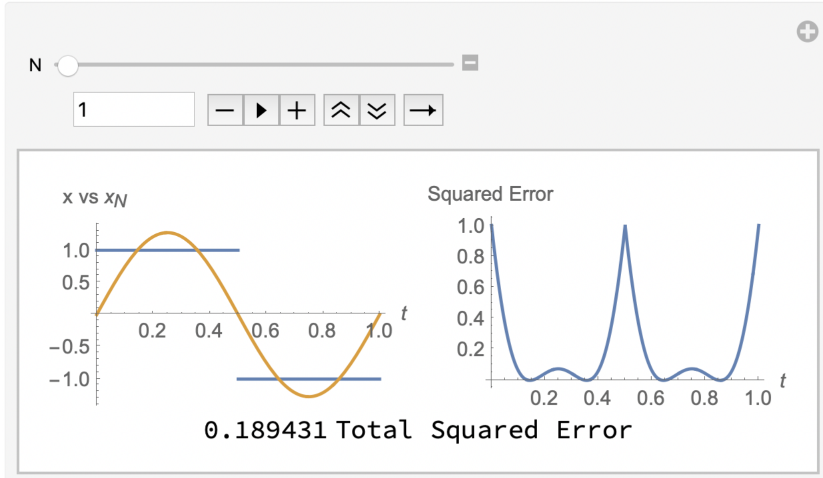
$$\angle a_k = \begin{cases} \pi & k < 0 \text{ and even} \\ -\pi & k > 0 \text{ and even} \\ \frac{\pi}{2} & k < 0 \text{ and odd} \\ -\frac{\pi}{2} & k > 0 \text{ and odd} \end{cases}$$

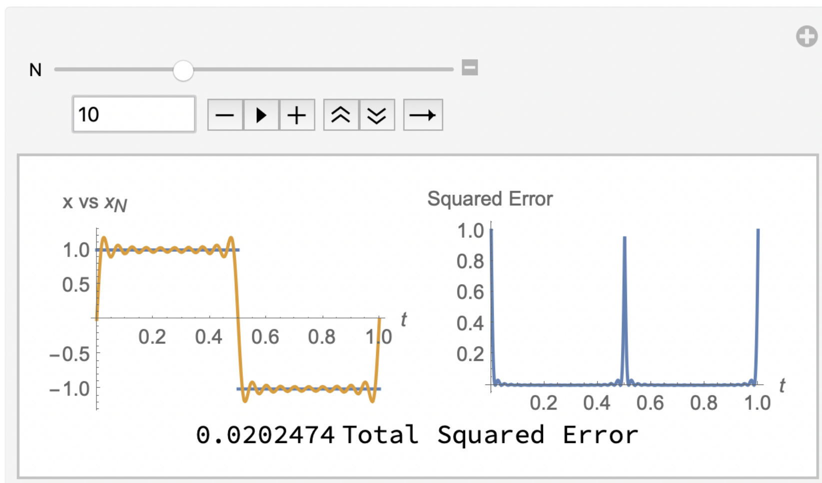
This is plotted below for $A = 1$.

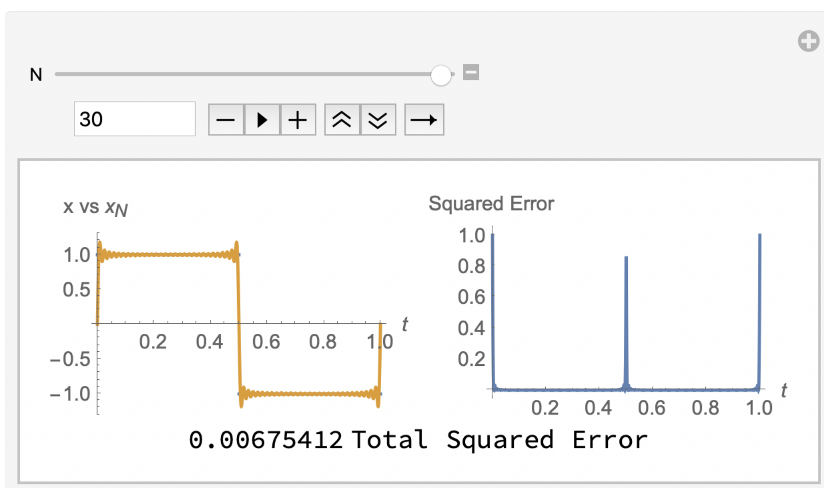




We can plot the truncated approximation for increasing number of terms N, the squared error, and the total error.







Note as N increases the approximation gets closer to the square wave, except at the discontinuities. This is called *Gibbs Ringing*. As $N \to \infty$ the mean-square error goes to zero, so the CTFS approximation to the square wave converges in the mean-square sense. ■

14.4 Properties of the CT Fourier Series

Let a_k and b_k be the CTFS coefficients for the periodic signals $x(t)$ and $y(t)$ respectively.

- **Linearity**. The coefficients of the signal

  $$z(t) = Ax(t) + By(t) \text{ for constants } A, B$$

  are $Aa_k + Bb_k$

- **Time Shifting**. The coefficients of

  $$z(t) = x(t - t_0) \text{ are } e^{-jk\omega_0 t_0} a_k$$

  that is it adds a phase shift.

- **Time reversal**. The coefficients of

  $$z(t) = x(-t) \text{ are } a_{-k}$$

  that is the sequence reverses.

- **Time Scaling**. Let T_0 and ω_0 be the fundamental period and frequency of a periodic $x(t)$. The signal

  $$z(t) = x(\alpha t) \text{ for } \alpha > 0$$

  is periodic with period $\frac{T_0}{\alpha}$ and fundamental frequency $\alpha\omega_0$. The coefficients of $z(t)$ are the same as $x(t)$.

- **Multiplication**. The coefficients of

  $$z(t) = x(t) \cdot y(t) \text{ are } \sum_{m=-\infty}^{\infty} a_m \cdot b_{k-m}$$

  the discrete convolution of the individual signals' coefficients.

- **Conjugate Symmetry**. The coefficients of

  $$z(t) = x^*(t) = \operatorname{Re} x(t) - j \operatorname{Im} x(t) \text{ are } a^*_{-k}$$

  A consequence of this property and the time-reversal property is that real, even signals have real, even a_k; and real, odd signals have purely imaginary, odd a_k (see the examples above).

- **Parseval's Relation**. The power of the signal with Fourier series coefficients

  $$\frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Chapter 15

DT Fourier Series

Recall the complex exponential z^n is the Eigenfunction of DT LTI systems. If we can decompose an input into a (possibly infinite) sum of such signals, we can easily determine the output using the superposition principle. In this section we consider the decomposition when the input is periodic, called the DT *Fourier Series* (DTFS). The DTFS is similar, but not identical to the CTFS. Notably, the approximation requires only a finite number of terms, there are no convergence issues, and the resulting spectrum is a periodic function.

Recall a DT signal $x[n]$ is periodic, with fundamental frequency $\omega_0 = \frac{2\pi}{N}$ rad/sec, if $x[n] = x[n + kN]$ for integer multiple k and fundamental period $N \in \mathbb{Z}$. As we shall see, in this case the complex base of the Eigenfunction becomes $z_k = e^{jk\omega_0}$, and the decomposition is a finite sum. This gives the input-output relationship for a stable DT LTI system as

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \longrightarrow y[n] = \sum_{k=N_0}^{N_0+N-1} H\left(e^{jk\omega_0}\right) a_k e^{jk\omega_0 n}$$

where $H\left(e^{jk\omega_0}\right)$ are the Eigenvalues or DT frequency response. We now turn to how to find the coefficients a_k.

15.1 Synthesis and Analysis Equation

Similar to the CTFS we wish to show that any periodic DT signal can be represented by the sum of complex exponentials whose frequencies are harmonics of the fundamental. This differs from the CTFS in that there are only N distinct harmonics, so that the sum is over a finite range

$$\boxed{x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n}}$$

where N is the period and N_0 is any starting index for the sum. Note the course text defines $< N >= \{N_0, N_0 + 1, \cdots (N_0 + N - 1)\}$. This is called the *synthesis equation* of the DT Fourier series.

One approach to find the coefficients a_k is to note that there are a finite number of terms in the summation and the signal has a finite number of values over one period. This gives a system of N linear equations in N

unknowns (the a_k's)

$$x[N_0] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 N_0}$$
$$x[N_0+1] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 (N_0+1)}$$
$$\vdots = \vdots$$
$$x[N_0+N-1] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 (N_0+N-1)}$$

which can be solved to find the coefficients using linear algebra.

Example 15.1.1. Consider the periodic DT signal $x[n] = \cdots - 1, 1, -1, 1, -1, 1, \cdots$ where $x[0] = 1$. The period is $N = 2$ so that $\omega_0 = \pi$. If we let $N_0 = 0$, the system of equations is

$$x[0] = \sum_{k=0}^{1} a_k = a_0 + a_1 = 1$$
$$x[1] = \sum_{k=0}^{1} a_k e^{jk\pi} = a_0 - a_1 = -1$$

which has the solution $a_0 = 0$ and $a_1 = 1$ and $x[n] = e^{j\pi n}$.
■

Another approach is similar to that taken when deriving the CT Fourier Series. Beginning with the synthesis equation

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n}$$

we multiply both sides by $e^{-j\frac{2\pi r}{N}n}$ for $r \in \mathbb{Z}$ and sum over N terms

$$\sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi r}{N}n} = \sum_{n=N_0}^{N_0+N-1} \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} e^{-j\frac{2\pi r}{N}n}$$

We then interchange to order of summation on the right-hand-side

$$\sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi r}{N}n} = \sum_{k=N_0}^{N_0+N-1} a_k \underbrace{\sum_{n=N_0}^{N_0+N-1} e^{jk\omega_0 n} e^{-j\frac{2\pi r}{N}n}}$$

Since $\omega_0 = \frac{2\pi}{N}$, the bracketed term is

$$\sum_{n=N_0}^{N_0+N-1} e^{j(k-r)\frac{2\pi}{N}n} = \begin{cases} N & \text{if } k-r = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{else} \end{cases} = N\delta[(k-r)+mN] \text{ for arbitrary } m \in \mathbb{Z}$$

and the right-hand side is

$$\sum_{k=N_0}^{N_0+N-1} a_k N\delta[(k-r)+mN] = Na_r$$

since $N_0 < mN < N_0 + 1$ for some m. Solving for a_r gives the *analysis equation* of the DT Fourier series:

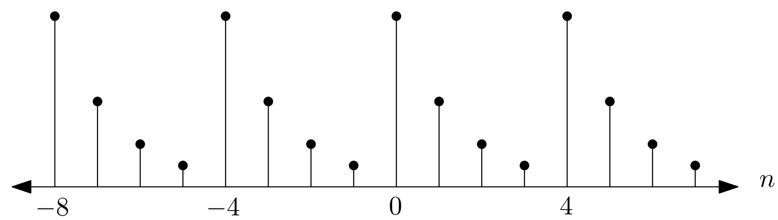
$$\boxed{a_r = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi}{N}rn}}$$

where the summation can be over any interval of length N and the symbol for the subscript (integer r) is arbitrary. The DT Fourier Series coefficients are also called the *spectrum* of the signal. In general the a_k are complex. *Note the spectrum is periodic in N.* The function of k, $|a_k|$ is called the *amplitude spectrum*. The function of k, $\angle a_k$ is called the *phase spectrum*. When plotting the coefficients it is common to plot the amplitude and phase spectrum together over a single interval of length N (since it is periodic).

Example 15.1.2. A simple way to construct a DT periodic signal is to use the modulus % operator. For example,

$$x[n] = \gamma^{n\%N} \text{ for any } \gamma \in \mathbb{C}$$

is periodic in N, e.g. $x[n] = \left(\frac{1}{2}\right)^{n\%4}$



The synthesis equation is given by

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n}$$

Where the coefficients are found using the analysis equation. Let $N_0 = 0$ arbitrarily, then

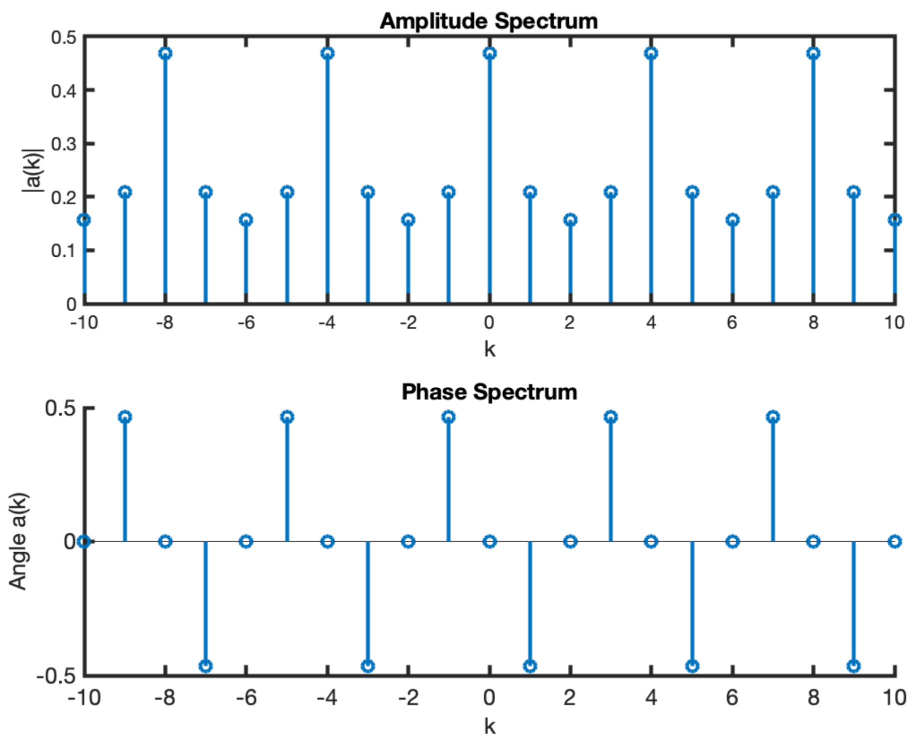
$$\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \gamma^n e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \left(\gamma e^{-j\frac{2\pi}{N}k}\right)^n \\
a_k &= \frac{1}{N} \frac{1 - \left(\gamma e^{-j\frac{2\pi}{N}k}\right)^N}{1 - \left(\gamma e^{-j\frac{2\pi}{N}k}\right)}
\end{aligned}$$

We can plot the spectrum of this signal (using for example Matlab)

```
gamma = 0.5;
N = 4;
k = -10:10;
a = (1-(gamma*exp(-j*2*pi*k/N)).^N)./(N*(1-gamma*exp(-j*2*pi*k/N)));

h0 = subplot(2,1,1);
```

```
h1 = stem(k, abs(a));
h2 = xlabel('k');
h3 = ylabel('|a(k)|');
h4 = title('Amplitude Spectrum');
h5 = subplot(2,1,2);
h6 = stem(k, angle(a));
h7 = xlabel('k');
h8 = ylabel('Angle a(k)');
h9 = title('Phase Spectrum');
```

Giving the amplitude and phase spectrum plot



■

Example 15.1.3. Find the DTFS of $x[n] = \cos[\frac{\pi}{4}n]$. Note $N = 8$ and $\omega_0 = \frac{\pi}{4}$. Using Euler's formula

$$x[n] = \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n}$$

The synthesis equation is

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = a_0 + a_1 e^{j\frac{\pi}{4}n} + a_2 e^{j\frac{2\pi}{4}n} + \cdots + a_7 e^{j\frac{7\pi}{4}n}$$

Comparing to the expansion above and noting that $e^{-j\frac{\pi}{4}n} = e^{j\frac{7\pi}{4}n}$ we see that

$$a_k = \begin{cases} \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = 7 \\ 0 & \text{else} \end{cases}$$

for $k \in [0,7]$ and $a_k = a_{k\%8}$ for all k.
■

15.2 Properties of the DT Fourier Series

Given two signals $x[n]$ and $y[n]$ periodic in N with $\omega_0 = \frac{2\pi}{N}$, having DT Fourier coefficients a_k and b_k respectively.

- Linearity. The coefficients of the signal

$$z[n] = Ax[n] + By[n] \text{ for constants } A, B$$

  are $Aa_k + Bb_k$

- Index Shifting. The coefficients of

$$z[n] = x[n - n_0] \text{ are } e^{-jk\omega_0 n_0} a_k$$

  that is, it adds a phase shift.

- Frequency Shift. The coefficients of

$$z[n] = x[n]e^{jm\omega_0 n} \text{ are } a_{k-m}$$

- Index Reversal. The coefficients of

$$z[n] = x[-n] \text{ are } a_{-k}$$

- Multiplication. The coefficients of

$$z[n] = x[n] \cdot y[n] \text{ are } \sum_{m=N_0}^{N_0+N-1} a_m \cdot b_{k-m}$$

  the discrete convolution of the individual signals' coefficients.

- Convolution. The coefficients of

$$z[n] = x[n] * y[n] \text{ are } Na_k b_k$$

- Conjugate Symmetry. The coefficients of

$$z[n] = x^*[n] = \operatorname{Re} x[n] - j \operatorname{Im} x[n] \text{ are } a^*_{-k}$$

  A consequence of this property is that real, even signals have real, even a_k; and real, odd signals have purely imaginary, odd a_k. Thus if $x[n]$ is real $|a_k|$ is an even periodic function of k and $\angle a_k$ is an odd periodic function of k.

- Parseval's Relation. The power of the signal with Fourier series coefficients is

$$\frac{1}{N} \sum_{n=N_0}^{N_0+N-1} |x[n]|^2 \, dt = \sum_{k=N_0}^{N_0+N-1} |a_k|^2$$

15.3 Comparison of CT and DT Fourier Series

A summary of the CT and DT Fourier Series is as follows.

In CT, a periodic signal $x(t)$ can be decomposed as a countably infinite combination of complex sinusoids at harmonic frequencies of the fundamental. The Fourier series coefficients are a discrete signal that is a-periodic.

$$x(t) \approx \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} \, dt$$

In DT, a periodic signal $x[n]$ can be decomposed as a finite combination of complex sinusoids at harmonic frequencies of the fundamental. The Fourier series coefficients are a discrete signal that is periodic.

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N} \sum_{n=N_0}^{N_0+N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

Chapter 16

CT Fourier Transform

Recall the complex exponential e^{st} for $s \in \mathbb{C}$ is the Eigenfunction of CT LTI systems. If we can decompose an input into a (possibly infinite) sum of such signals, we can easily determine the output using the superposition principle. In this section we consider the decomposition when the input is aperiodic, called the CT *Fourier Transform* (CTFT).

In contrast to the CT Fourier series, in this case the complex exponent of the Eigenfunction becomes $s = j\omega$ a continuous variable, and the decomposition is an uncountably infinite sum (integral). This gives the input-output relationship for a stable LTI system as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \;\longrightarrow\; y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) X(j\omega)\, e^{j\omega t}\, d\omega$$

where $H(j\omega)$ are the Eigenvalues, again called the *frequency response*. We now turn to determining under what circumstances the decomposition exists and how to find the function $X(j\omega)$.

Note: The difference in notation between $X(\omega)$ and $X(j\omega)$ is superficial. They generally are the same function. The latter just emphasizes that $s \to j\omega$. For example

$$H(j\omega) = \frac{1}{1 + (j\omega)^2} = \frac{1}{1 - \omega^2} = H(\omega)$$

are the same function since $j^2 = -1$.

16.1 Synthesis and Analysis Equation

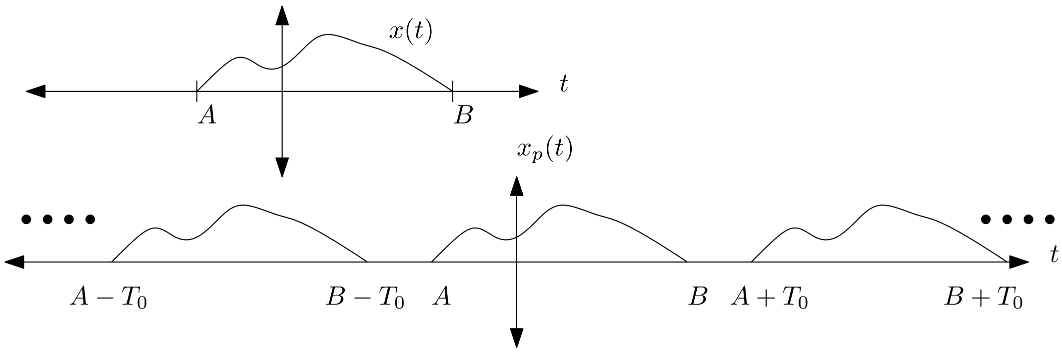
Consider the aperiodic signal

$$x(t) = \begin{cases} p(t) & A < t < B \\ 0 & \text{else} \end{cases}$$

and it's periodic extension with fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT_0)$$

where $T_0 > B - A$. For example:



The CT Fourier series coefficients are

$$\begin{aligned} a_k &= \frac{1}{T_0} \int_{T_0} x_p(t) e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t}\, dt \text{ since } x(t) = 0 \text{ outside the interval } (A, B) \end{aligned}$$

Define the *CT Fourier Transform* of $x(t)$ as

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}$$

so that

$$a_k = \frac{1}{T_0} X(k\omega_0)$$

are samples of $X(\omega)$ spaced at frequencies ω_0. By the CT Fourier series synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}$$

Now, let $T_0 \to \infty$ so that the periodic copies move toward ∞ and $x_p(t) \to x(t)$. At the same time the frequency sample spacing becomes infinitesimal and

$$X(k\omega_0) e^{jk\omega_0 t} \to X(\omega) e^{j\omega t}\, d\omega$$

To give the *Inverse Fourier Transform*

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega}$$

This gives the *Fourier Transform Pair*:

$$\underbrace{X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}_{\text{Forward Transform / Analysis Equation}} \qquad \underbrace{x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega}_{\text{Inverse Transform / Synthesis Equation}}$$

The forward transform decomposes $x(t)$ into an infinite number of complex sinusoids. The inverse transform synthesizes a signal as an infinite sum of the sinusoids. It is an example of an *Integral Transform.* Note the signal $x(t)$ and $X(\omega)$ are the same signal, just represented in different *domains*, the time-domain and frequency-domain respectively.

Similar to the CT Fourier series, the function $X(\omega)$ is called the *spectrum* of the signal $x(t)$. The magnitude spectrum is the function $|X(\omega)|$ and the phase spectrum is the function $\angle X(\omega)$. It is common to plot the spectrum as the combination of the magnitude and phase spectrum.

Example 16.1.1. Consider the signal $x(t) = \delta(t)$. The Fourier transform is

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\, dt \\ &= e^{-j\omega(0)} \text{ by the sifting property} \\ &= 1 \end{aligned}$$

■

Example 16.1.2. Consider the signal $x(t) = e^{at}u(t)$ for $a \in \mathbb{R}$. The Fourier transform is

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{at}\, e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{(a-j\omega)t}\, dt \\ &= \frac{1}{a-j\omega} e^{(a-j\omega)t}\Big|_0^{\infty} \\ &= \frac{1}{a-j\omega}\left[\lim_{T\to\infty} e^{(a-j\omega)T} - \underbrace{e^{(a-j\omega)(0)}}_{1}\right] \end{aligned}$$

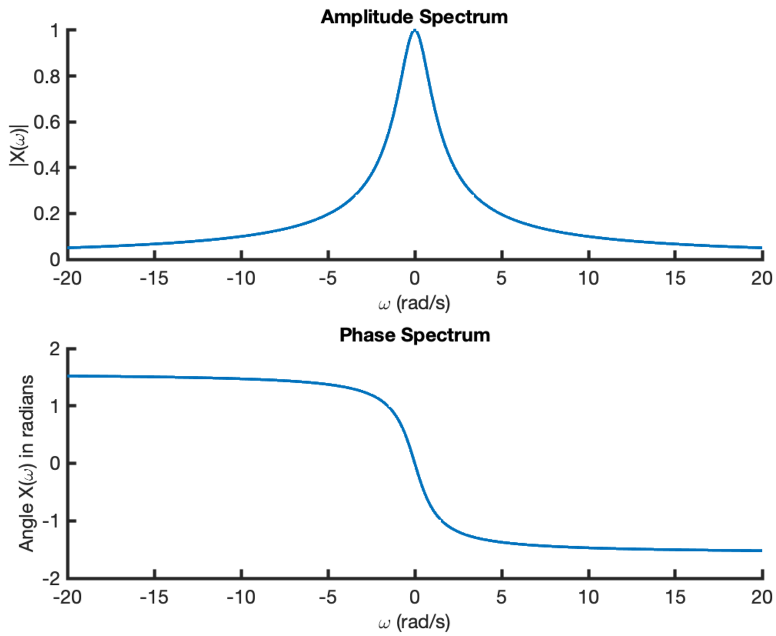
This example raises the question, of when does the Fourier Transform exist? Note if $a < 0$ then the limit above converges to zero, otherwise the integral diverges. In the former case we say the Fourier transform exists, and in the latter that it does not. Thus

$$X(\omega) = \frac{-1}{a-j\omega} = \frac{1}{j\omega - a} \text{ for } a < 0\,.$$

Note when $a < 0$, $x(t)$ is an energy signal. A sufficient, but not necessary condition for the Fourier transform to exist is that the signal be an energy signal. For this example, let's examine the spectrum, noting

$$|X(\omega)| = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}} \qquad \text{and} \qquad \angle X(\omega) = -\arctan\left(\frac{\omega}{-a}\right)$$

plotted below for $a = -1$.



■

Example 16.1.3. Consider the signal $x(t) = e^{j\omega_0 t}$ for $\omega_0 \in \mathbb{R}$. The Fourier transform is

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{-j(\omega_0 - \omega)t}\, dt \end{aligned}$$

For $\omega \neq \omega_0$ this integral evaluates to

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \cos((\omega - \omega_0)t)\, dt + j \int_{-\infty}^{\infty} \sin((\omega - \omega_0)t)\, dt \\ &= 0 \end{aligned}$$

since the average value of a sinusoid is zero. When $\omega = \omega_0$ this integral diverges

$$\int_{-\infty}^{\infty} e^{-j(\omega_0 - \omega)t} dt = \int_{-\infty}^{\infty} e^{-j(0)t} dt = \int_{-\infty}^{\infty} dt = \infty$$

What signal is zero everywhere, but infinite at one point (I am hand-waving a bit here)? The delta function

$$X(\omega) = A\delta(\omega - \omega_0) \text{ for some constant } A.$$

To find the constant we can use the inverse transform

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} A\delta(\omega-\omega_0)e^{j\omega t}\,d\omega \\
&= \frac{1}{2\pi}Ae^{j\omega_0 t} \\
&= e^{j\omega_0 t}
\end{aligned}$$

which implies $A = 2\pi$.
■

Example 16.1.4. Consider the signal $x(t) = \cos(\omega_0 t)$ for $\omega_0 \in \mathbb{R}$. The Fourier transform can be found using the result in the previous example by noting

$$\begin{aligned}
X(\omega) &= \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\
&= \int\limits_{-\infty}^{\infty} \cos(\omega_0 t)\,e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{j\omega_0 t}\,e^{-j\omega t}\,dt + \frac{1}{2}\int\limits_{-\infty}^{\infty} e^{-j\omega_0 t}\,e^{-j\omega t}\,dt \\
&= \frac{1}{2}2\pi\delta(\omega-\omega_0) + \frac{1}{2}2\pi\delta(\omega+\omega_0) \\
&= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)
\end{aligned}$$

This example highlights that the cosine signal is composed of exactly two frequencies.
■

Example 16.1.5. Consider the signal

$$X(\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & \text{else} \end{cases}$$

The Inverse Fourier transform is

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \\
&= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t}\, d\omega \\
&= \frac{1}{2\pi} \frac{1}{jt} \left[e^{j\omega_0 t} - e^{-j\omega_0 t} \right] \\
&= \frac{1}{\pi t} \left[\frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \right] \\
&= \frac{1}{\pi t} \sin(\omega_0 t) \\
&= \frac{\omega_0}{\pi} \frac{\sin(\omega_0 t)}{\omega_0 t} \\
&= \frac{\omega_0}{\pi} \mathrm{sinc}(\omega_0 t)
\end{aligned}$$

where sinc() is the (unnormalized) *sinc function.*
■

16.2 Existence of the CT Fourier Transform

The example of the real exponential above showed that for the Fourier transform to exist, the Fourier (analysis) integral must exist. Similar to the Fourier series some mild conditions, called the Dirichlet conditions, are a sufficient prerequisite for the Fourier transform of a signal $x(t)$ to exist:

- $x(t)$ is absolutely integrable

$$x(t) = \int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- $x(t)$ has a finite number of minima and maxima over any finite interval
- $x(t)$ has a finite number of finite-valued discontinuities over any finite interval

These conditions are not necessary however, and we can extend the Fourier transform to a broader class of signals, if we allow delta functions in the transform, as in the cosine example above.

16.3 Properties of the CT Fourier Transform

There are several useful properties of the CT Fourier Transform that, when combined with a table of transforms (see Table 4.2, page 329 of OW), allow us to take the Fourier transform of wide array of signals, and one, the convolution property, that allows us to determine the output of a system in the frequency domain easily. We state these here without proof in rough order of usefulness. See the course text for detailed derivations.

We use the notation $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$ to indicate the signals are related by a Fourier Transform pair.

- **Linearity:** if $x_1(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$ then

$$a x_1(t) + b x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} a X_1(\omega) + b X_2(\omega)$$

- **Convolution:** if $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$ then

$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega)$$

  Note in particular if one signal is the system input and the other is the impulse response, the output is the product of the Fourier transforms of each, where the Fourier transform of $h(t)$ is $H(\omega)$, the Eigenvalue or frequency response.

- **Differentiation:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$\frac{dx}{dt}(t) \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)$$

  This allows us to easily determine the Eigenvalues/Frequency Response from a stable differential equation.

- **Multiplication:** if $x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$ then

$$x_1(t) \cdot x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

  where $X_1(\omega) * X_2(\omega)$ is convolution in the frequency domain

$$X_1(\omega) * X_2(\omega) = \int_{-\infty}^{\infty} X_1(\gamma) \cdot X_2(\omega - \gamma)\, d\gamma$$

- **Time-Shift:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)e^{-j\omega t_0}$$

- **Frequency-Shift:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$e^{j\omega_0 t}x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_o)$$

- **Conjugate Symmetry:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-\omega)$$

  This implies that if $x(t)$ is real, then the magnitude spectrum is an even function, and the phase spectrum is an odd function.

- **Integration:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

- **Time and Frequency Scaling:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then if a is a real constant

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

- **Parseval's Relation:** if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$\int_{0-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{0-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

16.4 CT Fourier Transform of a Periodic Signal

Even though the Fourier transform was derived in the case of an a-periodic signal, the linearity property of the transform, combined with one of our examples above shows us that we can take the Fourier Transform of a periodic signal. Consider a periodic signal with Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Taking the Fourier Transform

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

Thus the discrete Fourier series coefficients become the weights of the corresponding delta functions centered at the harmonic frequency.

Chapter 17

DT Fourier Transform

Recall the complex exponential z^n for $z \in \mathbb{C}$ is the Eigenfunction of DT LTI systems. If we can decompose an input into a (possibly infinite) sum of such signals, we can easily determine the output using the superposition principle. In this section we consider the decomposition when the input is aperiodic, called the DT *Fourier Transform* (DTFT).

In contrast to the DT Fourier series, in this case the complex exponent of the Eigenfunction becomes $z = e^{j\omega}$ a continuous variable, and the decomposition is an uncountably infinite sum (integral). This gives the input-output relationship for a stable DT LTI system as

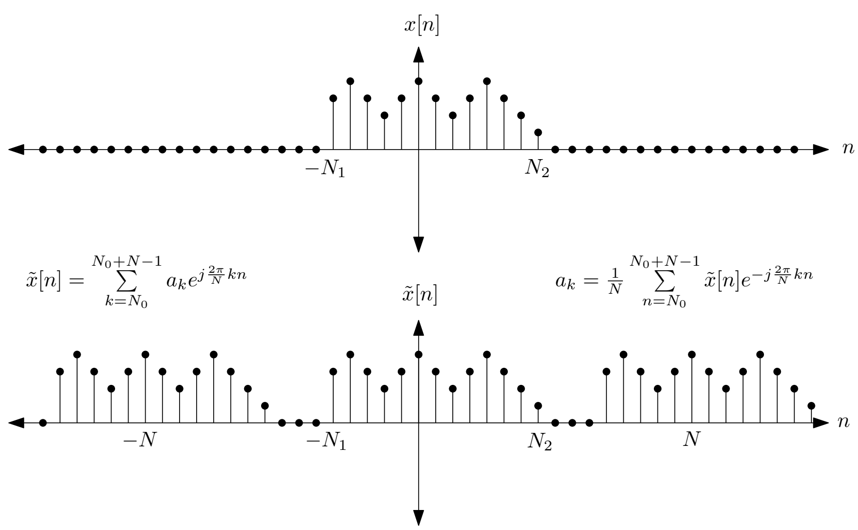
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \;\longrightarrow\; y[n] = \frac{1}{2\pi} \int_{2\pi} H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

where $H\left(e^{j\omega}\right)$ are the Eigenvalues, again called the *frequency response*. We now turn to determining under what circumstances the decomposition exists and how to find the function $X\left(e^{j\omega}\right)$.

Note: The notation $X\left(e^{j\omega}\right)$ can be confusing. It just emphasizes that $z \to e^{j\omega}$. The expressions are functions of the independent variable ω.

17.1 Analysis and Synthesis Equations

Consider the Fourier series of $x[n]$, a periodically extended finite-length DT signal $\tilde{x}[n]$, e.g.



where $\tilde{x}[n]$ is zero outside the range $[N_1, N_2]$. Since $x[n] = \tilde{x}[n]$ over the interval $-N_1$ to N_2

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi}{N}kn}$$

Define the function $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, then

$$a_k = \frac{1}{N} X\left(e^{jk\omega_0}\right)$$

are samples of $X\left(e^{j\omega}\right)$ at locations that are multiples of $\omega_0 = \frac{2\pi}{N}$. Substituting back into the synthesis equation

$$\tilde{x}[n] = \sum_{k=-N_1}^{N_2} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-N_1}^{N_2} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}$$

Now note that $N = \frac{2\pi}{\omega_0}$ so that

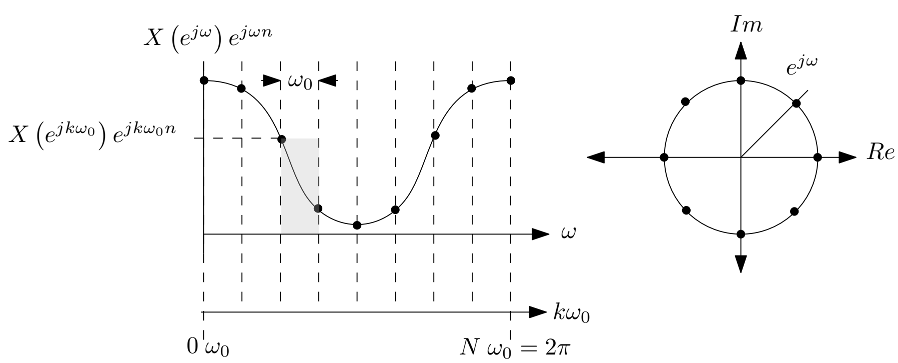
$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=-N_1}^{N_2} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}\, \omega_0$$

Now let $N \to \infty$.

$$\lim_{N\to\infty} \tilde{x}[n] = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{k=-N_1}^{N_2} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}\, \omega_0$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

This is shown graphically in the figure below. As N approaches infinity the sampling of the unit circle becomes infinite, and the summation approaches an integral.



This gives the *DT Fourier Transform Pair.* The Analysis Equation or Forward Transform is:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Note $X\left(e^{j\omega}\right)$ must be a periodic function with period 2π. The Synthesis Equation or Inverse Transform is:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

where the integral is over any 2π period of X.

Example 17.1.1. Let $x[n] = \delta[n]$

$$\begin{aligned} X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} \\ &= e^{-j\omega(0)} \\ &= 1 \end{aligned}$$

■

Example 17.1.2. Let $x[n] = (\gamma)^n \; u[n]$

$$\begin{aligned} X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (\gamma)^n \; e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\gamma e^{-j\omega}\right)^n \end{aligned}$$

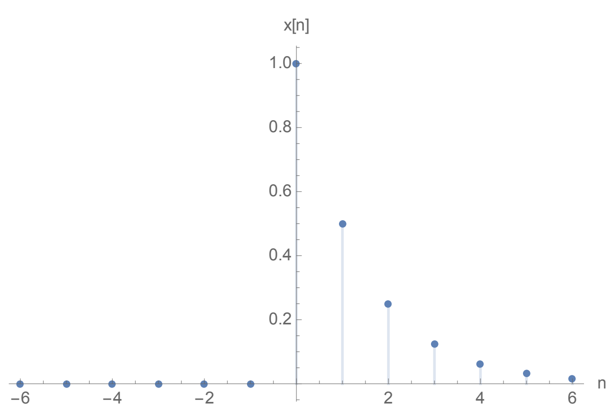
Using the geometric series $\sum\limits_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$ gives:

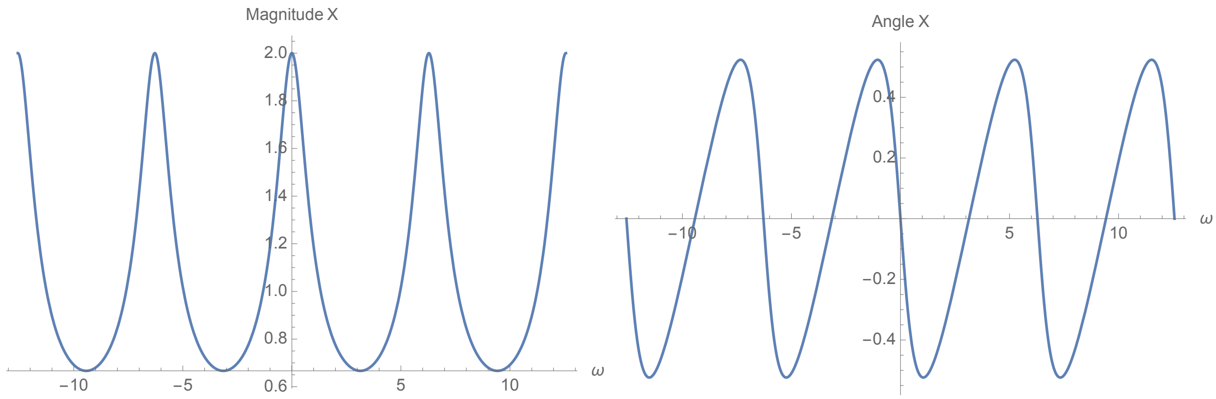
$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} \left(\gamma e^{-j\omega}\right)^n = \frac{1}{1-\gamma e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-\gamma}$$

If $\mid \gamma e^{-j\omega} \mid < 1$ or equivalently $\mid \gamma \mid < 1$.

$$(\gamma)^n \; u[n] \xrightarrow{\mathcal{F}} \frac{1}{1-\gamma e^{-j\omega}}$$

Below is a plot of the original signal and the magnitude and phase spectrum when $\gamma = \frac{1}{2}$.





■

Example 17.1.3. Let

$$X\left(e^{j\omega}\right) = \begin{cases} 1 & |\omega - 2\pi k| < \omega_c \\ 0 & \text{else} \end{cases} \quad \text{for } k \in \mathbb{Z} \text{ and } \omega_c < \pi$$

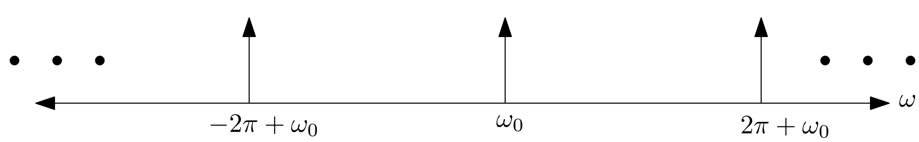
$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega \\ &= \frac{1}{2\pi} \frac{1}{jn} e^{j\omega n} \Big|_{-\omega_c}^{\omega_c} \\ &= \frac{1}{\pi n} \left(\frac{1}{2j} e^{j\omega_c n} - \frac{1}{2j} e^{-j\omega_c n} \right) \\ &= \frac{1}{\pi n} \sin(\omega_c n) = \frac{\omega_c}{\pi} \text{sinc}(\omega_c n) \end{aligned}$$

■

Example 17.1.4. Let

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$$

for $-\pi < \omega_0 < \pi$



$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega - \omega_0) e^{j\omega n} \, d\omega \\ &= \frac{1}{2\pi} e^{j\omega_0 n} \end{aligned}$$

■

17.2 Existence of the DT Fourier Transform

The example of the exponential $x[n] = (\gamma)^n \; u[n]$ above showed that for the DT Fourier transform to exist, the Fourier (analysis) sum must exist. Similar to the CT Fourier transform, a mild conditions is a sufficient prerequisite for the Fourier transform of a signal $x[n]$ to exist: it must be absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

This conditions is not necessary however, and we can extend the Fourier transform to a broader class of signals, if we allow delta functions in the transform, as in the sinusoidal examples above.

17.3 Properties of the DT Fourier Transform

There are several useful properties of the DT Fourier Transform that, when combined with a table of transforms (see Table 5.2, page 392 of OW), allow us to take the Fourier transform of wide array of signals, and one, the convolution property, that allows us to determine the output of a system in the frequency domain easily. We state these here without proof in rough order of usefulness. See the course text for detailed derivations.

We use the following notation

$$\mathcal{F}\left\{x[n]\right\} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\} = x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}\, d\omega$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

Important: $X\left(e^{j\omega}\right)$ is periodic in 2π such that

$$X\left(e^{j(\omega+2\pi k)}\right) = X\left(e^{j\omega}\right) \text{ for } k \in \mathbb{Z}$$

- Linearity Property. Let $x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1\left(e^{j\omega}\right)$ and $x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2\left(e^{j\omega}\right)$ then for $a, b \in \mathbb{C}$

  $$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$$

  Example:

  $$\mathcal{F}\left\{2\left(\frac{1}{2}\right)^n u[n] - 5\left(-\frac{1}{4}\right)^n u[n]\right\} = \frac{2}{1-\frac{1}{2}e^{-j\omega}} - \frac{5}{1+\frac{1}{4}e^{-j\omega}}$$

- Time-shift Property. Let

  $$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

  then

  $$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}X\left(e^{j\omega}\right)$$

  Example:

  $$\mathcal{F}\left\{\delta[n-5]\right\} = e^{-j5\omega}$$

- Frequency Shift Property. Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Example:

$$\mathcal{F}^{-1}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}e^{j\frac{\pi}{20}}}\right\} = e^{j\frac{\pi}{20}n}\left(\frac{1}{2}\right)^n u[n]$$

- Conjugation Property. Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X*\left(e^{-j\omega}\right)$$

Thus, if $x[n]$ is real $X\left(e^{j\omega}\right)$ has conjugate symmetry

$$X\left(e^{-j\omega}\right) = X^*\left(e^{j\omega}\right)$$

and the magnitude spectrum is an even function and the phase spectrum is an odd function.

- Differencing and Accumulation Property. Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$x[n]-x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right) - e^{-j\omega}X\left(e^{j\omega}\right) = \left(1-e^{-j\omega}\right)X\left(e^{j\omega}\right)$$

and

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X\left(e^{j\omega}\right) + \pi X\left(e^{j0}\right)\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

- Time Expansion Property. Let

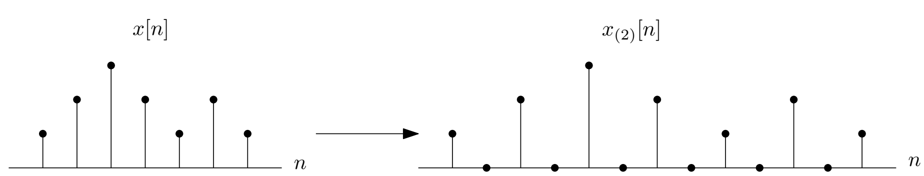
$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{jk\omega}\right)$$

where

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n = \text{ multiple of } k \\ 0 & \text{if } n \neq \text{ multiple of } k \end{cases}$$



- Frequency Differentiation Property Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X\left(e^{j\omega}\right)$$

Example:

$$\mathcal{F}\left\{n\left(\frac{1}{8}\right)^n u[n]\right\} = j\frac{d}{d\omega}\left\{\frac{1}{1-\frac{1}{8}e^{-j\omega}}\right\}$$
$$= j\frac{-\left(-\frac{1}{8}(-j)e^{-j\omega}\right)}{\left(1-\frac{1}{8}e^{-j\omega}\right)^2}$$
$$= \frac{\frac{1}{8}e^{-j\omega}}{\left(1-\frac{1}{8}e^{-j\omega}\right)^2}$$

- Parseval's Relation. Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\omega}\right)\right|^2 \, d\omega$$

The energy is also the integral over one period of the DTFT magnitude squared.

- Convolution Property. Recall for a DT LTI system with impulse response $h[n]$ the output is

$$y[n] = h[n] * x[n]$$

In the frequency domain this is equivalent to

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)\, X\left(e^{j\omega}\right)$$

As in CT systems, convolution in the discrete-time domain is equivalent to multiplication in the frequency domain. Example: suppose a DT system has impulse response

$$h[n] = \left(\gamma_1^n + \gamma_2^n\right)\, u[n]$$

and the input is $x[n] = n\,\gamma_3^n\, u[n]$ where $|\gamma_1| < 1$, $|\gamma_2| < 1$, $|\gamma_3| < 1$. The output in the frequency domain is

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)\, X\left(e^{j\omega}\right)$$
$$= \left[\frac{1}{1-\gamma_1 e^{-j\omega}} + \frac{1}{1-\gamma_2 e^{-j\omega}}\right]\frac{\gamma_3 e^{-j\omega}}{\left(1-\gamma_3 e^{-j\omega}\right)^2}$$
$$= \frac{\gamma_3 e^{-j\omega}}{\left(1-\gamma_1 e^{-j\omega}\right)\left(1-\gamma_3 e^{-j\omega}\right)^2} + \frac{\gamma_3 e^{-j\omega}}{\left(1-\gamma_2 e^{-j\omega}\right)\left(1-\gamma_3 e^{-j\omega}\right)^2}$$

- Multiplication (modulation) Property. Let

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

and

$$y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y\left(e^{j\omega}\right)$$

then

$$x[n]\, y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right)\, Y\left(e^{j(\omega-\theta)}\right)\, d\theta$$

17.4 DT Fourier Transform of a Periodic Signal

The DTFS allows us to write any periodic function with period N as

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{j\frac{2\pi}{N}kn}$$

taking the DT Fourier Transform

$$X\left(e^{j\omega}\right) = \sum_{k=N_0}^{N_0+N-1} a_k \mathcal{F}\left\{e^{j\frac{2\pi}{N}kn}\right\}$$

Using the previously derived transform shows, similar to CT, the DT Fourier Transform of a periodic signal is

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

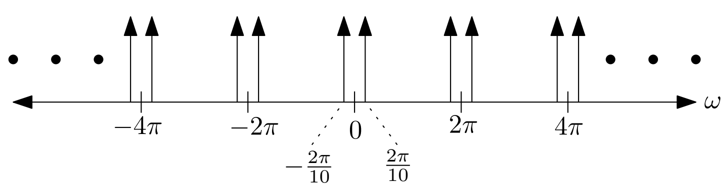
Example

$$x[n] = \cos\left(\frac{2\pi}{10}n\right) = \frac{1}{2}e^{j\frac{2\pi}{10}n} + \frac{1}{2}e^{-j\frac{2\pi}{10}n}$$

Using the previous transform

$$X\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty} \pi\delta\left(\omega - \frac{2\pi}{10} - 2\pi k\right) + \pi\delta\left(\omega + \frac{2\pi}{10} - 2\pi k\right)$$

Which looks like



Chapter 18

CT Frequency Response

In this lecture we are going to focus on the frequency response and highlight its importance in linear systems theory.

18.1 Determining the frequency response (FR) of a CT system

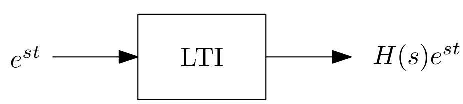
The frequency response of a CT LTI system can be thought of as arising in several equivalent ways. What follows is a common, but not exhaustive, list of ways the frequency response can be derived from other representations.

Using the Eigenvalues / Transfer Function

Recall if we apply the Eigenfunction e^{st} for the complex frequency $s \in \mathbb{C}$ as the input to a LTI system, the output is the Eigenfunction scaled by the Eigenvalue (transfer function) $H(s)$ for values of s in the region of convergence, where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\, dt\ .$$

is the bilateral Laplace transform of the impulse response.



If a system is stable, then the region of convergence includes the imaginary axis $s = j\omega$. In that case, evaluating the Eigenvalues on the imaginary axis $s = j\omega$ gives the CT frequency response $H(j\omega)$. This converts from a function of a complex variable, s, to one of a real variable ω.

Example 18.1.1. Consider a system with Eigenvalues (transfer function)

$$H(s) = \frac{2}{s+5} \text{ for } \operatorname{Re} s > -5$$

Determine the frequency response of the system, if possible.

Solution: We first need to check if the system is stable using the region-of-convergence. Since the real part of the region of convergence includes the imaginary axis ($\operatorname{Re} s = 0$), the system is stable. To find the frequency response we substitute $s = j\omega$ to give

$$H(j\omega) = \frac{2}{j\omega + 5}$$

■

Example 18.1.2. Consider an apparently similar system with Eigenvalues

$$H(s) = \frac{2}{s-5} \text{ for } \operatorname{Re} s > 5$$

Determine the frequency response of the system, if possible.

Solution: Again, we first need to check of the system is stable using the region-of-convergence. Since the real part of the region of convergence does not include the imaginary axis ($\operatorname{Re} s = 0$), the system is unstable. Thus, the frequency response does not exist.
■

Using the CTFT

Another way we can view the frequency response is as the CT Fourier Transform of the impulse response. If the system is stable, then the impulse response is absolutely integrable, and the Fourier transform exists giving $H(j\omega) = \mathcal{F}\{h(t)\}$. This is connected to the transfer function by noting the bilateral Laplace transform and the Fourier Transform are identical under the substitution $s = j\omega$, which is allowed if the system is stable.

Example 18.1.3. Suppose the impulse response of a CT LTI system is given by

$$h(t) = \left(e^{-t} - e^{-6t}\right) u(t)$$

Determine the frequency response of the system, if possible.

Solution: If the system is stable, the Fourier transform of the impulse response exists. Since

$$\int_0^\infty \left|e^{-t} - e^{-6t}\right| \, dt < \int_0^\infty e^{-t} \, dt < \infty$$

the system is stable and the Fourier Transform exists, giving

$$H(j\omega) = \mathcal{F}\left\{\left(e^{-t} - e^{-6t}\right) u(t)\right\} = \mathcal{F}\left\{\left(e^{-t} u(t)\right\} - \mathcal{F}\left\{e^{-6t}\right) u(t)\right\} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 6} = \frac{7}{6 - \omega^2 + j7\omega}$$

■

Directly from a LCCDE

By the convolution theorem of the CTFT, the frequency response is the ratio of the output to input in the frequency domain, i.e.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

We can easily determine this ratio from the LCCDE representation of the system using the derivative property of the Fourier Transform. Recall this property states if $\mathcal{F}\{x(t)\} = X(j\omega)$ then

$$\mathcal{F}\left\{\frac{d^n x}{dt^n}(t)\right\} = (j\omega)^n X(j\omega) \; .$$

If the system is stable (and thus the frequency response exists) then **all** roots of the characteriztic equation $Q(D)$ have real parts that are less than zero. If the system is stable we can take the Fourier transform of each term of the LCCDE using the derivative property, then algebrically solve for the ratio of output to input. Note this provides a signifigant savings in analysis effort since we do not have to first find the impulse response, then take its Fourier transform to arrive at the frequency response (although that approach is still valid).

Example 18.1.4. Consider a system decribed by the LCCDE

$$\frac{d^2y}{dt^2}(t) + 15\frac{dy}{dt}(t) + 50y(t) = 10x(t)$$

Determine the frequency response of the system, if possible.

Solution: We first need to check for stability. The characteristic equation is $Q(D) = D^2 + 15D + 50$ which has two real roots -10 and -5. Since both are less than zero, the system is stable. Next we take the Fourier transform of both sides and apply the derivative property

$$(j\omega)^2 Y(j\omega) + 15(j\omega)Y(j\omega) + 50Y(j\omega) = 10X(j\omega)$$

and rearrange to get the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{10}{(j\omega)^2 + 15(j\omega) + 50} = \frac{10}{50 - \omega^2 + j15\omega}$$

■

18.2 Magnitude-phase representation of the CTFR

Note that any complex valued function can be expressed in polar form using the magnitude and phase. Specifically the input and output can be put into this form

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)}$$

$$Y(j\omega) = |Y(j\omega)|e^{\angle Y(j\omega)}$$

By the convolution theorem then

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{|Y(j\omega)|e^{\angle Y(j\omega)}}{X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)}} = \frac{|Y(j\omega)|}{|X(j\omega)|}e^{\angle Y(j\omega) - \angle X(j\omega)} = |H(j\omega)|e^{\angle H(j\omega)}$$

Thus we see that

$$|H(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|}$$

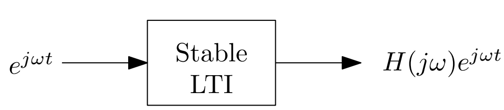
and

$$\angle H(j\omega) = \angle Y(j\omega) - \angle X(j\omega)$$

This is the magnitude and phase representation of the frequency response.

18.3 CTFR acting on sinusoids

The advantage of the magnitude and phase representation of the frequency response, is the ease with which we can find the output due to a sinusoidal input. If we apply a sinusoidal input $x(t) = Ae^{j\omega t}$, the output is a the same sinusoid scaled by the frequency response $y(t) = H(j\omega)Ae^{j\omega t}$.



Now using the magnitude and phase representation

$$y(t) = H(j\omega)Ae^{j\omega t} = |H(j\omega)|e^{\angle H(j\omega)}Ae^{j\omega t} = A|H(j\omega)|e^{j\omega t+\angle H(j\omega)}$$

Thus we can interpret the frequency response as telling us how the input sinsusoids are scaled in magnitude and phase shifted as they pass through the system.

By the linearity property this extends to real sinusoidal inputs since

$$\begin{aligned} x(t) &\longrightarrow y(t) \\ \sin(\omega t) &\longrightarrow \frac{1}{2j}|H(j\omega)|e^{j\omega t+\angle H(j\omega)} - \frac{1}{2j}|H(j\omega)|e^{-j\omega t+\angle H(j\omega)} \\ \sin(\omega t) &\longrightarrow |H(j\omega)|\sin(\omega t + \angle H(j\omega)) \end{aligned}$$

and

$$\begin{aligned} x(t) &\longrightarrow y(t) \\ \cos(\omega t) &\longrightarrow \frac{1}{2}|H(j\omega)|e^{j\omega t+\angle H(j\omega)} + \frac{1}{2}|H(j\omega)|e^{-j\omega t+\angle H(j\omega)} \\ \cos(\omega t) &\longrightarrow |H(j\omega)|\cos(\omega t + \angle H(j\omega)) \end{aligned}$$

Also by the linearity property this analysis extends to the CT Fourier representation of a signal (an infinite sum of sinusoids):

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \;\longrightarrow\; y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|\, X(j\omega)\, e^{j\omega t+\angle H(j\omega)}\, d\omega$$

Thus we arrive at the reason for the name *Frequency Response* – it specifies the the response of a stable system to any linear combination of sinusoidal inputs, i.e. any signal with a Fourier Transform.

18.3.1 Bode plots

We can visualize the frequency response as a plot of the real and imaginary part, or, of the magnitude and phase. Since the magnitude and phase allow us to directly see the system behavior at a given frequency, those plots are much more useful.

Rather than simply plot the magnitude and phase as a function of ω, it is common to change the abscissa (horizontal / ω-axis) to be on a logarithmic scale and so only plot the positive frequency portion of the spectrum (recall if the signal is real, the frequency response is even, so no information is lost). This is because the frequency response for physically realizable systems changes slowly as a function of frequency. Plotting on a log-scale compresses this information horizontally so that we can see how a wide range of frequency content is scaled. When plotting the magnitude spectrum it is also common to make the ordinate (vertical / gain axis) to be in decibels (dB). This is because of Weber's law, which states the humans perceive a doubling in strength of stimulus, when it is actually a ten-fold increase. Thus the magnitude of the frequency response in dB is $20\log_{10}|H(j\omega)|$. When the frequency response is plotted this particular way we get what is called a *Bode plot* (after the engineer Hendrik Wade Bode, an important figure in the development of control theory).

You will likely encounter Bode plots at several points in your career, so it is important to understand them well enough to create them on your own using software and read them. Also data-sheets and other documentation for CT devices generally use a Bode plot rather than giving an explicit mathematical model of the frequency response. It is also instructive to learn how to plot them manually (which was the traditional way to do it) since it gives you insight that can help with reverse engineering a model, however we do not cover this in detail in this course. Note that we will plot the spectrum as a function of frequency in units of rad/s, but it is also common to see it plotted in units of Hz. Take care to read the horizontal axis label as mixing up the two is a common source of error.

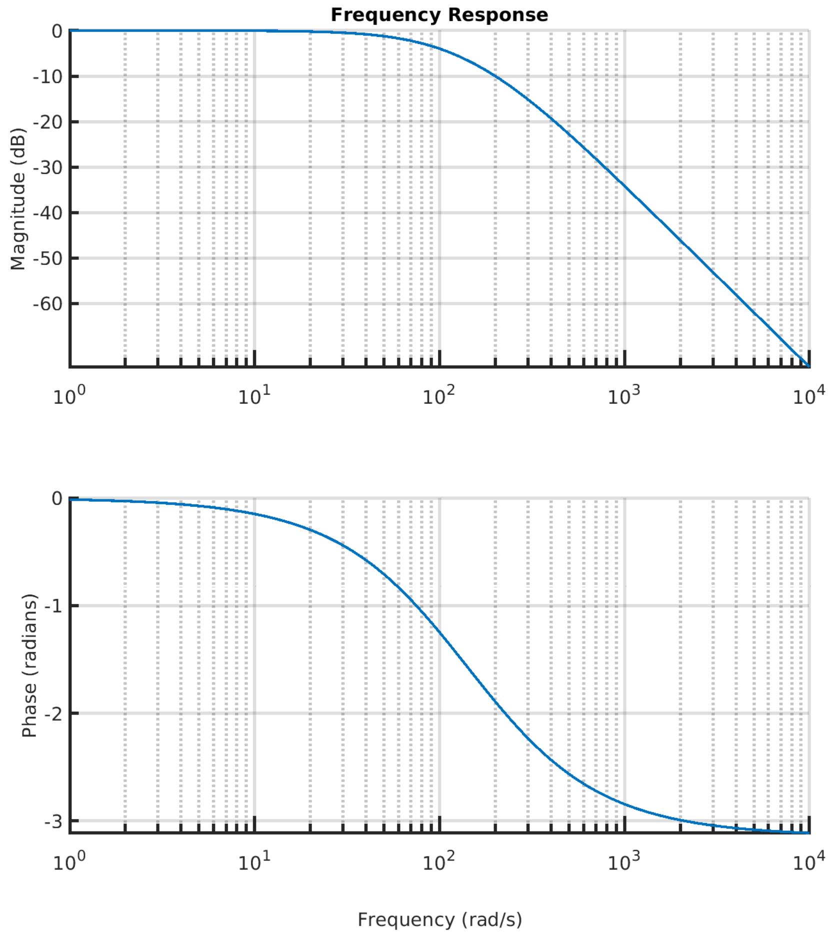
Example 18.3.1. Consider a frequency response given by

$$H(j\omega) = \frac{20000}{(j\omega)^2 + 300(j\omega) + 20000}$$

The following Matlab code shows you how to plot the spectrum as a Bode plot (with some extra code to make it look nicer). You should read the documentation for the `bode` command in Matlab. It is also easy to just compute the magnitude and phase yourself.

```
H = tf([20000],[1,300,20000]);
[mag,ph,w] = bode(H);

% Create a nice bode plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = semilogx(w,20*log10(squeeze(mag)));
grid on;
hTitle  = title ('Frequency Response');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, 'YTick', -60:10:20, ...
    'Box', 'off', 'LineWidth', 2);

subplot(2,1,2);
hp = semilogx(w,squeeze(ph*(pi/180)));
grid on;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/s)');
set(gca, 'FontSize', 14, 'Box', 'off', 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);
set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14          );
set( hTitle                      , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

This gives the following plot



■

To read a Bode plot to see the behavior of the system at a given frequency, one need only read the values off the plot and convert from dB to a unit-less gain. A common mistake is to not realize the horizontal axis is logarithmic.

Example 18.3.2. Suppose you are given the Bode plot (only) from the previous example and are asked what the output of the system is when the input is $x(t) = \cos(2\pi 32 t)$, i.e. a sinusoid at 32 Hz.
Solution: First we determine the frequency in rad/s, $\omega = 2\pi 32 \approx 200$ rad/s. We go to that frequency on the Bode plot and read off a value of about -10 dB for the magnitude and about -1.9 rad for the phase. To convert back from dB

$$|H(200)| = 10^{\frac{-10}{20}} \approx 0.3$$

so the output would be

$$y(t) \approx 0.3 \cos(2\pi 32 t - 1.9)$$

■

18.4 CTFR of first and second order systems

TODO

Chapter 19

DT Frequency Response

In this lecture we are going to focus on the frequency response of discrete-time systems and highlight its importance in linear systems theory.

19.1 Determining the frequency response (FR) of a DT system

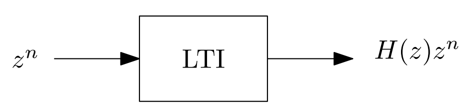
The frequency response of a DT LTI system can be thought of as arising in several equivalent ways. What follows is a common, but not exhaustive, list of ways the frequency response can be derived from other representations.

Using the Eigenvalues / Transfer Function

Recall if we apply the Eigenfunction z^n for $z \in \mathbb{C}$ as the input to a LTI system, the output is the Eigenfunction scaled by the Eigenvalue (transfer function) $H(z)$ for values of z in the region of convergence, where

$$H(z) = \sum_{-\infty}^{\infty} h[n] z^{-n} \ .$$

is the bilateral Z transform of the impulse response.



If a system is stable, then the region of convergence includes the unit circle $z = e^{j\omega}$. In that case, evaluating the Eigenvalues on the unit circle gives the DT frequency response $H\left(e^{j\omega}\right)$. This converts from a function of a complex variable, z, to one of a real variable ω.

Example 19.1.1. Consider a system with Eigenvalues (transfer function)

$$H(z) = \frac{z}{z + \frac{1}{2}} \text{ for } |z| > \frac{1}{2}$$

Determine the frequency response of the system, if possible.

Solution: We first need to check if the system is stable using the region-of-convergence. Since the region of convergence includes the unit circle, the system is stable. To find the frequency response we substitute $s = e^{j\omega}$ to give

$$H\left(e^{j\omega}\right) = \frac{e^{j\omega}}{e^{j\omega} + \frac{1}{2}}$$

■

Example 19.1.2. Consider an apparently similar system with Eigenvalues

$$H(z) = \frac{z}{z+2} \text{ for } |z| > 2$$

Determine the frequency response of the system, if possible.

Solution: Again, we first need to check of the system is stable using the region-of-convergence. Since the region of convergence does not include the unit circle, the system is unstable. Thus, the frequency response does not exist.

■

Using the DTFT

Another way we can view the frequency response is as the DT Fourier Transform of the impulse response. If the system is stable, then the impulse response is absolutely integrable, and the Fourier transform exists giving $H\left(e^{j\omega}\right) = \mathcal{F}\left\{h[n]\right\}$. This is connected to the transfer function by noting the bilateral Z transform and the DT Fourier Transform are identical under the substitution $z = e^{j\omega}$, which is allowed if the system is stable.

Example 19.1.3. Suppose the impulse response of a DT LTI system is given by

$$h[n] = \left(\frac{1}{4}\right)^n u[n] + 5\left(\frac{2}{3}\right)^n u[n]$$

Determine the frequency response of the system, if possible.

Solution: If the system is stable, the Fourier transform of the impulse response exists. Since $\left(\frac{1}{4}\right) < 1$ and $\left(\frac{2}{3}\right) < 1$

$$H\left(e^{j\omega}\right) = \mathcal{F}\left\{\left(\frac{1}{4}\right)^n u[n] + 5\left(\frac{2}{3}\right)^n u[n]\right\} = \mathcal{F}\left\{\left(\frac{1}{4}\right)^n u[n]\right\} + 5\mathcal{F}\left\{\left(\frac{2}{3}\right)^n u[n]\right\} = \frac{e^{j\omega}}{e^{j\omega} - \left(\frac{1}{4}\right)} + \frac{5e^{j\omega}}{e^{j\omega} - \left(\frac{2}{3}\right)}$$

■

Directly from a LCCDE

By the convolution theorem of the DTFT, the frequency response is the ratio of the output to input in the frequency domain, i.e.

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

We can easily determine this ratio from the LCCDE representation of the system using the shifting property of the DT Fourier Transform. Recall this property states if $\mathcal{F}\{x[n]\} = X\left(e^{j\omega}\right)$ then

$$\mathcal{F}\left\{x[n-m]\right\} = e^{-j\omega m} X\left(e^{j\omega}\right) \ .$$

for index shift $m \in \mathbb{Z}$.

If the system is stable (and thus the frequency response exists) then **all** roots of the characteristic equation $Q(E)$ have magnitude that are less than one. If the system is stable we can take the Fourier transform of each term of the LCCDE using the shift property, then algebraically solve for the ratio of output to input. Note this provides a significant savings in analysis effort since we do not have to first find the impulse response, then take its Fourier transform to arrive at the frequency response (although that approach is still valid).

Example 19.1.4. Consider a system described by the LCCDE

$$3y[n+1] - y[n] = x[n+1]$$

Determine the frequency response of the system, if possible.

Solution: We first need to check for stability. The characteristic equation is $Q(E) = 3E - 1$ which has a single root of $\frac{1}{3}$. Since it is less than one, the system is stable. Next we take the Fourier transform of both sides and apply the derivative property

$$3e^{j\omega} Y\left(e^{j\omega}\right) - Y\left(e^{j\omega}\right) = e^{j\omega} X\left(e^{j\omega}\right)$$

and rearrange to get the frequency response

$$H\left(e^{j\omega}\right) = \frac{e^{j\omega}}{3e^{j\omega} - 1}$$

■

19.2 Magnitude-phase representation of the DTFR

Note that any complex valued function can be expressed in polar form using the magnitude and phase. Specifically the input and output can be put into this form

$$X\left(e^{j\omega}\right) = |X\left(e^{j\omega}\right)| e^{\angle X\left(e^{j\omega}\right)}$$

$$Y\left(e^{j\omega}\right) = |Y\left(e^{j\omega}\right)| e^{\angle Y\left(e^{j\omega}\right)}$$

By the convolution theorem then

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{|Y\left(e^{j\omega}\right)| e^{\angle Y\left(e^{j\omega}\right)}}{|X\left(e^{j\omega}\right)| e^{\angle X\left(e^{j\omega}\right)}} = \frac{|Y\left(e^{j\omega}\right)|}{|X\left(e^{j\omega}\right)|} e^{\angle Y\left(e^{j\omega}\right) - \angle X\left(e^{j\omega}\right)} = |H\left(e^{j\omega}\right)| e^{\angle H\left(e^{j\omega}\right)}$$

Thus we see that

$$|H\left(e^{j\omega}\right)| = \frac{|Y\left(e^{j\omega}\right)|}{|X\left(e^{j\omega}\right)|}$$

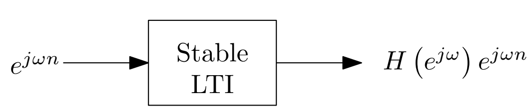
and

$$\angle H\left(e^{j\omega}\right) = \angle Y\left(e^{j\omega}\right) - \angle X\left(e^{j\omega}\right)$$

This is the magnitude and phase representation of the frequency response.

19.3 DTFR acting on sinusoids

The advantage of the magnitude and phase representation of the frequency response, is the ease with which we can find the output due to a sinusoidal input. If we apply a sinusoidal input $x[n] = Ae^{j\omega n}$, the output is a the same sinusoid scaled by the frequency response $y[n] = H\left(e^{j\omega}\right) Ae^{j\omega n}$.



Now using the magnitude and phase representation

$$y[n] = H\left(e^{j\omega}\right) Ae^{j\omega n} = |H\left(e^{j\omega}\right)| e^{\angle H\left(e^{j\omega}\right)} Ae^{j\omega n} = A|H\left(e^{j\omega}\right)| e^{j\omega n + \angle H\left(e^{j\omega}\right)}$$

Thus we can interpret the frequency response as telling us how the input sinusoids are scaled in magnitude and phase shifted as they pass through the system.

By the linearity property this extends to real sinusoidal inputs since

$$\begin{aligned} x[n] &\longrightarrow y[n] \\ \sin(\omega n) &\longrightarrow \frac{1}{2j}|H\left(e^{j\omega}\right)| e^{j\omega n + \angle H\left(e^{j\omega}\right)} - \frac{1}{2j}|H\left(e^{j\omega}\right)| e^{-j\omega n + \angle H\left(e^{j\omega}\right)} \\ \sin(\omega n) &\longrightarrow |H\left(e^{j\omega}\right)| \sin(\omega n + \angle H\left(e^{j\omega}\right)) \end{aligned}$$

and

$$\begin{aligned} x[n] &\longrightarrow y[n] \\ \cos(\omega n) &\longrightarrow \frac{1}{2}|H\left(e^{j\omega}\right)| e^{j\omega n + \angle H\left(e^{j\omega}\right)} + \frac{1}{2}|H\left(e^{j\omega}\right)| e^{-j\omega n + \angle H\left(e^{j\omega}\right)} \\ \cos(\omega n) &\longrightarrow |H\left(e^{j\omega}\right)| \cos(\omega n + \angle H\left(e^{j\omega}\right)) \end{aligned}$$

Also by the linearity property this analysis extends to the DT Fourier representation of a signal (an infinite sum of sinusoids):

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \;\longrightarrow\; y[n] = \frac{1}{2\pi} \int_{2\pi} H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{2\pi} \left|H\left(e^{j\omega}\right)\right| X\left(e^{j\omega}\right) e^{j\omega n + \angle H\left(e^{j\omega}\right)}\, d\omega$$

Thus we arrive at the reason for the name DT *Frequency Response* – it specifies the response of a stable system to any linear combination of DT sinusoidal inputs, i.e. any signal with a Fourier Transform.

19.4 Plotting the DT frequency response

As in CT, we can visualize the frequency response as a plot of the real and imaginary part, or, of the magnitude and phase. Since the magnitude and phase allow us to directly see the system behavior at a given frequency, those plots are much more useful.

In contrast to CT, where the unique Bode plot format is used, for the DTFR it is most common to plot the magnitude spectrum in dB and the phase spectrum in rad over just $\omega = [0, \pi]$. Since the DTFR is periodic there is no need to compress the information using a logarithmic frequency scale. Further, if $x[n]$ is real the DTFR magnitude spectrum is even, so that the magnitude from $\omega = [\pi, 2\pi]$ is the same as from $\omega = [-\pi, 0]$. Also if $x[n]$ is real the DTFR phase spectrum is odd, so that the phase from $\omega = [\pi, 2\pi]$ is the same as the negative from $\omega = [-\pi, 0]$. Note we would not call this kind of plot a Bode plot as that is typically reserved for the CTFR.

As with the CTFR it is important to understand these plots well enough to create them on your own using software and read them.

Example 19.4.1. Consider a frequency response given by

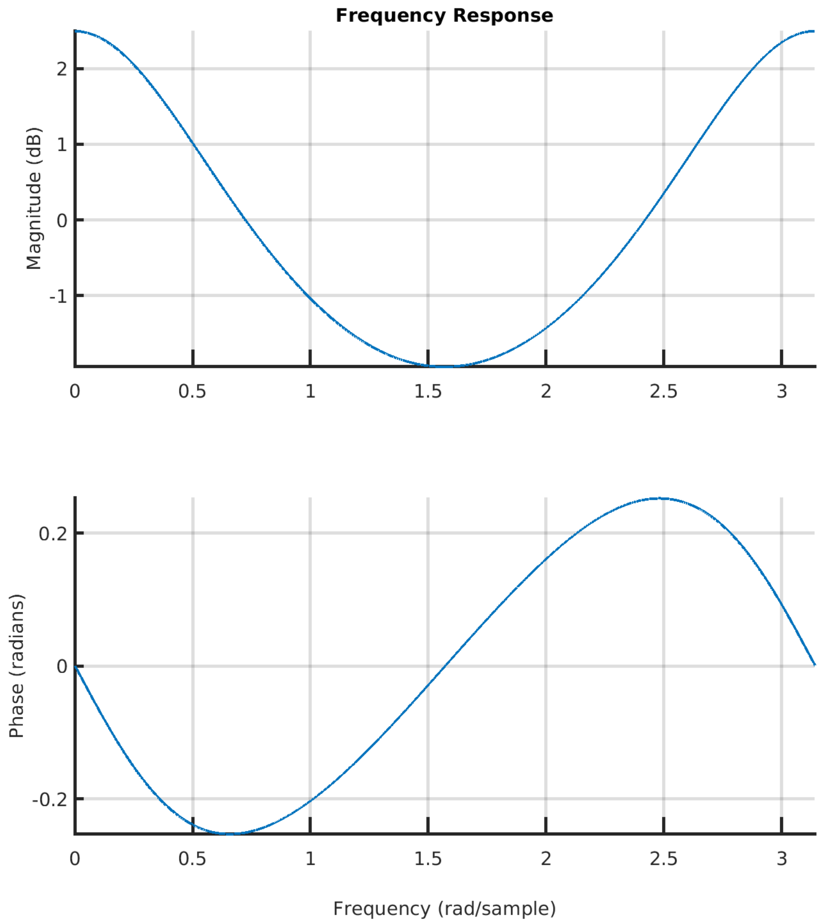
$$H\left(e^{j\omega}\right) = \frac{4e^{j2\omega}}{4e^{j2\omega} - 1}$$

The following Matlab code shows you how to plot the spectrum (with some extra code to make it look nicer).

```
% compute the FTFR
w = 0:0.001:pi;
```

```
H = 4.*exp(j*2*w)./(4*exp(j*2*w) - 1);

% Create a nice DTFR plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = plot(w,20*log10(abs(H)));
axis tight;
grid on;
hTitle  = title ('Frequency Response');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, ...
    'Box', 'off', 'LineWidth', 2);

subplot(2,1,2);
hp = plot(w,angle(H));
axis tight;
grid on;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/sample)');
set(gca, 'FontSize', 14, 'Box', 'off', 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);
set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14          );
set( hTitle                    , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

This gives the following plot



■

To read a Bode plot to see the behavior of the system at a given frequency, one need only read the values off the plot and convert from dB to a unit-less gain.

Example 19.4.2. Suppose you are given the DTFR plot (only) from the previous example and are asked: what the output of the system is when the input is $x[n] = \cos\left(\frac{\pi}{4}n\right)$?
Solution: We go to the frequency $\frac{\pi}{4} \approx 0.78$ on the plot and read off a value of about -0.1 dB for the magnitude and about -0.25 rad for the phase. To convert back from dB

$$\left|H\left(e^{j\frac{\pi}{4}}\right)\right| = 10^{\frac{-0.1}{20}} \approx 0.988$$

so the output would be

$$y[n] \approx 0.988 \cos\left(\frac{\pi}{4}n - 0.25\right)$$

■

Chapter 20

Frequency Selective Filters in CT

Recall the response of stable CT LTI systems to periodic inputs. Given a stable LTI system with frequency response $H(j\omega)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Note the output is equivalent to a signal with Fourier series coefficients $b_k = a_k H(jk\omega_0)$. That is the Fourier coefficients are scaled by the frequency response at the harmonic frequency $k\omega_0$.

Similarly for aperiodic signals, given a stable LTI system with frequency response $H(j\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega \longrightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t} \, d\omega$$

Note the output is equivalent to a signal with Fourier Transform $Y(j\omega) = X(j\omega)H(jk\omega_0)$. That is the Fourier transform at each continuous frequency ω is scaled by the frequency response at that frequency.

We can use this behavior to our advantage. In many applications we want to modify the values of a_k or $X(j\omega)$ selectively, passing them unmodified, increasing (amplifying) them, or decreasing (attenuating) them. This is accomplished by designing a frequency response. Such systems are called frequency selective *filters* and come in 4 basic types:

- Low-pass Filters attenuate high frequencies while passing through lower frequencies. They are often used to reduce the effects of high-frequency noise in a signal and to prepare it for sampling (so-called anti-aliasing filters). They are the most common filter.

- High-pass Filters attenuate lower frequencies while passing through higher frequencies. While less common, they are often used to remove the DC component ($\omega = 0$) of a signal and to compute the derivative of a signal.

- Bandpass Filters attenuate frequencies outside a band of frequencies. They can be viewed as a combination of a high-pass and low-pass filter. They are commonly used to select a range of frequencies for further processing and are central to many communication technologies.

- Notch or Bandstop Filters attenuate frequencies inside an often narrow band of frequencies. Common applications are the removal of one or more corrupting signals mixed into another signal.

While the design of such filters is outside the scope of this course, you are now equipped to understand and apply them based on your knowledge of the Fourier methods covered over the past several weeks.

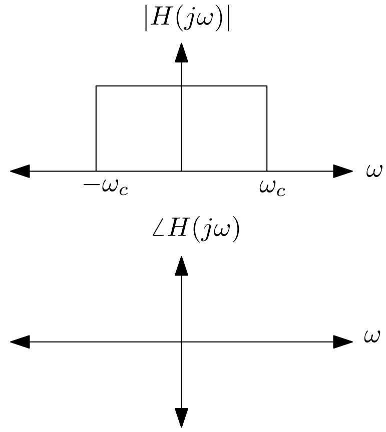
20.1 Ideal Filters

The above filter types each have an ideal (although unrealizable) form.

Low-pass filters remove frequency content above a threshold, ω_c, called the *cutoff frequency.* They have an ideal frequency response

$$H(j\omega) = \begin{cases} 1 & -\omega_c < \omega < \omega_c \\ 0 & \text{else} \end{cases}$$

with magnitude and phase plot

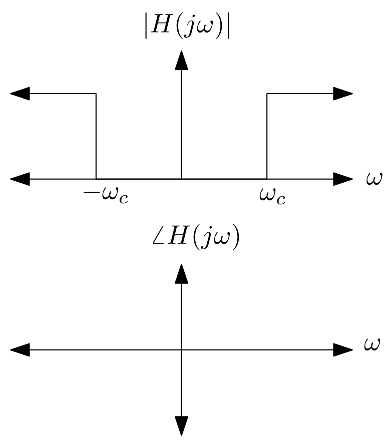


The range of frequencies below $|\omega_c|$ are called the pass-band. The range of frequencies above $|\omega_c|$ are called the stop-band.

High-pass filters remove frequency content below the cutoff frequency ω_c. They have an ideal frequency response

$$H(j\omega) = \begin{cases} 0 & -\omega_c < \omega < \omega_c \\ 1 & \text{else} \end{cases}$$

with magnitude and phase plot

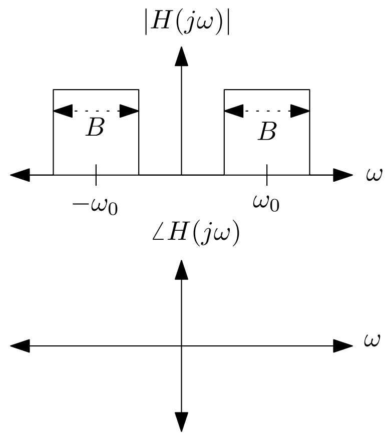


The range of frequencies above $|\omega_c|$ are called the pass-band. The range of frequencies below $|\omega_c|$ are called the stop-band.

Bandpass filters remove frequency content outside a band of frequencies called the pass-band. They have an ideal frequency response

$$H(j\omega) = \begin{cases} 1 & -\omega_0 - \frac{B}{2} < \omega < -\omega_0 + \frac{B}{2} \\ 1 & \omega_0 - \frac{B}{2} < \omega < \omega_0 + \frac{B}{2} \\ 0 & \text{else} \end{cases}$$

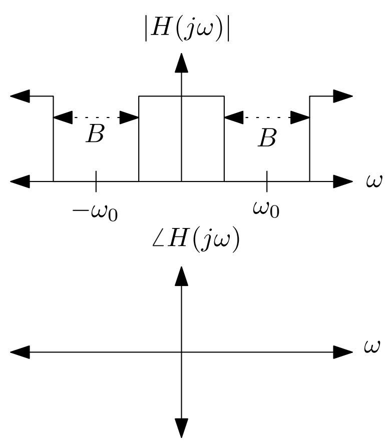
where ω_0 is the *center frequency* and B is the *bandwidth*. The frequencies outside this range are in the stop-band. The magnitude and phase plot looks like



Finally, notch or bandstop filters remove frequency content inside a band of frequencies (the stop band) defined by the center frequency ω_0 and bandwidth B. The ideal frequency response is

$$H(j\omega) = \begin{cases} 0 & -\omega_0 - \frac{B}{2} < \omega < -\omega_0 + \frac{B}{2} \\ 0 & \omega_0 - \frac{B}{2} < \omega < \omega_0 + \frac{B}{2} \\ 1 & \text{else} \end{cases}$$

with magnitude and phase plot



Often the bandstop filter has a very narrow bandwidth, thus it "notches" out a frequency component of the input signal.

20.2 Practical Filters

Ideal CT filters cannot be implemented in practice because they are non-causal and thus physically impossible. To see why consider the impulse response of the ideal low-pass filter:

$$h(t) = \mathcal{F}^{-1}\left\{H(j\omega)\right\} = \frac{1}{2\pi}\int_{\omega_c}^{\omega_c} e^{j\omega t}\, d\omega = \frac{1}{\pi t}\sin(\omega_c t)$$

which has nonzero values for $t < 0$, and thus corresponds to a non-casual system. Ideal filters also have zero phase which cannot be achieved in practice.

Practical filters are described by a frequency response that is a ratio of two polynomials in $j\omega$, i.e.

$$H(j\omega) = \frac{K \cdot (j\omega - \beta_1) \cdot (j\omega - \beta_2) \cdots (j\omega - \beta_M)}{(j\omega - \alpha_1) \cdot (j\omega - \alpha_2) \cdots (j\omega - \alpha_N)}$$

where K is a constant that controls the gain at DC, and the zero or more complex coefficients β_k and the one or more complex coefficients α_k are called the *zeros* and *poles* of the filter respectively. Such systems correspond to differential equations as we have covered before and are physically realizable as circuits if all poles and zeros are real or come in conjugate pairs. The processes of designing filters consists of choosing the poles and zeros, or equivalently choosing the coefficients of the numerator and dominator polynomials. This is covered in ECE 3704, ECE 4624, and other upper-level courses.

Practical filters differ from ideal filters in that they cannot be zero over any finite range of frequencies and cannot transition discontinuously between stop and pass bands. Instead they must vary over the bands and transition smoothly, with a degree of variation and sharpness that is a function of the order of the filter and the exact form of the frequency response polynomials. Thus practical filters are described by additional parameters that define the stop and pass-bands.

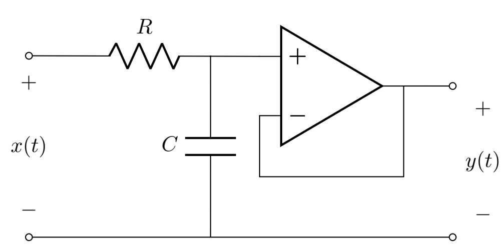
The overall gain of the filter is the magnitude of the frequency response at a frequency that depends on the filter type, zero for a low-pass filter and the center frequency for a band-pass filter. The pass-band is defined by the frequency at which the magnitude of the frequency response drops below the overall gain, often -3dB $= \frac{\sqrt{2}}{2}$. The stop-band is defined similarly, as the frequency at which the magnitude of the frequency response drops further below the overall gain, often -20dB $= 0.1$ or -40dB $= 0.01$. The *transition bandwidth* is defined as the difference in the stop-band and pass-band frequencies. The *pass-band ripple* is defined as the maximum deviation from the overall gain, over the pass-band.

20.3 First-order and second-order systems as filters

Given the equivalence of LTI systems and linear, constant-coefficient differential equations, block diagrams, impulse responses, and frequency responses, filters can be represented in any of these ways.

We have covered extensively first-order and second-order CT systems and seen how they can be represented variously as circuits, differential equations, block diagrams, and as frequency responses. We now see how they can describe simple filters and serve as building blocks for higher-order filters.

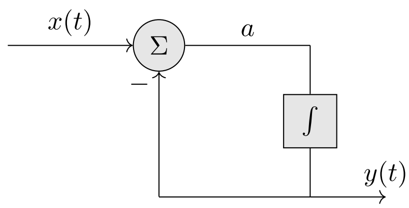
Example 20.3.1. Consider a low-pass filter with the desired characteristics of having a pass-band of -3dB at 1kHz, and a stop-band of -20dB at 10kHz. Suppose this is implemented as a first-order "Butterworth" filter, which can be realized by an RC circuit.



where $R = 99.2k\Omega$ and $C = 1.6$nF. This is equivalent to the differential equation

$$\frac{dy}{dt}(t) + ay(t) = ax(t)$$

where $a = \frac{1}{RC}$, or the block diagram



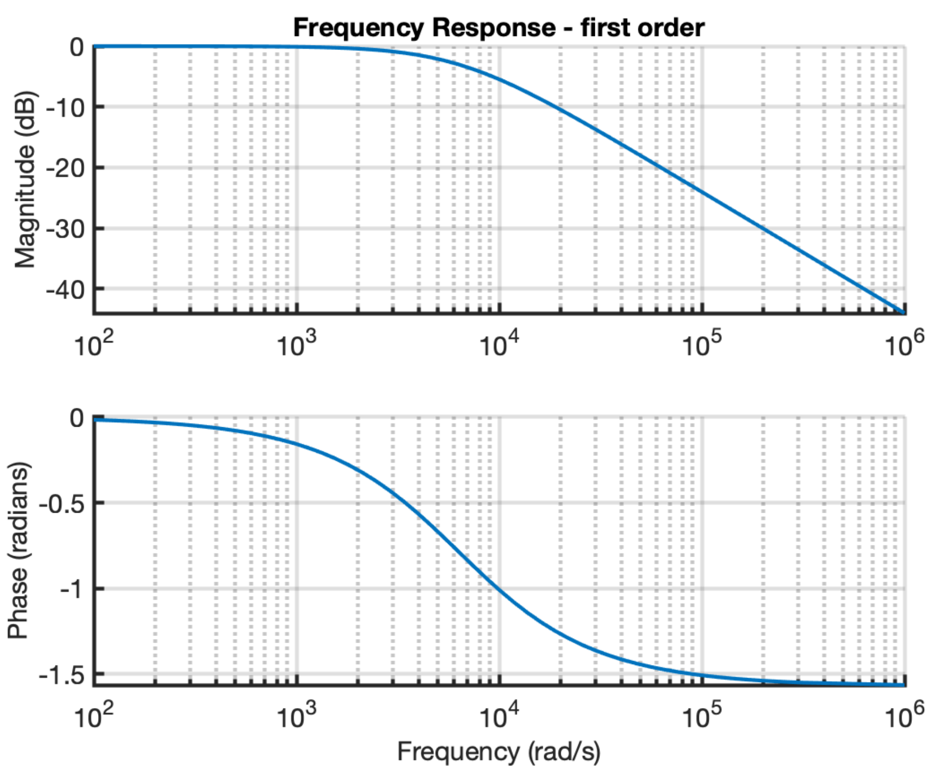
This system has the impulse response

$$h(t) = ae^{-at}u(t)$$

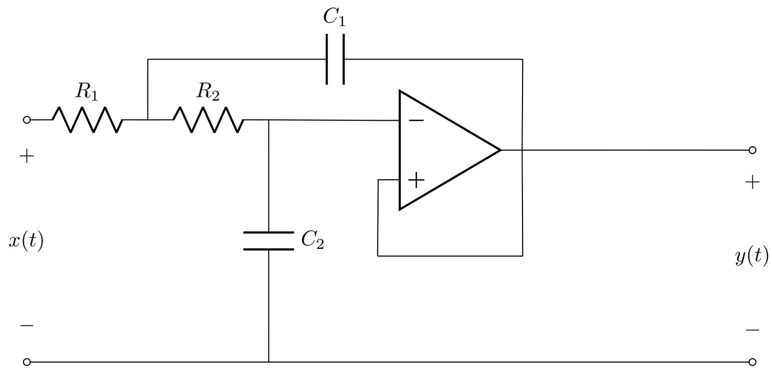
and the frequency response

$$H(j\omega) = \frac{a}{j\omega + a}$$

If we plot the frequency response as a Bode plot, we see the DC gain is 0dB, and the response passes through -3dB and -20dB at the expected frequencies $2\pi * 1000 \approx 6.3 \times 10^3$ rad/s and $2\pi * 10000 \approx 6.3 \times 10^4$ rad/s. Thus the transition bandwidth is 9kHz.



Example 20.3.2. Suppose we wish to sharpen the transition band for the previous example so that has a pass-band of -3dB at 1kHz, and a narrower stop-band of -20dB at 5kHz. This requires a second-order filter, and can be realized by a circuit called the Sallen-Key.



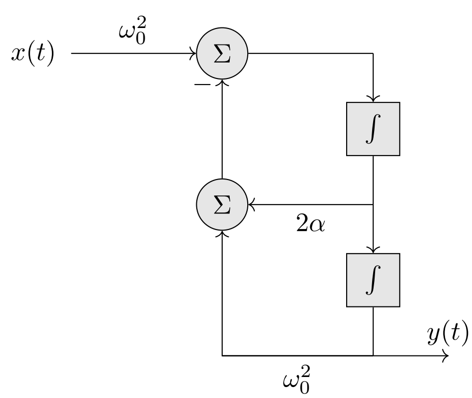
where $R_1 = 74.2k\Omega$, $R_2 = 91.3M\Omega$, $C_1 = 1.6$nF and $C_2 = 160$pF. This is equivalent to the differential equation

$$\frac{d^2y}{dt^2}(t) + 2\alpha\frac{dy}{dt}(t) + \omega_0^2 y(t) = \omega_0^2 x(t)$$

where

$$\alpha = \frac{R_1 + R_2}{2R_1R_2C_1} \text{ and } \omega_0^2 = \frac{1}{R_1R_2C_1C_2}$$

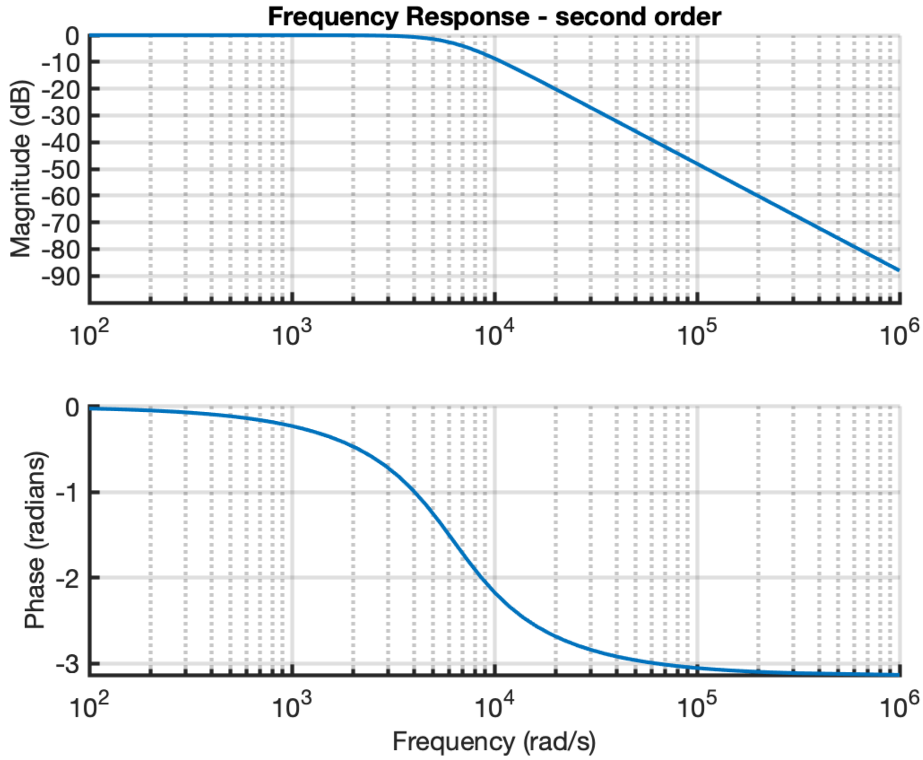
or the block diagram



This system has the frequency response

$$H(j\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\alpha\omega}$$

If we plot the frequency response as a Bode plot using the resistor and capacitor values above, we see the DC gain is 0dB, and the response passes through -3dB at the expected frequency $2\pi * 1000 \approx 6.3 \times 10^3$ rad/s. At the frequency $2\pi * 5000 \approx 3.14 \times 10^4$ rad/s the response passes through about -28dB. Thus this circuit has a transition bandwidth even narrower than that designed (it is slightly better).



Note the price we pay for this decreased transition bandwidth is a larger phase shift (and a two more components).

20.4 Higher-Order Filters

We can continue to increase the steepness of the passband to stop-band transitions by increasing the order of the filter. This is typically accomplished using a serial connection of systems, called *stages* in filter parlance, where each stage is a first-order or second-order system.

Recall in a series connection of systems the overall impulse response is the convolution of the individual responses. If we assume each stage is stable then, by the convolution property, the overall frequency response is given by the product of their individual frequency responses.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = H_1(j\omega) \cdot H_2(j\omega)$$

Writing each response in polar form

$$H_1(j\omega) \cdot H_2(j\omega) = |H_1(j\omega)| \cdot |H_2(j\omega)| e^{\angle H_1(j\omega) + \angle H_2(j\omega)}$$

we note that the magnitudes multiply and the phases add. That means we can use additional stages to reinforce the attenuation of previous stages. Note this requires in the circuit that the stages be impedance isolated, thus the use of the opamps at the end of CT filters. Again the price we pay for increasing the order of the filter and decreasing the transition frequency is increased phase shift in the signal.

Matlab code for plotting the first-order example Bode plot:

```
R = 99.2e3;
C = 1.6e-9;
a = 1/(R*C);

H = tf([a],[1,a]);
[mag,ph,w] = bode(H);

% Create a nice bode plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = semilogx(w,20*log10(squeeze(mag)));
grid on;
hTitle  = title ('Frequency Response - first order');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, 'YTick', -60:10:20, ...
    'Box', 'off', 'LineWidth', 2);

subplot(2,1,2);
hp = semilogx(w,squeeze(ph*(pi/180)));
grid on;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/s)');
set(gca, 'FontSize', 14, 'Box', 'off', 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);
set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14          );
set( hTitle                     , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

Matlab code for plotting the second-order example Bode plot:

```
R1 = 74.2e3;
R2 = 1.33e6;
C1 = 1.6e-9;
C2 = 160e-12;

a = (R1+R2)/(R1*R2*C1);
b = 1/(R1*R2*C1*C2);

H = tf([b],[1,a,b]);
[mag,ph,w] = bode(H);

% Create a nice bode plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = semilogx(w,20*log10(squeeze(mag)));
grid on;
hTitle  = title ('Frequency Response - second order');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, 'YTick', -90:10:20, ...
    'Box', 'off', 'LineWidth', 2);

subplot(2,1,2);
hp = semilogx(w,squeeze(ph*(pi/180)));
grid on;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/s)');
set(gca, 'FontSize', 14, 'Box', 'off', 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);

set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14         );
set( hTitle                       , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

Chapter 21

Frequency Selective Filters in DT

Recall the response of stable DT LTI systemsi to periodic inputs. Given a stable LTI system with frequency response $H\left(e^{j\omega}\right)$ the input-output relationship is

$$x[n] = \sum_{k=N_0}^{N_0+N-1} a_k e^{jk\omega_0 n} \longrightarrow y[n] = \sum_{k=N_0}^{N_0+N-1} a_k H\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}$$

Note the output is equivalent to a signal with Fourier series coefficients $b_k = a_k H\left(e^{jk\omega_0}\right)$. That is the Fourier coefficients are scaled by the frequency response at the harmonic frequency $k\omega_0$.

Similarly for aperiodic signals, given a stable LTI system with frequency response $H\left(e^{j\omega}\right)$ the input-output relationship is

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \longrightarrow y[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

Note the output is equivalent to a signal with DT Fourier Transform $Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$. That is the Fourier transform at each continuous frequency ω is scaled by the frequency response at that frequency.

As in CT, we can design the frequency response to modify the values of a_k or $X\left(e^{j\omega}\right)$ selectively, passing them unmodified, increasing (amplifying) them, or decreasing (attenuating) them. Such systems are called DT filters (or more colloquially digital filters). As in CT there are 4 basic types:

- Low-pass Filters attenuate high frequencies while passing through lower frequencies. They are often used to reduce the effects of high-frequency noise in a signal.
- High-pass Filters attenuate lower frequencies while passing through higher frequencies. They are used, for example, to select high-frequency audio components in high-end audio systems.
- Bandpass Filters attenuate frequencies outside a band of frequencies. They can be viewed as a combination of a high-pass and low-pass filter. They are commonly used to select a range of frequencies for further processing and are central to many communication technologies.
- Notch or Bandstop Filters attenuate frequencies inside an often narrow band of frequencies. Common applications are the removal of one or more corrupting signals mixed into another signal.

While the design of such filters is outside the scope of this course, you are now equipped to understand and apply them based on your knowledge of the Fourier methods covered over the past several weeks. This is similar to CT filtering, with the important exception that the frequency domain is periodic in 2π for DT systems, so the filter frequency responses are periodic as well, with all the work being done in a range of frequencies from $(0, 2\pi)$ or equivalently from $(-\pi, \pi)$.

Digital filters have a number of advantages over CT filters and are widely used now in place of CT filters in audio, communication, and control applications. Audio (and video) in particular is now almost exclusively processed, stored, and transmitted digitally, converting to CT only at the point of an amplifier and speaker.

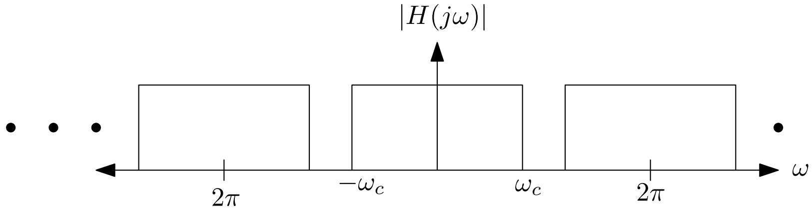
21.1 Ideal Filters

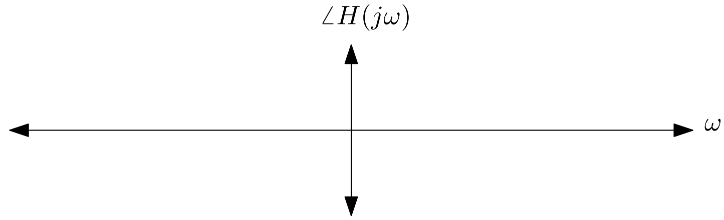
The above filter types each have an ideal form.

Low-pass filters remove frequency content above a threshold, $\omega_c \in [0, \pi]$, called the *cutoff frequency*. They have an ideal frequency response ,for any integer multiple k, given by:

$$H\left(e^{j\omega}\right) = \begin{cases} 1 & 2\pi k - \omega_c < \omega < 2\pi k + \omega_c \\ 0 & \text{else} \end{cases}$$

with magnitude and phase plot



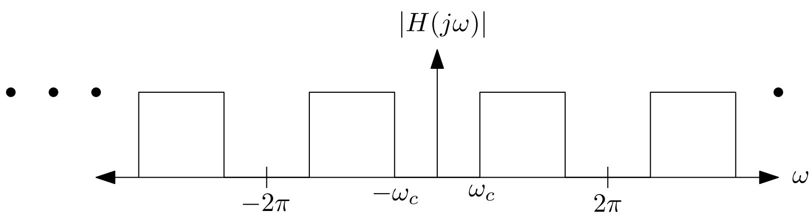


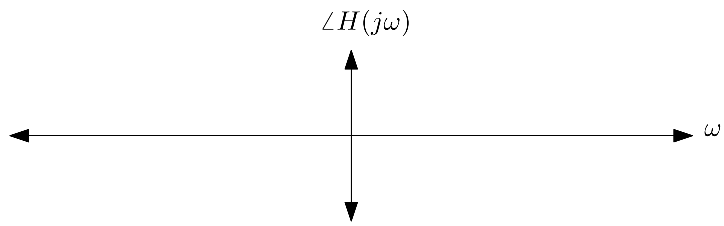
As in CT filters, the range of frequencies $-\omega_c \leq \omega \leq \omega_c$ are called the pass-band. The range of frequencies outside the pass-band are called the stop-band.

High-pass filters remove frequency content below the cutoff frequency $\omega_c \in [0, \pi]$. They have an ideal frequency response

$$H\left(e^{j\omega}\right) = \begin{cases} 0 & 2\pi k - \omega_c < \omega < 2\pi k + \omega_c \\ 1 & \text{else} \end{cases}$$

with magnitude and phase plot

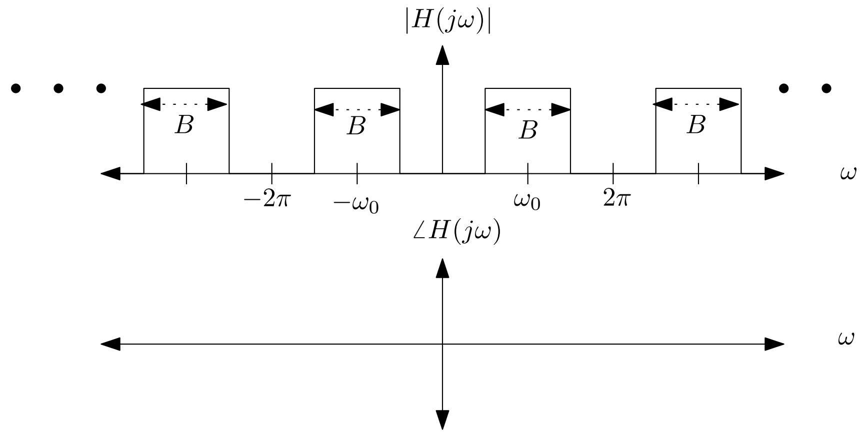




Bandpass filters remove frequency content outside a band of frequencies called the pass-band. They have an ideal frequency response

$$H\left(e^{j\omega}\right) = \begin{cases} 1 & 2\pi k - \omega_0 - \frac{B}{2} < \omega < 2\pi k - \omega_0 + \frac{B}{2} \\ 1 & 2\pi k + \omega_0 - \frac{B}{2} < \omega < 2\pi k + \omega_0 + \frac{B}{2} \\ 0 & \text{else} \end{cases}$$

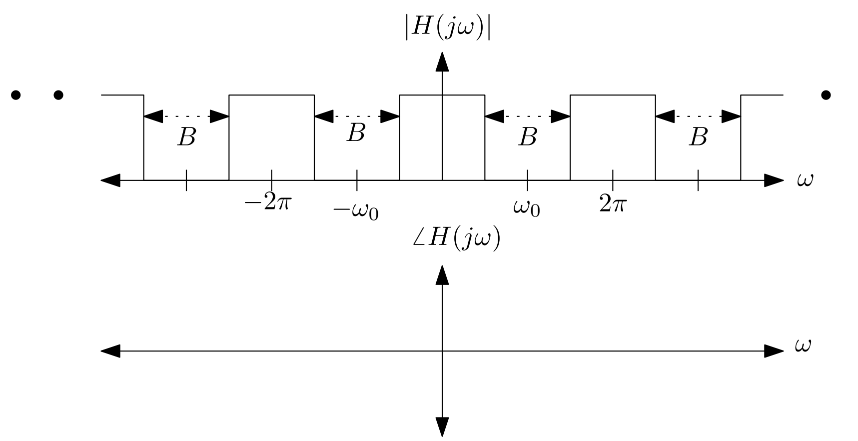
where $\omega_0 \in [0, \pi]$ is the *center frequency* and $B < \pi - \omega_0$ is the *bandwidth.* The magnitude and phase plot looks like



Finally, notch or bandstop filters remove frequency content inside a band of frequencies (the stop band) defined by the center frequency $\omega_0 \in [0, \pi]$ and bandwidth $B < \pi - \omega_0$. The ideal frequency response is

$$H\left(e^{j\omega}\right) = \begin{cases} 0 & -\omega_0 - \frac{B}{2} < \omega < -\omega_0 + \frac{B}{2} \\ 0 & \omega_0 - \frac{B}{2} < \omega < \omega_0 + \frac{B}{2} \\ 1 & \text{else} \end{cases}$$

with magnitude and phase plot



Often the bandstop filter has a very narrow bandwidth, thus it ”notches” out a frequency component of the input signal.

21.2 Practical Filters

While ideal CT filters cannot be implemented in practice because they are non-causal, this more nuanced in DT systems. We have to make a distinction between *off-line* and *real-time* DT filters. Off-line DT filters, which we will discuss next time, can store as many samples as needed to arbitrarily approximate a non-causal filter, leading to an output that is delayed relative to the input by a significant amount. Real-time filters on the other hand must produce an output $y[n]$ for every input $x[n]$ with no delay. The distinction is important in some applications, controls in particular, and less so in other areas like audio or video where a delay is not noticeable. In the remainder of this lecture we assume a real-time filter implementation, which should be causal.

Practical filters are described by a frequency response that is a ratio of two polynomials in $e^{j\omega}$, i.e.

$$H\left(e^{j\omega}\right) = \frac{K \cdot \left(e^{j\omega} + b_1\right) \cdot \left(e^{j\omega} + b_2\right) \cdots \left(je^{j\omega} + b_M\right)}{\left(e^{j\omega} + a_1\right) \cdot \left(e^{j\omega} + a_2\right) \cdots \left(e^{j\omega} + a_N\right)}$$

where K is a constant that controls the gain at DC, and the zero or more complex coefficients b_k and the one or more complex coefficients a_k are called the *zeros* and *poles* of the filter respectively. Such systems correspond to difference equations as we have covered before and are realizable in real arithmetic if all poles and zeros are real or come in conjugate pairs. The processes of designing DT filters consists of choosing the poles and zeros, or equivalently choosing the coefficients of the numerator and dominator polynomials. This is covered in ECE 3704, ECE 4624, and other upper-level courses.

The general DT frequency response corresponds to a difference equation that when written in recursive form looks like

$$y[n] = \underbrace{-\frac{c_{N+1}}{c_1}y[n-N] - \frac{c_N}{c_1}y[n-N+1] + \cdots - \frac{c_2}{c_1}y[n-1]}_{\text{auto-regressive}} + \underbrace{\frac{d_{N+1}}{c_1}x[n-N] + \frac{d_N}{c_1}x[n-N+1] + \cdots + \frac{d_1}{c_1}x[n]}_{\text{moving-average}}$$

The terms corresponding to the weighted sums of previous outputs are called the *auto-regressive* portion of the filter. The terms corresponding to the weighted sums of previous inputs are called the *moving-average* portion of the filter.

Filters without auto-regressive terms ($c_i = 0$ for $i > 1$) are called *finite impulse response* (FIR) filters, because their impulse response has only a finite number of non-zero values. Filters with auto-regressive terms are called *infinite impulse response* (IIR) filters, because their impulse response is non-zero for $n > 0$ (although they do approach zero as $n \to \infty$).

Practical (real-time) DT filters differ from the ideal in that they cannot be zero over any finite range of frequencies and cannot transition discontinuously between stop and pass bands. Similar to CT filters, they must vary over the bands and transition smoothly, with a degree of variation and sharpness that is a function of the order of the filter and the exact form of the frequency response polynomials. Thus practical filters are described by additional parameters that define the stop and pass-bands.

The overall gain of the filter is the magnitude of the frequency response at a frequency that depends on the filter type, zero for a low-pass filter and the center frequency for a band-pass filter. The pass-band is defined by the frequency at which the magnitude of the frequency response drops below the overall gain, often -3dB = $\frac{\sqrt{2}}{2}$. The stop-band is defined similarly, as the frequency at which the magnitude of the frequency response drops further below the overall gain, often -20dB = 0.1 or -40dB = 0.01. The *transition bandwidth* is defined as the difference in the stop-band and pass-band frequencies. The *pass-band ripple* is defined as the maximum deviation from the overall gain, over the pass-band. For DT filters the frequencies are specified in radians per sample. After we discuss sampling we will see how to convert this to an equivalent CT frequency using the sample time.

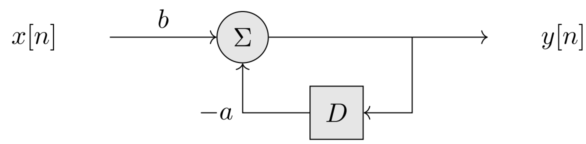
21.3 First-order and second-order systems as filters

Given the equivalence of stable LTI systems and linear, constant-coefficient difference equations, block diagrams, impulse responses, and frequency responses, filters can be represented in any of these ways. We have covered extensively first-order and second-order DT systems and seen how they can be represented variously as state machines, difference equations, block diagrams, and as frequency responses. We now see how they can describe simple filters and serve as building blocks for higher-order filters.

Example 21.3.1. Consider a first-order DT system

$$y[n+1] + ay[n] = bx[n+1]$$

It can be represented as a block diagram



or as a recursive difference equation

$$y[n] = -ay[n-1] + bx[n]$$

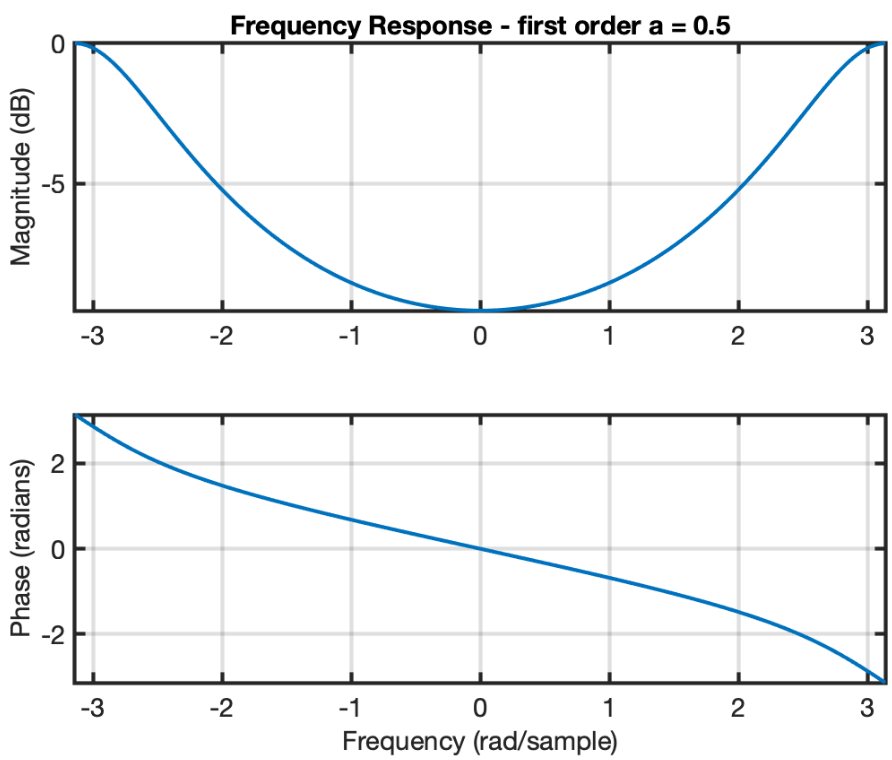
or as an impulse response

$$h[n] = b(-a)^n u[n]$$

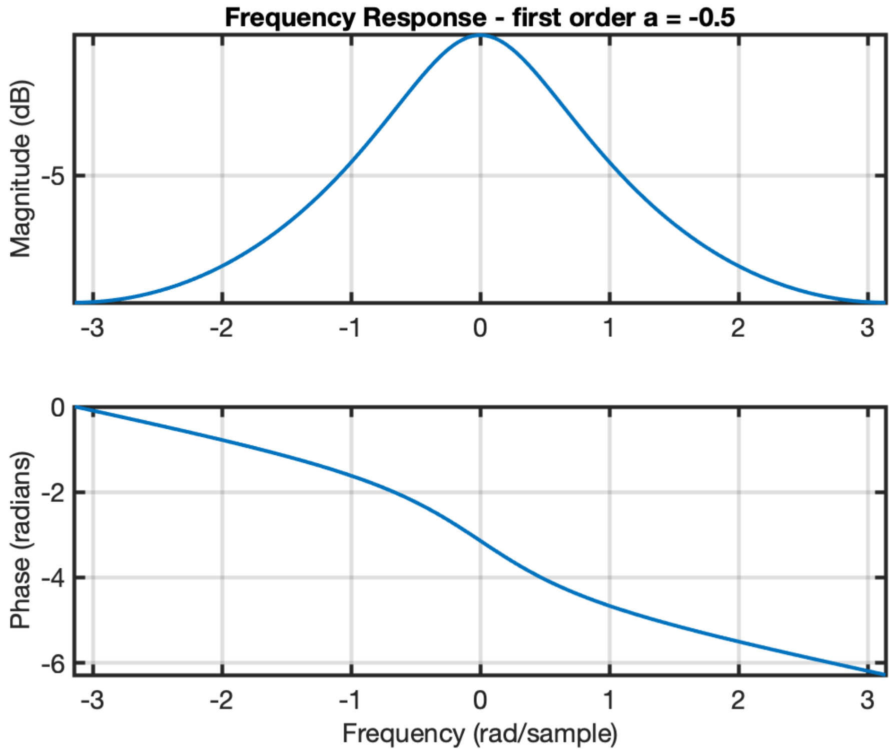
or as a frequency response if $|a| < 1$

$$H\left(e^{j\omega}\right) = \frac{b}{1 + ae^{-j\omega}} = \frac{be^{j\omega}}{e^{j\omega} + a}$$

Let us examine two cases, where $a = \frac{1}{2}, b = 1 - a = \frac{1}{2}$ and $a = -\frac{1}{2}, b = 1 + a = \frac{1}{2}$. If we plot the frequency response when $a = \frac{1}{2}, b = \frac{1}{2}$ we see the DC gain is about -9.5 dB, and the response passes through -3dB at ± 2.4 rad/sample. Thus this corresponds approximately to a high-pass filter.



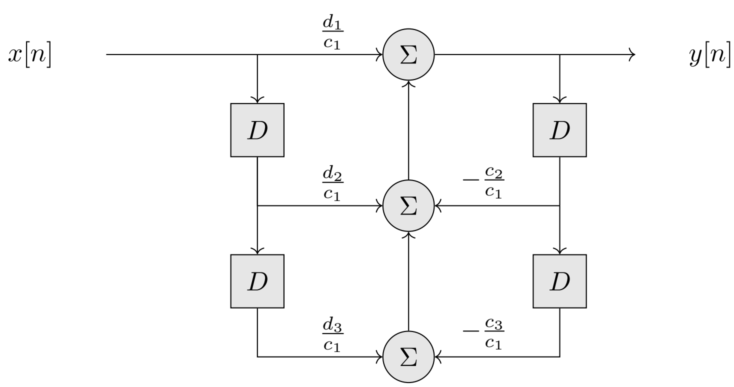
If we plot the frequency response when $a = -\frac{1}{2}, b = \frac{1}{2}$ we see the DC gain is 0 dB, and the response passes through -3dB at ± 0.7 rad/sample. Thus this corresponds approximately to a low-pass filter.



Example 21.3.2. As with CT filters we can increase the sharpness of the filter by increasing the order. Consider a second-order DT system

$$c_1 y[n+2] + c_2 y[n+1] + c_3 y[n] = d_1 x[n+2] + d_2 x[n+1] + d_3 x[n]$$

It can be represented as a block diagram



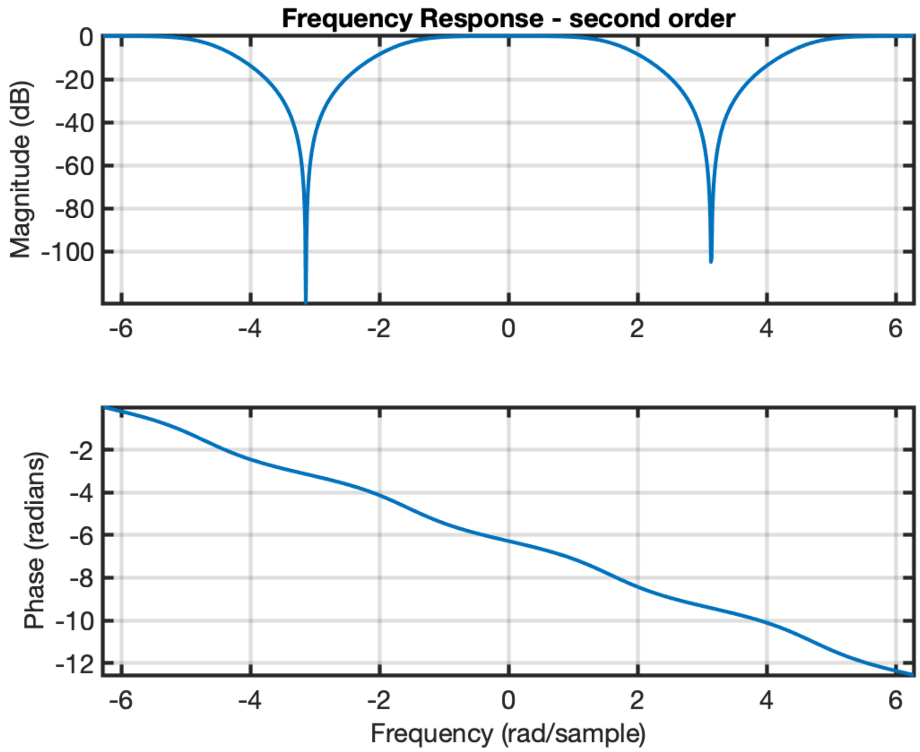
or as a recursive difference equation

$$y[n] = -\frac{c_3}{c_1} y[n-2] - \frac{c_2}{c_1} y[n-1] + \frac{d_3}{c_1} x[n-2] + \frac{d_2}{c_1} x[n-1] + \frac{d_1}{c_1} x[n]$$

or as a frequency response if $|\gamma_i| < 1$ for all i where γ_i are the roots of the characteristic equation $c_1 E^2 + c_2 E + c_3 = 0$.

$$H\left(e^{j\omega}\right) = \frac{d_1 e^{j2\omega} + d_2 e^{j\omega} + d_3}{c_1 e^{j2\omega} + c_2 e^{j\omega} + c_3}$$

As a concrete example, consider this system when c = [$c_1 = 1$, $c_2 = 0$, $c_3 = 0.1716$, $d_1 = 0.2929$, $d_2 = 0.5858$, and $d_3 = 0.2929$. If we plot the frequency response we see the filter gain is 0 dB at DC and passes through -3 dB at approximately ±1.57 rad/sample. Thus it corresponds to a low-pass filter.



One thing to note is the attenuation drops off quickly after the passband. This is an advantage of DT filters; they can have small transition bands. Comparing them directly to an equivalent CT filter will have to be deferred until we discuss sampling.

Note the autoregressive part of the filter is to the right of the summations, while the moving average part is to the left. The output of the delay blocks multiplied by non-zero coefficients are called *filter taps* in signal processing parlance. This repeating structure can be taken advantage of in the creation of general-purpose digital signal processing hardware.

21.4 Higher-Order Filters

We can continue to increase the steepness of the passband to stop-band transitions by increasing the order of the filter. While this can be accomplished using a serial connection of stages as in CT filters, since the implementation of DT filters requires just memory (delay blocks) and adders/multipliers it is common to just implement the filter with a larger number of delay blocks.

Matlab code for plotting the first-order frequency response plot:

```
a = 1/2;
w = -pi:0.01:pi;
H = a./(exp(j*w) + a);

% Create a nice FR plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = plot(w,20*log10(abs(H)));
grid on;
axis tight;
hTitle  = title ('Frequency Response - first order a = 0.5');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, 'YTick', -20:5:5, 'LineWidth', 2);

subplot(2,1,2);
hp = plot(w,unwrap(angle(H)));
grid on;
axis tight;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/sample)');
set(gca, 'FontSize', 14, 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);

set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14          );
set( hTitle                     , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

Matlab code for plotting the second-order frequency response plot:

```
c = [1,0,0.1716];
d = [0.2929, 0.5858, 0.2929];
w = -pi:0.01:pi;
H = (d(1)*exp(j*2*w) + d(2)*exp(j*w) + d(3))./(c(1)*exp(j*2*w) + c(2)*exp(j*w) + c(3));

% Create a nice FR plot
hFig = figure();
hold on;

subplot(2,1,1);
hm = plot(w,20*log10(abs(H)));
grid on;
axis tight;
hTitle  = title ('Frequency Response - second order');
hYLabel1 = ylabel('Magnitude (dB)');
set(gca, 'FontSize', 14, 'YTick', -100:20:0, 'LineWidth', 2);

subplot(2,1,2);
hp = plot(w,unwrap(angle(H)));
grid on;
axis tight;
hYLabel2 = ylabel('Phase (radians)');
hXLabel = xlabel('Frequency (rad/sample)');
set(gca, 'FontSize', 14, 'LineWidth', 2);

set(hm, 'linewidth', 2);
set(hp, 'linewidth', 2);

set([hXLabel, hYLabel1, hYLabel2]  , ...
     'FontSize'   , 14         );
set( hTitle                    , ...
     'FontSize'   , 14          , ...
     'FontWeight' , 'bold'      );
```

Chapter 22

Discrete Fourier Transform

The Discrete Fourier Transform or DFT is the Fourier Transform of a finite length DT signal. As we shall see, the DFT/FFT is mathematically equivalent to the Discrete-Time Fourier Series. It can be viewed as a way to numerically approximate the CT Fourier transform. Lets first just state the transform and then derive it and see how to interpret and apply it.

Given a finite-length sequence of real or complex numbers $x[n]$, indexed from 0 to $N-1$, the *Discrete Fourier Transform* or DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

for $k = 0, 1, 2, \cdots, N-1$. When N is a power of 2, an efficient algorithm to compute this result exists and is called the *Fast Fourier Transform* or FFT.

22.1 Numerically Approximating the CT Fourier Transform

Recall the CT Fourier transform pair $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$:

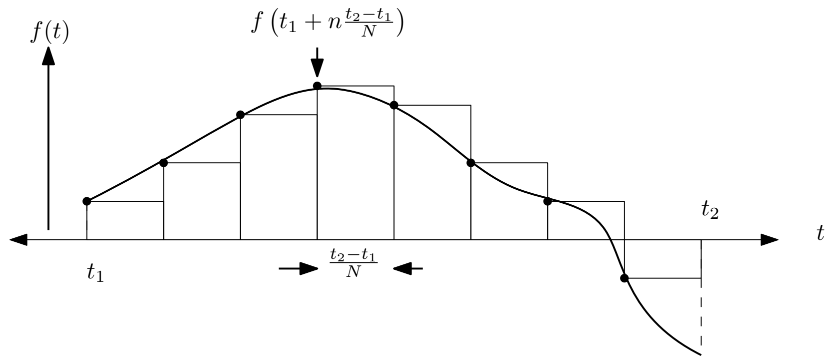
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

How could we compute these when we have a physical signal, rather than just a mathematical model?

Recall from calculus the (left) Riemann sum approximation of a definite integral

$$\int_{t_1}^{t_2} f(t)\, dt = \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{t_2 - t_1}{N} f\left(t_1 + n\frac{t_2 - t_1}{N}\right)$$



In the case of the CTFT, if the signal $x(t)$ is non-zero only over some interval (t_1, t_2), then

$$\mathcal{F}\{x(t)\} = \int_{t_1}^{t_2} x(t)e^{-j\omega t}\, dt \approx \sum_{n=0}^{N-1} \frac{t_2 - t_1}{N} x\left(t_1 + n\frac{t_2 - t_1}{N}\right) e^{-j\omega\left(t_1 + n\frac{t_2-t_1}{N}\right)}$$

for large N.

If we define the time sample spacing as $T = \frac{t_2-t_1}{N}$, then

$$X(j\omega) \approx \sum_{n=0}^{N-1} T\, x\,(t_1 + nT)\, e^{-j\omega(t_1+nT)}$$

Note $x\,(t_1 + nT)$ corresponds to *samples* of $x(t)$ starting at t_1 with sampling interval T. This information is equivalent to the triad $t_1, T, x[n]$, where $x[n]$ is a finite length sequence of numbers, i.e. a DT signal where $x[n] = 0$ for $n < 0$ and $n \geq N$. Thus,

$$x\,(t_1 + nT) = x[n]$$

Substituting into our approximation

$$X(j\omega) \approx \sum_{n=0}^{N-1} T\, x\,(t_1 + nT)\, e^{-j\omega(t_1+nT)} = Te^{-j\omega t_1} \sum_{n=0}^{N-1} x[n]e^{-j\omega nT}$$

Now, consider a sampling of the frequency axis $\omega = \frac{2\pi}{NT}k$. Then

$$\omega nT = \frac{2\pi}{NT}knT = \frac{2\pi}{N}kn$$

and

$$X\left(j\frac{2\pi}{NT}k\right) = Te^{-j\omega t_1} \underbrace{\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}}_{\text{DFT}} = Te^{-j\omega t_1}\, X[k]$$

Thus we see the DFT corresponds to the Fourier transform of a sampled CT signal over a limited time-interval, at samples of the frequency axis.

Similarly, in the case of the Inverse CTFT, if the signal $X(j\omega)$ is non-zero only over some interval (ω_1, ω_2), then

$$\mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} X(j\omega)e^{j\omega t}\,dt$$
$$\approx \frac{1}{2\pi}\sum_{k=0}^{M-1}\frac{\omega_2-\omega_1}{M}X\left(\omega_1 + k\frac{\omega_2-\omega_1}{M}\right)e^{j\left(\omega_1+k\frac{\omega_2-\omega_1}{M}\right)t}$$

for large M. If we define the frequency sample spacing as $W = \frac{\omega_2-\omega_1}{M}$, then

$$x(t) \approx \sum_{m=0}^{M-1}\frac{W}{2\pi}X\left(\omega_1 + kW\right)e^{j(\omega_1+kW)t}$$

Note $X\left(\omega_1 + mW\right)$ corresponds to samples of $X(j\omega)$ starting a ω_1 with sampling interval W. This information is equivalent to the triad ω_1, W, $X[k]$, where $X[k]$ is a finite length sequence of numbers where

$$X\left(\omega_1 + mW\right) = X[k]$$

Substituting

$$x(t) \approx \sum_{m=0}^{M-1}\frac{W}{2\pi}X\left(\omega_1 + kW\right)e^{j(\omega_1+kW)t} = \frac{W}{2\pi}e^{j\omega_1}\sum_{m=0}^{M-1}X[k]e^{jkWt}$$

Consider the sampling of the time axis in the derivation of the DFT, $t = nT$. Let $\omega_1 = 0$ and $\omega_2 = \frac{2\pi}{T}$ and $M = N$. Then

$$kWt = kWnT = k\frac{2\pi}{NT}nT = \frac{2\pi}{N}kn$$

Since $W = \frac{2\pi}{NT}$

$$x(nT) = \frac{1}{T}\underbrace{\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}}_{\text{Inverse DFT}} = \frac{1}{T}x[n]$$

Thus we see the IDFT corresponds to the Inverse Fourier transform of a sampled Fourier Transform over a limited bandwidth, at samples of the time axis.

This gives us the DFT pair

$$X[k] = \text{DFT}\{x[n]\} = \sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}$$

$$x[n] = \text{IDFT}\{X[k]\} = \frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}$$

Note the similarity to the DT Fourier Series when $N_0 = 0$

$$x[n] = \sum_{k=N_0}^{N_0+N-1}a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1}a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N}\sum_{n=N_0}^{N_0+N-1}x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}$$

22.2 Efficient Computation of DFT (FFT)

Given the DFT pair and an input signal, it is easy to compute the DFT. For example in C++ we can define a signal as an array of complex numbers

```
#include <complex>
#include <vector>

typedef Signal std::vector<std::complex<double>>;
```

and implement the DFT in a straightforward translation of the expressions above:

```
Signal dft(const Signal & in){

  Signal out = in;

  const int N = in.size();

  for(int k = 0; k < N; ++k){
    out[k] = 0;
    for(int n = 0; n < N; ++n){
      out[k] += in[n]*exp(-j*2.*PI*double(k)*double(n)/double(N));
    }
  }

  return out;
}
```

Because of the nested for loops the number of multiplies and adds required to compute the DFT is proportional to the number of samples in the signal, squared. However, by expanding the complex exponential we see

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} x[n] \left\{ \cos\left(-\frac{2\pi}{N}kn\right) + j\sin\left(-\frac{2\pi}{N}kn\right) \right\} \\
&= \sum_{n=0}^{N-1} x[n] \left\{ \cos\left(\frac{2\pi}{N}kn\right) - j\sin\left(\frac{2\pi}{N}kn\right) \right\}
\end{aligned}
$$

Which can be compactly written as

$$X = \mathcal{W}x$$

where $x \in \mathbb{C}^N$ is the sampled signal treated as a complex-valued vector, and $\mathcal{W} \in \mathbb{C}^{N\times N}$ is a complex-valued matrix with entries

$$\mathcal{W}_{k\,n} = \cos\left(\frac{2\pi}{N}kn\right) - j\sin\left(\frac{2\pi}{N}kn\right) = \left(e^{-j\frac{2\pi}{N}}\right)^{kn}$$

Similarly for the inverse DFT

$$
\begin{aligned}
x[n] &= \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \\
&= \sum_{k=0}^{N-1} X[k] \left\{ \cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}kn\right) \right\}
\end{aligned}
$$

Which can be compactly written as

$$x = \frac{1}{N}\mathcal{W}^* X$$

This implies $\frac{1}{N}\mathcal{W}\mathcal{W}^* = I$ and that $\mathcal{W}$ is orthogonal.

The Fast Fourier Transform algorithm computes the DFT/IDFT in $O(N \log_2 N)$ multiply/adds. The most common algorithm for implemeting the FFT is called the Cooley–Tukey radix-2 algorithm. This algorithm can be implemented using C++ as:

```
Signal fft(const Signal & in){

  Signal out;

  std::size_t n = in.size();
  double logn = log2(n);

  for (unsigned int i = 0; i < n; ++i) {
    int rev = bitReverse(i, logn);
    out[i] = in[rev];
  }

  // make sure logn is positive integer > 1
  std::size_t temp = static_cast<std::size_t>(logn);
  assert(static_cast<double>(temp) == logn);

  for(std::size_t s = 1; s <= logn; ++s){
    std::complex<double> w(1,0);

    int m = 1 << s; // 2 power s
    int m2 = m >> 1; // m2 = m/2 -1

    std::complex<double> wm = exp(-PI*j/static_cast<double>(m2));

    for(std::size_t j = 0; j < m2; ++j){
      for(std::size_t k = j; k < n; k+=m){
        std::complex<double> t = w*out[k+m2];
        std::complex<double> u = out[k];
        out[k] = u + t;
        out[k+m2] = u - t;
      }
      w = w*wm;
    }
  }

  return out;
}
```

where the function `bitReverse` reverses the bitwise representation of the index argument

```
unsigned int bitReverse(unsigned int x, int log2n){
  int n = 0;
  for (int i = 0; i < log2n; i++){
    n <<= 1;
    n |= (x & 1);
```

```
    x >>= 1;
  }
 return n;
}
```

22.3 DFT/FFT in Matlab

In Matlab (and other languages/platforms) you can use the Fast Fourier Transform to compute the DFT. For example:

```
>> T = 0.001;
>> t = 0:T:100;
>> x = cos(2*pi*t);
>> X = fft(x);
>> plot(abs(X))
```

To give a plot consistent with the CTFT of $\cos(2\pi t)u(t-100)$

```
>> w = (-pi/0.001):(2*pi/100):(pi/0.001);
>> stem(w, T*fftshift(abs(X)))
```

or

```
>> stem(w, T*fftshift(angle(X)))
```

22.4 Summary of Fourier Transforms

- Discrete-time Fourier Series: periodic DT signal $x[n] \mapsto a_k$ periodic discrete frequencies
- Discrete-time Fourier Transform: aperiodic DT signal of indefinite length $x[n] \mapsto X(e^{j\omega})$ periodic continuous frequencies
- Continuous-time Fourier Series: periodic CT signal $x(t) \mapsto a_k$ discrete frequencies of indefinite length
- Continuous-time Fourier Transform: aperiodic CT signal of indefinite length $x(t) \mapsto X(j\omega)$ continuous frequencies

And now we have the DFT

- Discrete Fourier Transform: aperiodic DT signal of finite length $x[n] \mapsto X[k]$ periodic discrete frequencies

22.5 Applications of the DFT

Our discussion of the DFT raises some important questions:

- For what values of sampling interval T does this hold?
- What are the effects of time and frequency sampling on $x(t)$ and $X(j\omega)$?
- What if $x(t)$ or $X(j\omega)$ is non-zero outside the interval?

These will be answered in the last two lectures. It also admits some important applications:

- Numerical computation of Fourier transform of physical signals

- Simulation or approximation of stable CT systems
- Implementation of CT systems using DT systems

As an example application, suppose you have a physical signal, say an audio signal from a microphone. How would you estimate its Fourier Transform? Sample $x(t)$ at a frequency of $\frac{2\pi}{T}$ for NT seconds.

$$x[n] = x(nT)$$

$$X[k] = \text{DFT}\left\{x[n]\right\}$$

$$X\left(j\frac{2\pi}{NT}k\right) = TX[k]$$

Note, in practice this requires multiplication by a windowing function to get good results unless there is silence on either side of the audio.

Example 22.5.1. Consider a CT signal $x(t) = \cos(2\pi t)\left[u(t) - u(t-100)\right]$ sampled at a frequency of $\frac{2\pi}{0.001}$ for $NT = 100$ seconds to obtain $x[n]$. Given the DFT of $x[n]$, $X[k]$, what values of k correspond to $\omega = \pm 2\pi$?

$$\omega = 2\pi = \frac{2\pi}{NT}k \implies k = 100$$

$$\omega = -2\pi = \frac{2\pi}{NT}k \implies k = -100$$

However $k \in (0, N-1)$ where $N = 100000$. Thus $k = -100 = N - 100 = 99900$. Note, the Matlab command `fftshift` does this unwrapping for you.

As another application, suppose you have a CT frequency response, $H(j\omega)$, for example a CT filter. How could you simulate the response to a physical signal, such as an audio signal from a microphone? Again, sample $x(t)$ at a frequency of $\frac{2\pi}{T}$ for NT seconds.

$$x[n] = x(nT)$$

$$X[k] = \text{DFT}\left\{x[n]\right\}$$

Using the convolution property

$$Y[k] = H\left(j\frac{2\pi}{NT}k\right)X[k]$$

$$y(nT) = \frac{1}{T}\text{IDFT}\left\{Y[k]\right\}$$

As a final application example we consider the case of filtering. DT implementations of CT systems have a number of benefits over CT implementations. The previous application hints at a method to implement a CT system using a DFT. We sample $x(t)$ at a frequency of $\frac{2\pi}{T}$ for NT seconds into a buffer, called a *frame*.

$$x[n] = x(nT)$$

$$X[k] = \text{DFT}\left\{x[n]\right\}$$

$$Y[k] = H\left(e^{j\frac{2\pi}{NT}k}\right)X[k]$$

$$y(t) \approx y(nT) = \frac{1}{T}\text{IDFT}\left\{Y[k]\right\}$$

This last step is called *reconstruction*. Note this can be done in real time using three frames, one being sampled, one being processing, and one being reconstructed.

Note the DT filter in the previous application adds a two frame delay. This delay can be removed using an FIR or IIR filter implementation, as we saw in lecture 25. We can sample $x(t)$ at a frequency of $\frac{2\pi}{T}$ continuously into a *ring buffer*.

$$x[n] = x(nT)$$

Compute $y[n]$ using a delay difference equation, e.g. for second order

$$y[n] = a_1 y[n-2] + a_2 y[n-1] + a_3 x[n-2] + a_4 x[n-1] + a_5 x[n]$$

Reconstruct the current output

$$y(t) \approx y(nT) = y[n]$$

Chapter 23

Sampling CT Signals

Up until now in the course we have focused on either CT or DT signals and systems. Practical systems though often are hybrid and require conversion between DT and CT signals. For example a CT audio signal might be converted to a DT audio signal for storage and/or transmission, and at a later time or location converted back to a CT signal for playback through a speaker.

It is also common to design a CT system and then implement it as a DT system. Advantages of this approach are e.g. such implementations are less susceptible to component variations, require no tuning a build time, are easier to change (firmware or software update), easier to prototype, and more easily use encryption.

In this lecture we focus on *sampling* of CT signals to produce a DT signal $x[n] = x(nT)$ with sample index n and sample time T. In the next lecture we consider the case of converting from a DT to CT signal.

23.1 Sampling Theory

The process of sampling is to produce a DT signal $x[n]$ from a CT signal $x(t)$ by sampling time at regular intervals $T \in \mathbb{R}^+$ called the *sample-time*, or equivalently sampling at a frequency of $\frac{1}{T}$ Hz or $\frac{2\pi}{T}$ rad/s. Mathematically this is simple to express in the time domain as $x[n] = x(nT)$, however we seek a system that can perform this task.

Recall the impulse train is the periodic signal

$$x_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

with period T_0 and frequency $\omega_0 = \frac{2\pi}{T_0}$. The exponential CT Fourier series of the impulse train is given by

$$x_1(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\frac{2\pi}{T_0}nt}$$

where the Fourier series coefficients are

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}$$

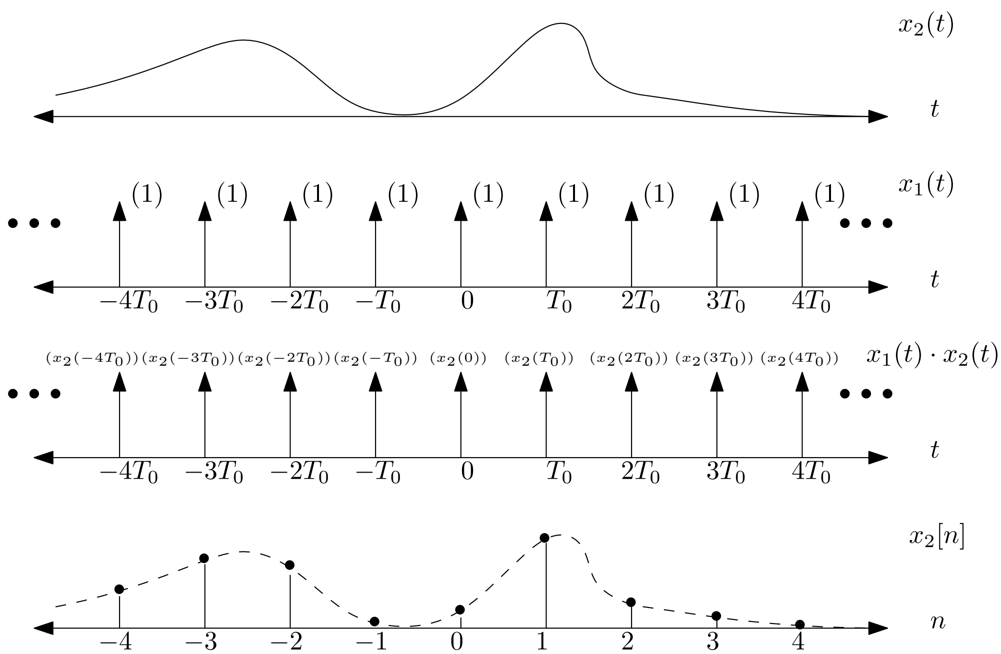
Now, lets take the Fourier Transform of the Fourier series representation

$$X_1(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{j\frac{2\pi}{T_0}nt} e^{j\omega t}\, dt$$
$$= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\frac{2\pi}{T_0}nt} e^{j\omega t}\, dt$$
$$= \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 n)$$

also an impulse train in the frequency domain. Now suppose we have another signal $x_2(t)$ and we multiply $x_1(t)$ and $x_2(t)$ to get a signal $y(t)$.

$$y(t) = x_1(t) \cdot x_2(t) = \sum_{n=-\infty}^{\infty} x_2(t)\delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x_2(nT_0)\delta(t - nT_0)$$

Since $y(t)$ is non-zero only at the locations of the delta functions, we can treat $y(nT_0) = x_2(nT_0)$ as the DT signal $x_2[n]$. This is illustrated below



Equivalently in the frequency domain the modulation theorem gives

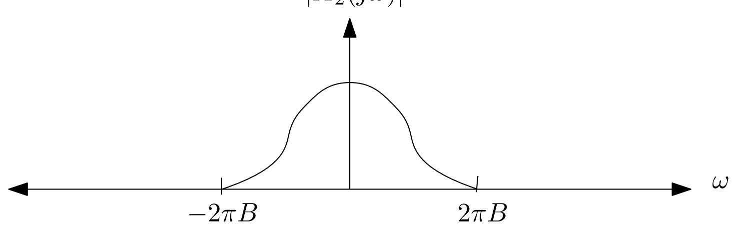
$$y(t) = x_1(t) \cdot x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega) = Y(j\omega)$$

Lets do the convolution

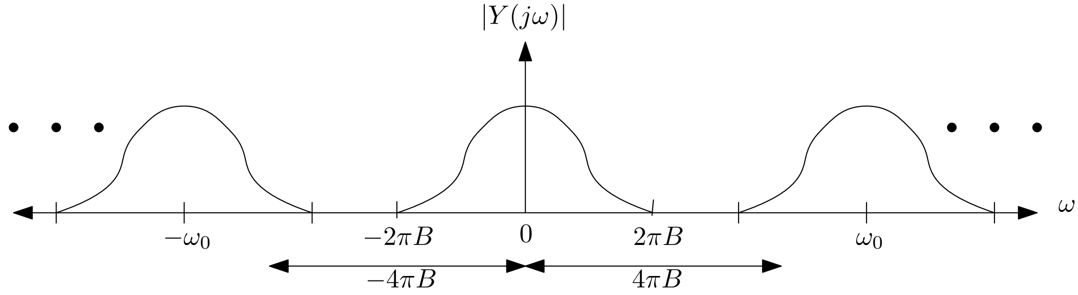
$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega) \\
&= \frac{1}{2\pi}\left[\frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty}\delta(\omega-\omega_0 n)\right] * X_2(j\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty}\delta(\omega-\omega'-\omega_0 n)X_2(j\omega')\,d\omega' \\
&= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X_2(j(\omega-n\omega_0))
\end{aligned}
$$

Thus the sampling process in the frequency domain causes periodic replication of the Fourier transform of the signal being sampled, $x_2(t)$, which are sometimes called *images*. This signal $Y(j\omega)$ is periodic in $\omega_0 = \frac{2\pi}{T_0}$ *radians per second* and corresponds to the DT Fourier Transform of $x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_2\left(e^{j\omega}\right)$, which is periodic in 2π *radians per sample time*.

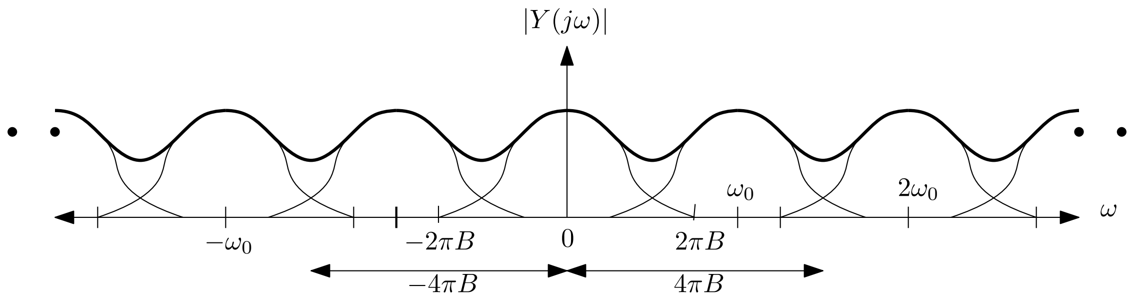
To help us visualize this, suppose that the signal $x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(j\omega)$ is *band-limited* to B Hz, that is $X_2(j\omega) = 0$ for all frequencies outside the band $-2\pi B < \omega < 2\pi B$. This is shown schematically as the magnitude spectrum below:



After sampling ($y(t) = x_1(t) * x_2(t)$) and assuming $\omega_0 > 4\pi B$ the spectrum of the sampled signal is:



If instead $\omega_0 < 4\pi B$ the images overlap and we get *aliasing*, where high frequency content gets added to the lower frequency content. This is shown below with the lighter lines showing the images and the heavier line showing their sum.



As we will see next time, to reconstruct the signal $x_2[n]$ back to $x_2(t)$ we need to ensure that $\omega_0 > 4\pi B$ rad/s or equivalently $f_0 > 2B$ Hz, which requires the sample time $T_0 < \frac{1}{2B}$ seconds. This is called the *Nyquist* sample rate/frequency.

Example 23.1.1. Consider a signal representing a musical chord (an additive mixture of three notes)

$$x(t) = \sin(2\pi \cdot (261)t) + \sin(2\pi \cdot (329)t) + \sin(2\pi \cdot (392)t)$$

Suppose it is sampled at a frequency of $f_0 = 1$ kHz. Then there is no aliasing into the frequency range $(0, 500)$ Hz. After reconstruction $x(t)$ would be unmodified. Suppose instead it is sampled at $f_0 = 500$ Hz. Then the signal component at 261 Hz aliases to $239 = 500 - 261$ Hz, the signal component at 329 Hz aliases to $171 = 500 - 329$ Hz, and the signal component at 392 Hz aliases to $108 = 500 - 392$ Hz. When reconstructed, the signal now has an additional 3 tones mixed in at audible frequencies, but do not correspond to (Western) musical notes, i.e.

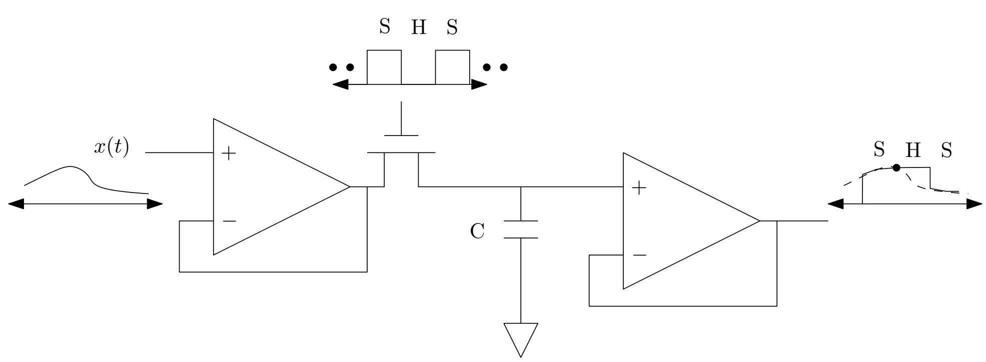
$$x(t) = \sin(2\pi \cdot (108)t) + \sin(2\pi \cdot (171)t) + \sin(2\pi \cdot (239)t) + \sin(2\pi \cdot (261)t) + \sin(2\pi \cdot (329)t) + \sin(2\pi \cdot (392)t)$$

23.2 Practical Sampling

Sampling in practice requires addressing three issues. First, we cannot generate the impulse train, but can only approximate it. Second, digital signals must have a fixed bit width so we have to convert the real signal value to a *quantized* one. Lastly, since in general we have no control over the input signal means we need to ensure the signal is approximately band-limited before sampling.

23.2.1 Sample and Hold

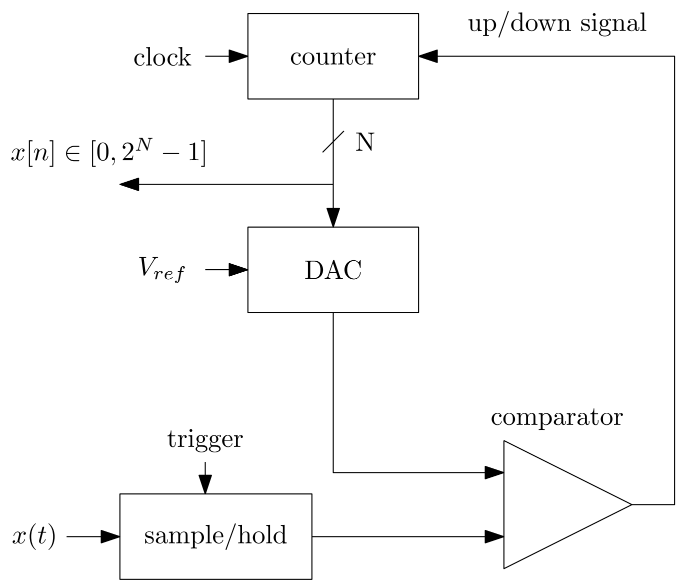
Sampling is typically accomplished using a circuit called a *sample-and-hold*, schematically illustrated below.



The CT signal is applied to the input of the first op-amp buffer. The output of this first buffer is switched into a charging capacitor for the *sample time*, then disconnected (high impedance) at regular intervals for the *hold time*, typically using a MOSFET switch. The effect is the capacitor is charged to the current value of $x(t)$ during the sample-time, which it maintains during the hold-time, the value of which is bufered by the second op-amp. This can be mathematically modeled as a pulse train with a width equal to the sample time rather than as an impulse train.

23.2.2 Quantization

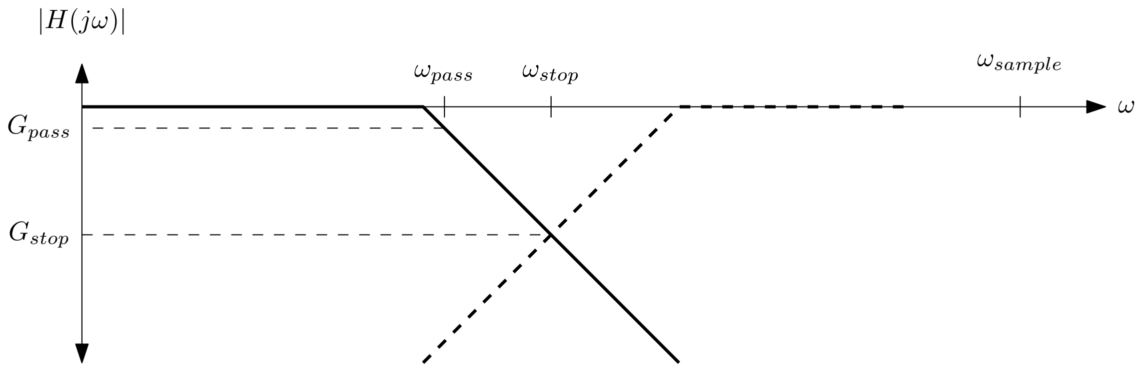
To quantize the signal after the sample-and-hold into N bits, several strategies can be used. One popular approach is called *successive approximation*, illustrated below



The current quantized digital value is held in a counter connected to a clock signal. The direction of the counter (up or down) is controlled by a comparator connected to the output of the sample and hold and the current counter output and a digital-to-analog converter (DAC, usually a resistor ladder) that converts it back to an analog value. If the DAC value is less than the held value, the counter counts up, if the DAC value is greater than the held value the counter counts down. In this fashion the counter output tracks the held value after a settling time required for convergence, at which point the counter value is clocked into a register for storage.

23.2.3 Anti-aliasing

Before the sample and hold we need to include a filter to limit the bandwidth. This can be accomplished by a CT low-pass filter called an *anti-aliasing* filter whose cutoff frequency in the ideal case is $\omega_c = 2\pi B$. As we saw in lecture 24 ideal filters cannot be implemented, thus we specify the anti-aliasing filter as a pass-band gain/frequency and a stop-band gain/frequency. Since the transition band is non-zero for a practical filter, this means we have to either lower the pass-band relative to the ideal or increase the sample rate. In the best case, the filter should have a stop-band frequency at half the sampling frequency with the order of the filter and pass-band frequency adjusted as needed. Alternatively the gain that defines the stop-band can be relaxed. This gives a desired frequency response magnitude that looks like the following.



The bold dotted line shows the maximum frequency response of the first image.

Chapter 24

Reconstructing CT Signals

In the previous lecture we focused on sampling of CT signals to produce a DT signal $x[n] = x(nT)$ with sample index n and sample time T. In this lecture we consider *reconstruction*, converting from a DT signal $x[n]$ to a CT signal $x(t)$ using a sample time T as the spacing between samples. Ideally a conversion from $x(t)$ to $x[n]$ and back again would result in identical signal.

24.1 Reconstruction Theory

Given a DT signal $x[n]$ and a sample spacing T, we can define a corresponding CT signal as

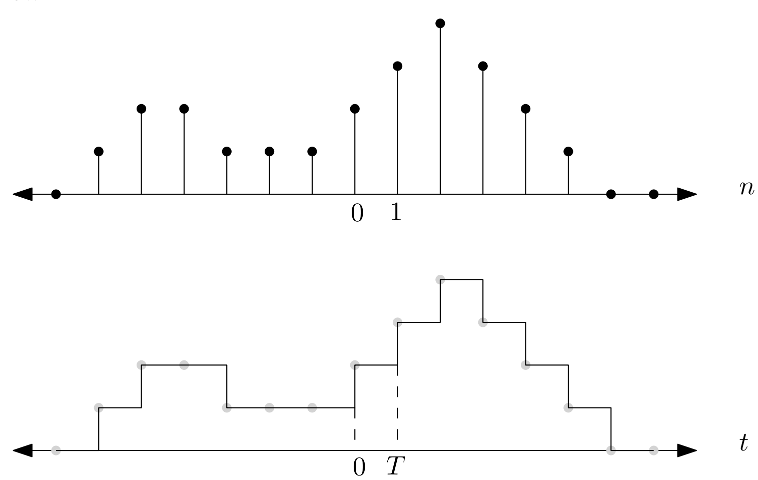
$$x_p(t) = \sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT)$$

the impulse train with each impulse weighted by the DT signal.

CT signal reconstruction can be viewed from two different (but equivalent) perspectives. In the time domain perspective, the CT signal $x(t)$ corresponding to a DT signal $x[n]$ can be viewed as *interpolation*, where the values of the CT signal are equal to the DT signal at intervals of the sample time, i.e. $x(nT) = x[n]$, and in between the value of $x(t)$ is interpolated. If the interpolation is of zero-order, the value at $x(nT)$ is held constant until $x(nT + T)$. This is called a *zero-order hold*, and can mathematically modeled as convolution of the weighted impulse train with a pulse $p(t) = u(t) - u(t - T)$ whose width is the sample time, called the interpolation function.

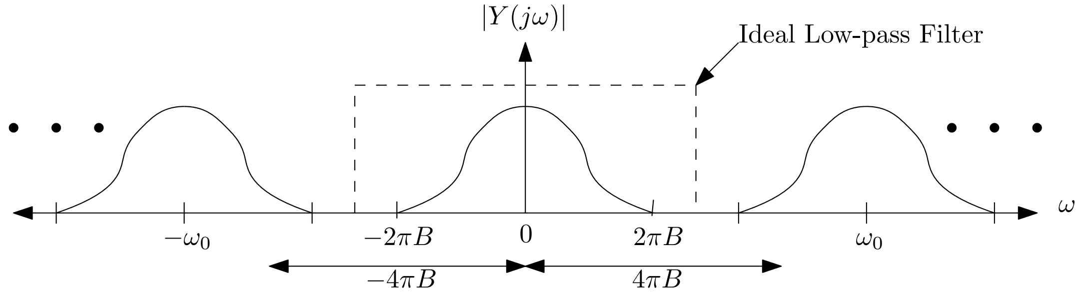
$$y(t) = p(t) * x_p(t)$$

This is illustrated below



The zero-order hold is not a very accurate representation of a band-limited signal. So, what interpolation function is optimal?

To answer this question we can turn to the alternative perspective on reconstruction, that of the frequency domain. Recall the sampled signal $x(nT)$ in the frequency domain can be viewed as the summation of the Fourier transform of $x(t)$, $X(j\omega)$, and periodic replicas or images centered at multiples of the sampling frequency. If we assume the original signal was band-limited and sampled appropriately (using the Nyquist criteria), then if we ideal low-pass filter the sampled signal we will preserve the central portion of the Fourier spectrum that corresponds to the original signal, and chop off the images. For this reason the reconstruction filter is also called an anti-imaging filter.



Recall filtering is multiplication the frequency domain and convolution in the time domain, so the optimal interpolation function corresponds to the impulse response of the ideal low-pass filter with cutoff frequency $\omega_c = 2\pi B$, a sinc function.

$$h(t) = \mathcal{F}^{-1}\left\{H(j\omega)\right\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega = \frac{1}{\pi t}\sin(\omega_c t)$$

Thus the ideal ideal interpolation function is the sinc function, and reconstruction is low-pass filtering of the weighted impulse train $x_p(t)$ [1].

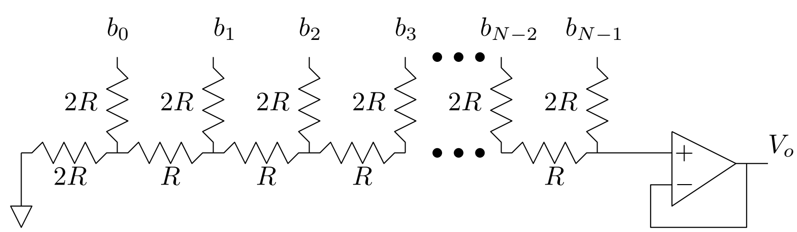
24.2 Practical Reconstruction

As we have seen before we cannot physically represent the impulse train nor the ideal low-pass filter. Thus practical reconstruction uses an approximation of the ideal reconstruction filter by a digital-to-analog converter (DAC), followed by a causal (and thus physically possible) low-pass filter.

24.2.1 Zero-order hold using an R-2R ladder

A zero-order hold DAC can be implemented by a circuit called a resistor ladder. Consider a digital output with N bits and a reference voltage V_{ref} (for example an 8-bit output port on a micro-controller using CMOS 3.3v logic).

If this port is connected to a resistor network consisting of resistor values R and $2R$ as follows



[1] As an aside this also gives an intuitive view of convolution with an impulse train, as interpolation

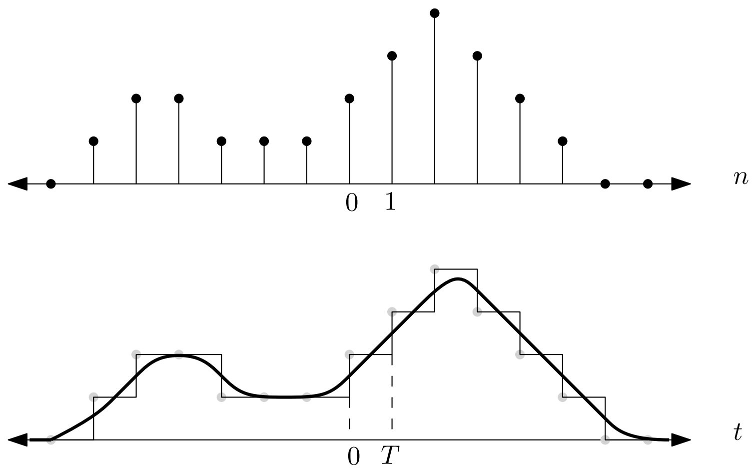
then depending on the bit pattern at the output port V, the output of the buffer op-amp will be

$$V_o = V_{ref} \frac{V}{2^N}$$

If the port value is changed every sample time T, then the resister ladder and buffer op-amp combine to implement a zero-order hold circuit.

24.2.2 Reconstruction(anti-imaging) filter

The zero-order hold is followed by the reconstruction (anti-imaging) filter which low-pass filters the output and smooths-out the jumps from value to value.



In general the reconstruction filter is of a similar, or identical form to the anti-aliasing filter.

Appendix A

Prerequisite and Otherwise Useful Math

This course uses many concepts from prerequisite courses, particularly those from calculus and circuits. While we assume you know this material, the following sections offer a review of the most pertinent and establish some notation. If you have trouble with any of them seek assistance – the sooner the better.

A.1 Complex Numbers

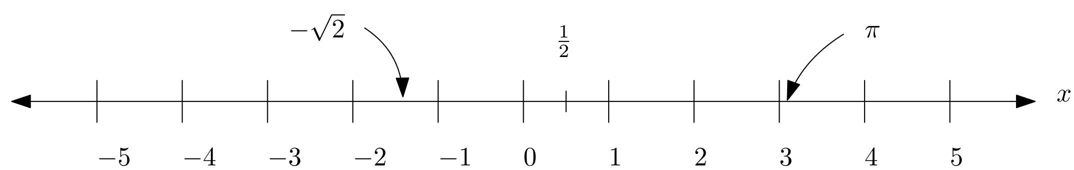
Complex numbers are used extensively throughout the course. You need to be very adept at manipulating them.

The Number System

By way of review and to motivate the discussion of complex numbers, recall the following basic facts.

- The *Natural Numbers* $\mathbb{N}$ are the positive integers $1, 2, 3, 4, \cdots$. Given two natural numbers a and b the sum $a+b$ and the product $a\,b$ are also natural numbers, that is the set of natural numbers is *closed* under addition and multiplication.
- Solving equations of the form $x + a = b$ for any natural numbers a, b requires the introduction of the negative integers $\cdots, -4, -3, -2 - 1$ and 0. These plus the natural numbers give the *integers* $\mathbb{Z}$. Note $\mathbb{N} \subset \mathbb{Z}$. Zero (0) is called the identity element with respect to addition, while 1 is the identity with respect to multiplication, that is $a + 0 = a$ and $a \cdot 1 = a$. The *inverse* of an integer a is $-a$, such that their sum gives the identity for addition, i.e. $a + -a = 0$.
- The *rational numbers* $\mathbb{Q}$ are of the form $\frac{b}{a}$ for integers a, b with $a \neq 0$. They solve problems of the form $ax = b$ and provide the inverse for multiplication since $\frac{1}{a} \cdot a = 1$. Note $\mathbb{Z} \subset \mathbb{Q}$
- The *irrational numbers* are those that cannot be written as a rational number, for example $\sqrt{2} = 1.414\ldots$ and $\pi = 3.14159\ldots$
- The union of the rational and irrational numbers give the *real numbers* denoted $\mathbb{R}$.

Graphically, the numbers and their ordering can be expressed using the number line:



Complex numbers as extension of reals

Continuing the pattern of the basic number system we can ask what are solutions of equations of the form $x^2 + a = 0$ or $x^2 + 2ax + a^2 + b^2 = 0$ for $a, b \in \mathbb{R}$? As above, finding such solutions requires moving to a larger set of numbers, the *complex numbers* denoted by $\mathbb{C}$.

A complex variable $z \in \mathbb{C}$ can be written as $z = a + j\,b$ for $a, b \in \mathbb{R}$, where j is the imaginary unit and $j^2 = -1$. Note in mathematics the imaginary unit is denoted i; this difference is purely historical. Some basic definitions:

- the *real part* $\mathrm{Re}(z) = a$
- the *imaginary part* $\mathrm{Im}(z) = b$
- two complex numbers $z_1, z_2 \in \mathbb{C}$ are equal if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$
- $\mathbb{R} \subset \mathbb{C}$, when $b = 0$ and we say that the number is purely real
- if $a = 0$ we say the number is purely imaginary
- the *complex conjugate* of $z = a + jb$ is $z^* = a - jb$.

Operations on complex numbers

Arithmetic operations on complex numbers are defined using the algebra of real numbers, replacing $j^2 = -1$. Given complex numbers $a + j\,b$ and $c + j\,d$

addition $(a + jb) + (c + jd) = (a + c) + j(b + d)$

subtraction $(a + jb) - (c + jd) = (a - c) + j(b - d)$

multiplication $(a + jb) \cdot (c + jd) = ac + jbc + jad + j^2bd = (ac - bd) + j(bc + ad)$

division $\frac{(a+jb)}{(c+jd)} = \frac{ac+jbc-jad-j^2bd}{c^2-j^2d^2} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}$

Basic properties of complex numbers

Let $z_1, z_2, z_3 \in \mathbb{C}$, then:

closure property $z_1 + z_2 \in \mathbb{C}$ and $z_1 \cdot z_2 \in \mathbb{C}$

commutative property $z_1 + z_2 = z_2 + z_1$ and $z_1 \cdot z_2 = z_2 \cdot z_1$

associative property $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ and $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

identity elements $0 = (0 + j0) \in \mathbb{C}$ is the identity element for addition since $z_1 + 0 = z_1$ and $1 = 1 + j0 \in \mathbb{C}$ is the identity element for multiplication since $z_1 \cdot 1 = z_1$

inverse elements for any z_1 there exists an inverse $z_2 = -z_1$ such that $z_1 + z_2 = 0$, and for any $z_1 \neq 0$ there exists an inverse $z_2 = z_1^{-1} = \frac{1}{z_1}$ such that $z_1 \cdot z_2 = 1$

Absolute Value (Magnitude) of complex numbers

The absolute value or *magnitude* of a complex number $z = a + jb$ is denoted $|z| = |a + jb|$ and is given by

$$|a + jb| = \sqrt{a^2 + b^2}$$

For complex numbers $z_1, z_2, \ldots, z_n$, the following useful properties hold

- $|z_1 \cdot z_2 \cdots z_N| = |z_1| \cdot |z_2| \cdots |z_N|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ for $z_2 \neq 0$

Argument (Phase Angle) of complex numbers

The argument or *angle* of a complex number $z = a + jb$ is denoted $\angle z = \angle(a + jb)$ and is given by

$$\angle(a + jb) = \arctan \frac{b}{a}$$

Take care when computing this number on your calculator (or in a programming language) so that it produces an angle in radians and in the correct quadrant. For example $\angle(-1-j1) = \arctan \frac{-1}{-1} = \frac{5\pi}{4} = -\frac{3\pi}{4}$ is different from $\angle(-1 - j1) = \arctan \frac{-1}{-1} = \arctan \frac{1}{1} = \frac{\pi}{4}$, the later being incorrect.

For complex numbers $z_1, z_2, \ldots, z_n$, the following useful properties hold

- $\angle\left(z_1 \cdot z_2 \cdots z_N\right) = \angle z_1 + \angle z_2 + \cdots + \angle z_N$
- $\angle z_i^{-1} = -\angle z_i$

Cartesian and Polar representation of complex numbers

A complex number z can be represented in Cartesian form as a pair of numbers in the *Complex Plane*, $(\operatorname{Re} z, \operatorname{Im} z)$. The same z can be represented in polar form as $z = |z| \cdot e^{j\angle z}$. We can convert between the representations using $\operatorname{Re} z = |z| \cos(\angle z)$ and $\operatorname{Im} z = |z| \sin(\angle z)$. The following relations hold

- Multiplication by j is equivalent to rotation by $\frac{\pi}{2}$

$$j \cdot z = e^{j\frac{\pi}{2}} \cdot |z| \cdot e^{j\angle z} = |z| \cdot e^{j(\angle z + \frac{\pi}{2})}$$

- Division by j is equivalent to rotation by $-\frac{\pi}{2}$

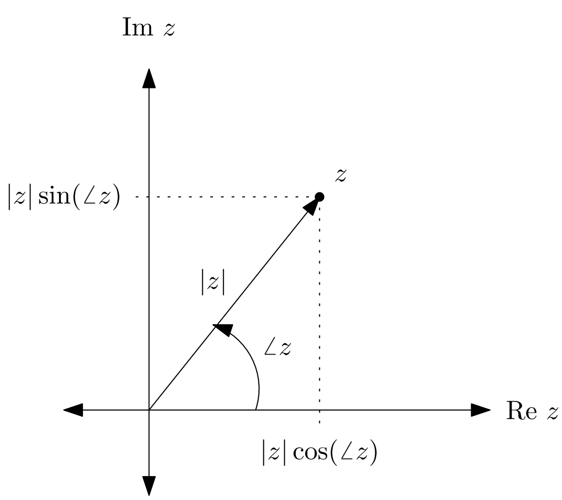
$$\frac{1}{j} \cdot z = e^{-j\frac{\pi}{2}} \cdot |z| \cdot e^{j\angle z} = |z| \cdot e^{j(\angle z - \frac{\pi}{2})}$$

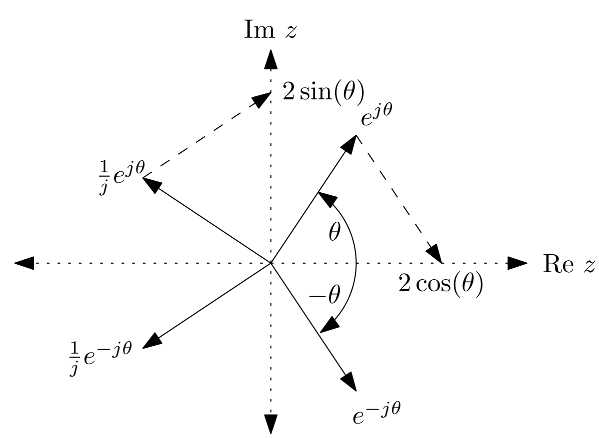
A related expression that will be very useful to us is *Eulers formula*: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. From this we can derive the relations:

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

$$\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$$

These representations and relations can be visualized as follows





Complex numbers as roots of polynomial equations

Recall our original motivation for complex numbers, as solutions to polynomials. Consider the N^{th} order polynomial

$$z^N + a_N z^{N-1} + \cdots + a_2 z + a_1$$

where in cases of interest to us in this course the N coefficients $a_N, \cdots, a_1$ are real. In such cases the polynomial can be factored into

$$z^N + a_N z^{N-1} + \cdots + a_2 z + a_1 = (z - z_1) \cdot (z - z_2) \cdots (z - z_N)$$

where the z_i are the N *roots* of the polynomial. These are complex numbers in general with two cases:

- the root is real
- the root is complex or purely imaginary, in which case they come in conjugate pairs

Note: the `roots` function in Matlab can be used to find the roots of any order polynomial given a vector of coefficients.

A.2 Functions

As we will see in the first few lectures, signals are modeled as functions. Recall a *function* is a mapping between sets

$$f : A \to B$$

where A is a set called the *domain* and B is a set called the *co-domain*. In this course we are primarily concerned with four kinds of functions

- the real-valued functions of an integer variable $f : \mathbb{Z} \mapsto \mathbb{R}$
- the complex-valued functions of an integer variable $f : \mathbb{Z} \mapsto \mathbb{C}$
- the real-valued functions of a real variable $f : \mathbb{R} \mapsto \mathbb{R}$
- the complex-valued functions of a real variable $f : \mathbb{R} \mapsto \mathbb{C}$

We will also briefly discuss the the complex-valued functions of a complex variable $f : \mathbb{C} \mapsto \mathbb{C}$.

Functions can be defined using an expression operating on the *independent variable* representing a value from the domain. For example a function $f : \mathbb{R} \mapsto \mathbb{R}$ might be defined by the expression

$$f(t) = 2t^2$$

where $t \in \mathbb{R}$ is the independent variable. Common operations are sums, difference, products, quotients, powers, and application of trigonometric and transcendental functions. Functions with different expressions for different intervals of the domain are called *piecewise*. For example

$$f(t) = \begin{cases} e^t & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

Functions can also be defined using composition. Given two functions $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ we can define a new function

$$(f \circ g)(t) = f(g(t))$$

where we first apply g then use it's value as the input to f. This will be used to define several transformations of signals.

Visualizing Functions

You are certainly familiar with the graph of functions $f : \mathbb{R} \mapsto \mathbb{R}$. To graph a complex-valued function of a single variable we need to plot two functions. Consider a function $z(t) \in \mathbb{C}$ for $t \in \mathbb{R}$ expressed in Cartesian form:

$$z(t) = z_r(t) + jz_i(t)$$

where $z_r(t) = \text{Re}(z(t))$ and $z_r(t) = \text{Im}(z(t))$ are the real and imaginary parts of the complex value at a given t. We can plot these two real-valued functions to visualize the complex function. Similarly consider a function $z(t) \in \mathbb{C}$ for $t \in \mathbb{R}$ expressed in polar form:

$$z(t) = z_m(t)e^{jz_a(t)}$$

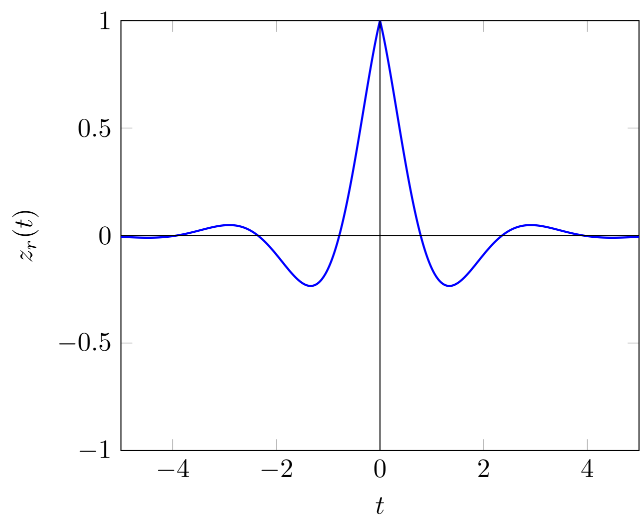
where $z_m(t) = |z(t)|$ and $z_a(t) = \angle z(t)$ are the magnitude and angle of the complex value at a given t. We can plot these two real-valued functions to visualize the complex function.

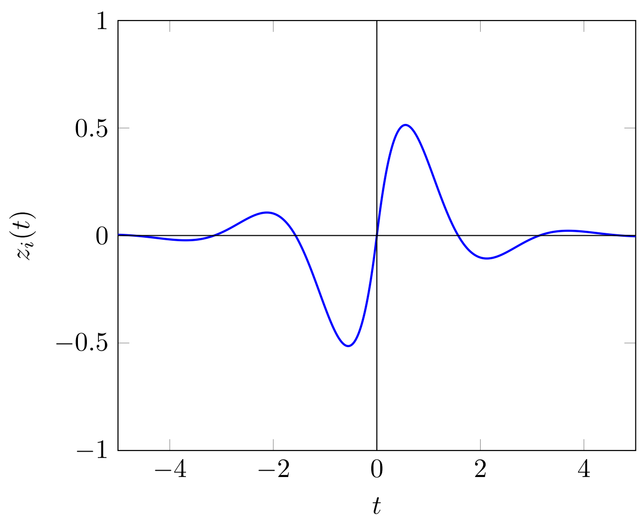
Another approach to visualizing a complex number is to plot it as the tip of a vector that moves as a function of the independent variable.

Example A.2.1. Consider the function $z(t) = e^{-|t|+j2t}$. Lets convert it to polar and Cartesian form

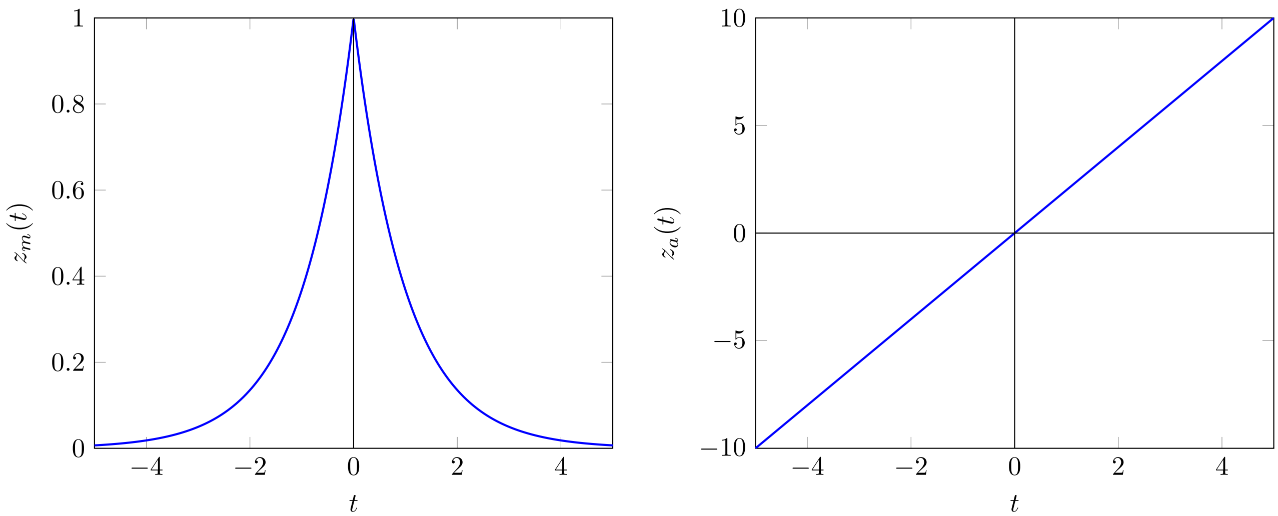
$$\begin{aligned} z(t) &= e^{-|t|+j2t} \\ &= \underbrace{e^{-|t|}}_{z_m(t)} e^{j\overbrace{2t}^{z_a(t)}} \\ &= e^{-|t|}(\cos(2t) + j\sin(2t)) \\ &= \underbrace{e^{-|t|}\cos(2t)}_{z_r(t)} + j\underbrace{e^{-|t|}\sin(2t)}_{z_i(t)} \end{aligned}$$

We can then visualize the function as plots of the real and imaginary functions,





or the magnitude and angle functions,



A.3 Algebra

There is a fair amount of algebra involved in the manipulation of expressions in this course. Beyond the standard items there are a few techniques you might have missed or don't remember that come in handy.

Solving a simple system of trigonometric equations

Consider the system of equations

$$\begin{aligned} x\cos(y) &= A \\ x\cos(B+y) &= C \end{aligned}$$

for constants $A, B, C \in \mathbb{R}$ and variables $x, y \in \mathbb{R}$. How do you solve this for x, y?

First expand the second equation using the trigonometric identity for the cosine of addition of two angles:

$$x\cos(B+y) = x\cos(B)\cos(y) - x\sin(B)\sin(y) = C$$

Then divide the first equation above by the previous:

$$\frac{x\cos(B)\cos(y) - x\sin(B)\sin(y)}{x\cos(y)} = \frac{C}{A}$$

which simplifies to

$$\cos(B) - \sin(B)\frac{\sin(y)}{\cos(y)} = \frac{C}{A}$$

Recognizing the tangent is sin/cos we can solve for y:

$$y = \arctan\left(\frac{\frac{C}{A} - \cos(B)}{-\sin(B)}\right)$$

Then we can substitute back to get x:

$$x = \frac{A}{\cos(y)}$$

A.4 Calculus

Calculus is used heavily in the course. Here we remind ourselves of some basic facts. Consult your calculus text for more details.

Limits

The *limit* L of a function $f(t)$ is the function value as the independent variable approaches a constant $t \to c$, written as

$$\lim_{t \to c} f(t) = L$$

The one-sided limit is defined as the limit as the independent variable approaches the constant from below or above

$$\lim_{t \to c^-} f(t) = L \text{ (limit from below) or } \lim_{t \to c^+} f(t) = L \text{ (limit from above)}$$

Derivatives of real-valued functions

For functions $f : \mathbb{R} \mapsto \mathbb{R}$ recall the derivative is the instantaneous rate of change in the value as a function of the independent variable, and can be defined using a limit of a difference. Consider such a function $f(t)$ for $t \in \mathbb{R}$, it's derivative is given using a limit of a forward difference:

$$\frac{df}{dt}(t) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$$

Higher-order derivative are defined recursively. For example, the second derivative is

$$\frac{d^2 f}{dt^2}(t) = \lim_{h \to 0^+} \frac{\frac{df}{dt}(t+h) - \frac{df}{dt}(t)}{h}$$

In the general case the N^{th} order derivative is

$$\frac{d^N f}{dt^N}(t) = \lim_{h \to 0^+} \frac{\frac{d^{N-1} f}{dt^{N-1}}(t+h) - \frac{d^{N-1} f}{dt^{N-1}}(t)}{h}$$

Note there are several different notations for derivatives, e.g. $\frac{df}{dt}(t) = f'(t) = \dot{f}(t)$, but we will use the former (Leibniz) in most cases. We will also use the derivative operator notation $\frac{d^N f}{dt^N} = (D^N f)(t)$, which is convenient for higher-order derivatives.

A function with finite derivatives (in the limit) for all values of the independent variable over an interval is called *continuous* over that interval. Values of the independent variable where the derivative is not finite (in the limit) are called *discontinuities*. A function with a finite number of discontinuities is called *piecewise continuous*. If the limit is one-sided then we say the function is piecewise continuous from the left or right (or from below/above).

Integrals of real-valued functions

The *indefinite integral* $F(t)$ is the anti-derivative of a function $f(t)$ if $\frac{dF}{dt}(t) = f(t)$ up to a constant term, written as

$$F(t) + C = \int f(t)\ dt$$

where C is an arbitrary constant.

The *definite integral* is the area under a function between the *lower limit* a and the *upper limit* b, defined as

$$\int_a^b f(t)\ dt = F(b) - F(a)$$

In cases where one or both of the limits is infinite, the definition changes to use limits

$$\int_{-\infty}^{b} f(t)\;dt = F(b) - \lim_{a\to-\infty} F(a)$$

$$\int_{a}^{\infty} f(t)\;dt = \lim_{b\to\infty} F(b) - F(a)$$

$$\int_{-\infty}^{\infty} f(t)\;dt = \lim_{b\to\infty} F(b) - \lim_{a\to-\infty} F(a)$$

A.5 Differential Equations

This course assumes a background in basic differential equations (e.g. as taught in Math 2214). However, we only consider linear, constant-coefficient differential equations.

A linear, constant coefficient (LCC) differential equation is of the form

$$a_0\, y + a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} + \cdots + a_N \frac{d^Ny}{dt^N} = b_0\, x + b_1 \frac{dx}{dt} + b_2 \frac{d^2x}{dt^2} + \cdots + b_M \frac{d^My}{dt^M}$$

which can be written compactly as

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

It is helpful to clean up this notation using the derivative operator $D^n = \frac{d^n}{dt^n}$. For example $D^2y = \frac{d^2y}{dt^2}$ and $D^0y = y$. To gives the form:

$$\sum_{k=0}^{N} a_k\, D^k y = \sum_{k=0}^{M} b_k\, D^k x$$

We can factor out the derivative operators

$$a_0y + a_1Dy + a_2D^2y + \cdots + a_ND^Ny = b_0\,x + b_1\,Dx + b_2\,D^2x + \cdots + b_M\,D^Mx$$

$$\underbrace{\left(a_0 + a_1D + a_2D^2 + \cdots + a_ND^N\right)}_{\text{Polynomial in } D, Q(D)} y = \underbrace{\left(b_0 + b_1D + b_2D^2 + \cdots + b_MD^M\right)}_{\text{Polynomial in } D, P(D)} x$$

to give:

$$Q(D)y = P(D)x$$

You learned how to solve these in differential equations as

$$y(t) = y_\text{h}(t) + y_\text{p}(t)$$

The term $y_\text{h}(t)$ is the solution of the homogeneous equation

$$Q(D)y = 0$$

Given the $N-1$ auxiliary conditions $y(t_0) = y_0$, $Dy(t_0) = y_1$, $D^2y(t_0) = y_2$, up to $D^{N-1}y(t_0) = y_{N-1}$.

The term $y_\text{p}(t)$ is the solution of the particular equation

$$Q(D)y = P(D)x$$

for a given $x(t)$.

Rather than recapitulate the solution to $y_{\mathrm{h}}(t)$ and $y_{\mathrm{p}}(t)$ in the general case, we focus on the homogeneous solution $y_{\mathrm{h}}(t)$ only. The reason is that we will use the homogeneous solution to find the impulse response in future lectures and take a different approach to solving the general case for an arbitrary input using the impulse response and convolution.

To solve the homogeneous system:

Step 1: Find the *characteristic equation* by replacing the derivative operators by powers of an arbitrary complex variable s.

$$Q(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_N D^N$$

becomes

$$Q(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_N s^N$$

a polynomial in s with N roots s_i for $i = 1, 2, \cdots, N$ such that

$$(s - s_1)(s - s_2) \cdots (s - s_N) = 0$$

Step 2: Select the form of the solution, a sum of terms corresponding to the roots of the characteristic equation.

- For a real root $s_1 \in \mathbb{R}$ the term is of the form
$$C_1 e^{s_1 t}.$$
- For a pair of complex roots (they will always be in pairs) $s_{1,2} = a \pm jb$ the term is of the form
$$C_1 e^{s_1 t} + C_2 e^{s_2 t} = e^{at}\left(C_3 \cos(bt) + C_4 \sin(bt)\right) = C_5 e^{at} \cos(bt + C_6).$$
- For a repeated root s_1, repeated r times, the term is of the form
$$e^{s_1 t}(C_0 + C_1 t + \cdots + C_{r-1} t^{r-1}).$$

Step 3: Solve for the unknown constants in the solution using the auxiliary conditions.

We now examine two common special cases, when $N = 1$ (first-order) and when $N = 2$ (second-order).

First-Order Homogeneous LCCDE

Consider the first order homogeneous differential equation

$$\frac{dy}{dt}(t) + ay(t) = 0 \text{ for } a \in \mathbb{R}$$

The characteristic equation is given by

$$s + a = 0$$

which has a single root $s_1 = -a$. The solution is of the form

$$y(t) = Ce^{s_1 t} = Ce^{-at}$$

where the constant C is found using the auxiliary condition $y(t_0) = y_0$.

Example: Consider the homogeneous equation

$$\frac{dy}{dt}(t) + 3y(t) = 0 \text{ where } y(0) = 10$$

The solution is

$$y(t) = Ce^{-3t}$$

To find C we use the auxiliary condition

$$y(0) = Ce^{-3 \cdot 0} = C = 10$$

and the final solution is

$$y(t) = 10e^{-3t}$$

Second-Order Homogeneous LCCDE

Consider the second-order homogeneous differential equation

$$\frac{d^2y}{dt^2}(t) + a\frac{dy}{dt}(t) + by(t) = 0 \text{ for } a, b \in \mathbb{R}$$

The characteristic equation is given by

$$s^2 + as + b = 0$$

Let's look at several examples to illustrate the functional forms.

Example 1:

$$\frac{d^2y}{dt^2}(t) + 7\frac{dy}{dt}(t) + 10y(t) = 0$$

The characteristic equation is given by

$$s^2 + 7s + 10 = 0$$

which has roots $s_1 = -2$ and $s_2 = -5$. Thus the form of the solution is

$$y(t) = C_1e^{-2t} + C_2e^{-5t}$$

Example 2:

$$\frac{d^2y}{dt^2}(t) + 2\frac{dy}{dt}(t) + 5y(t) = 0$$

The characteristic equation is given by

$$s^2 + 2s + 5 = 0$$

which has complex roots $s_1 = -1 + j2$ and $s_1 = -1 - j2$. Thus the form of the solution is

$$y(t) = e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right)$$

Example 3:

$$\frac{d^2y}{dt^2}(t) + 2\frac{dy}{dt}(t) + y(t) = 0$$

The characteristic equation is given by

$$s^2 + 2s + 1 = 0$$

which has a root $s_1 = -1$ repeated $r = 2$ times. Thus the form of the solution is

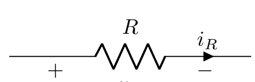
$$y(t) = e^{-t}\left(C_1 + C_2t\right)$$

In each of the above cases the constants, C_1 and C_2, are found using the auxiliary conditions $y(t_0)$ and $y\prime(t_0)$.

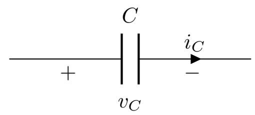
A.6 Circuits

ECE 2024 is required for knowledge of continuous signals representation as voltages and currents, and the analysis and construction of circuits containing resistors, capacitors, inductors, and operational amplifiers. We will assume you can derive the differential governing equation for simple circuits using the voltage and current relationships for the circuit elements and Kerchoffs laws.

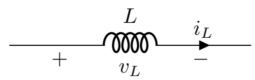
- Resistor: $v_R = R\,i_R$



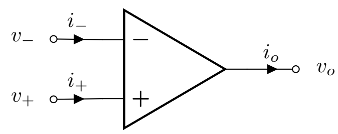
- Capacitor: $i_C = C\, v_C'$



- Inductor: $v_L = L\, i_L'$



- Ideal Op-Amp: the ideal op-amp operates so as to keep $i_+ = i_- = 0$ and $v_- = v_+$.



These elements are the building blocks of most continuous-time signal processing implementations.

KVL

Kerchoff's Voltage Law (KVL) says that the sum of the voltages around any closed loop must be zero.

KCL

Kerchoff's Current Law (KVL) says that the sum of the currents into a node must be zero.

Ideal OpAmps

An op-amp is a device that has two inputs, labeled the inverting (−) and non-inverting (+) input respectively, and a single output. The ideal op-amp is an approximation to simplify analysis. This approximation assumes:

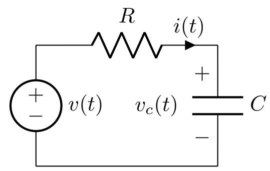
1. the inputs have an infinite impedance such that no current flows into the inputs, $i_+ = i_- = 0$
2. the op-amp operates such that the input voltages are made equal $v_- = v_+$
3. the output can source as much current as needed

While this is just an approximation, within the frequency regimes we are concerned with in this course, it works well.

Governing Equations

Given the component descriptions above and Kerchoff's laws with some algebra and calculus we can derive the input output equation for a wide variety circuits, including those with op-amps. When the circuit contains N energy storage elements (inductors or capacitors) the resulting equation will be an N^{th} order linear, constant-coefficient differential equation. Note, the cleanest route to the governing equation is not always clear at the start and it takes some trial and error on your part.

Example A.6.1. Consider the following RC circuit where $v(t)$ is the time-varying source voltage and we wish to know the resulting voltage across the capacitor $v_C(t)$.



We can analyze it using either a KVL or a KCL. Using a KVL we note

$$v(t) = R\,i(t) + v_c(t)$$

and note that the current through the resistor and capacitor is the same. We use the voltage-current relationship for the capacitor to obtain

$$i(t) = C\,\frac{dv_c}{dt}\ .$$

Substituting into the KVL we get

$$v(t) = RC\,\frac{dv_c}{dt} + v_c(t)$$

which we can rearrange into the standard form for a differential equation

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}\,v(t)$$

Alternatively we could have used a KCL at the top of the capacitor

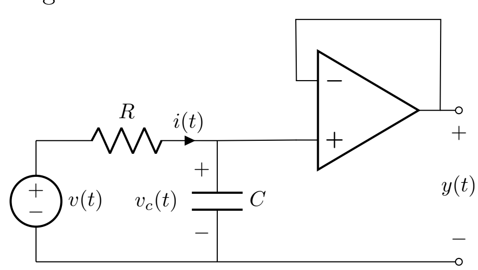
$$i(t) = C\frac{dv_C(t)}{dt}$$

The current through the resistor using Ohm's law is

$$i(t) = \frac{v(t) - v_C(t)}{R}$$

Equating the two expressions and putting into standard form we get the same governing equation as before.

Example A.6.2. We can take the previous circuit and connect to an op-amp in the voltage follower form to arrive at a circuit that isolates the circuit from other parts of a larger circuit. This strategy, where we form these *stages* will be used throughout the course.



The analysis proceeds very similar to the previous one. Using the ideal op-amp conditions, a KCL at the non-inverting input gives

$$i(t) = C\frac{dv_C(t)}{dt} + 0$$

and

$$y(t) = v_c(t)$$

Again, the current through the resistor using Ohm's law is

$$i(t) = \frac{v(t) - v_C(t)}{R}$$

Equating the two expressions, substituting $v_C(t) \to y(t)$, and putting into standard form we get

$$\frac{dy}{dt}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}\,v(t)$$

Building and Characterizing Circuits

We will be building and characterizing physical circuits to better connect the course to the real world. We assume you know how to build relatively simple circuits on a protoboard, as well as use a power supply, function generator and oscilloscope (i.e. the Digilent Analog Discovery). We also assume you have taken or are currently taking the embedded course, so that toward the end of the semester you know how to interface and program with the TI MSP432 "Red Board". (MSP-EXP432P401R Evaluation board).

A.7 Programming

ECE 2514 is required for the ability to model and simulate physical systems using computational tools, and basic programming ability.

- Matlab for general computation and plotting
- C++ (a small subset) for implementing digital filters

For general computation we don't require Matlab and Mathematica, Python, or Julia work as well. Matlab is required for access to filter design functionality using the signal processing toolbox.

Plotting and Visualization

We assume you can plot real and complex functions using Matlab/Python/Julia/Mathematica, label axes appropriately, and generate readable graphics for inclusion in problem set solutions and the project report.

A.8 Digital Systems

ECE 2544 is required for for knowledge of digital signal representation and the analysis and construction of circuits containing combinatorial and sequential logic.

Binary Representation of Integers vs Floating Point

TODO

shift registers

TODO

adders and multipliers

TODO

Appendix B

Deeper Dives into Particular Topics

This is an introductory course and so ommits many interesting and enlightening aspects of the mathematics involved. Some students need or want more details, which is what this appendix attempts to provide.

References:

- Fourier Analysis General Functions, by M.J. LightHill

B.1 Energy Signals and $L^2(\mathbb{R})$ Functions

B.2 The Impulse Function and Distributions

Bibliography

[1] Oppenheim, A. V., Willsky, A. S., and Nawab, S. H, *Signals and Systems*, 2nd Edition, Essex UK: Prentice Hall Pearson, 1996.

Topic Index