Source-Free Series

RLC Circuits
Objective of Lecture

- Derive the equations that relate the voltages across and currents flowing through a resistor, an inductor, and a capacitor in series as:
  - the unit step function associated with voltage or current source changes from 1 to 0 or
  - a switch disconnects a voltage or current source into the circuit.
- Describe the solution to the 2\textsuperscript{nd} order equations when the condition is:
  - Overdamped
  - Critically Damped
  - Underdamped
Series RLC Network

- With a step function voltage source.
Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at \( t < t_o \) and then find the final conditions at \( t = \infty \) s.
  - Since the voltage source has a magnitude of 0V at \( t < t_o \)
    - \( i(t_o^-) = i_L(t_o^-) = 0 \) A and \( v_C(t_o^-) = V_s \)
    - \( v_L(t_o^-) = 0 \) V and \( i_C(t_o^-) = 0 \) A
  - Once the steady state is reached after the voltage source has a magnitude of \( V_s \) at \( t > t_o \), replace the capacitor with an open circuit and the inductor with a short circuit.
    - \( i(\infty s) = i_L(\infty s) = 0 \) A and \( v_C(\infty s) = 0 \) V
    - \( v_L(\infty s) = 0 \) V and \( i_C(\infty s) = 0 \) A
Selection of Parameter

- **Initial Conditions**
  - \( i(t_0^-) = i_L(t_0^-) = 0 \text{A} \) and \( v_C(t_0^-) = V_s \)
  - \( v_L(t_0^-) = 0 \text{V} \) and \( i_C(t_0^-) = 0 \text{A} \)

- **Final Conditions**
  - \( i(\infty) = i_L(\infty) = 0 \text{A} \) and \( v_C(\infty) = 0 \text{V} \)
  - \( v_L(\infty) = 0 \text{V} \) and \( i_C(\infty) = 0 \text{A} \)

- Since the voltage across the capacitor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for \( v_C(t) \).
Kirchhoff’s Voltage Law

\[ \sum v(t) = 0 \]
\[ v_C(t) + v_L(t) + v_R(t) = 0 \]
\[ v_L(t) = L \frac{di_L(t)}{dt} \]
\[ v_R = Ri_R \]
\[ i_C(t) = C \frac{dv_C(t)}{dt} \]
\[ i_L(t) = i_C(t) = i_R(t) \]
\[ LC \frac{d^2v_C(t)}{dt^2} + RC \frac{dv_C(t)}{dt} + v_C(t) = 0 \]
\[ \frac{d^2v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = 0 \]
General Solution

Let $v_C(t) = Ae^{s\Delta t}$ where $\Delta t = t - t_o$

\[ As^2e^{s\Delta t} + \frac{AR}{L}se^{s\Delta t} + \frac{A}{LC}e^{s\Delta t} = 0 \]

\[ Ae^{s\Delta t} \left( s^2 + \frac{R}{L} s + \frac{1}{LC} \right) = 0 \]

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]
General Solution (con’t)

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \]

\[ s_1 = -\frac{R}{2L} + \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}} \]

\[ s_2 = -\frac{R}{2L} - \sqrt{\left( \frac{R}{2L} \right)^2 - \frac{1}{LC}} \]
General Solution (con’t)

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \]
\[ \alpha = \frac{R}{2L} \]
\[ \omega_o = \frac{1}{\sqrt{LC}} \]

\[ s^2 + 2\alpha s + \omega_o^2 = 0 \]
General Solution (con’t)

\[ v_{c_1}(t) = A_1 e^{s_1 \Delta t} \]
\[ v_{c_2}(t) = A_2 e^{s_2 \Delta t} \]
\[ v_c(t) = v_{c_1}(t) + v_{c_2}(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t} \]
Solve for Coefficients $A_1$ and $A_2$

- Use the boundary conditions at $t_o^-$ and $t = \infty$ s to solve for $A_1$ and $A_2$.
  
  \[
  v_c(t_o^-) = V_S \]

- Since the voltage across a capacitor must be a continuous function of time.

  \[
  v_c(t_o^-) = v_c(t_o^+) = v_{c1}(t_o^+) + v_{c2}(t_o^+) = V_S \\
  A_1 e^{s_1(0s)} + A_2 e^{s_2(0s)} = A_1 + A_2 = V_S 
  \]

- Also know that

  \[
  i_c(t_o) = C \frac{dv_c(t_o)}{dt} = \frac{d}{dt} [v_{c1}(t_o) + v_{c2}(t_o)] = 0 \\
  s_1 A_1 e^{s_1(0s)} + s_2 A_2 e^{s_2(0s)} = s_1 A_1 + s_2 A_2 = 0
  \]
Overdamped Case

- $\alpha > \omega_o$
- implies that $C > 4L/R^2$
- $s_1$ and $s_2$ are negative and real numbers

\[ v_C(t) = A_1 e^{s_1 \Delta t} + A_2 e^{s_2 \Delta t} \]

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \]

\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \]
Critically Damped Case

- $\alpha = \omega_0$
  - implies that $C = 4L/R^2$
  - $s_1 = s_2 = -\alpha = -R/2L$

$$v_C(t) = A_1 e^{-\alpha \Delta t} + A_2 \Delta t e^{-\alpha \Delta t}$$
Underdamped Case

- $\alpha < \omega_o$
  - implies that $C < 4L/R^2$
  
  
  \[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + j\omega_d \]
  
  \[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j\omega_d \]
  
  \[ \omega_d = \sqrt{\omega_o^2 - \alpha^2} \]

- $j = \sqrt{-1}$, i is used by the mathematicians for imaginary numbers
\[ v_c(t) = e^{-\alpha \Delta t} \left( B_1 e^{j \omega_d \Delta t} + B_2 e^{-j \omega_d \Delta t} \right) \]

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

\[ v_c(t) = e^{-\alpha \Delta t} \left[ B_1 (\cos \omega_d \Delta t + j \sin \omega_d \Delta t) + B_2 (\cos \omega_d \Delta t - j \sin \omega_d \Delta t) \right] \]

\[ v_c(t) = e^{-\alpha \Delta t} \left[ (B_1 + B_2) \cos \omega_d \Delta t + j(B_1 - B_2) \sin \omega_d \Delta t \right] \]

\[ v_c(t) = e^{-\alpha \Delta t} \left[ A_1 \cos \omega_d \Delta t + A_2 \sin \omega_d \Delta t \right] \]

\[ A_1 = B_1 + B_2 \quad \quad A_2 = j(B_1 - B_2) \]
Angular Frequencies

- $\omega_o$ is called the undamped natural frequency
  - The frequency at which the energy stored in the capacitor flows to the inductor and then flows back to the capacitor. If $R = 0\Omega$, this will occur forever.

- $\omega_d$ is called the damped natural frequency
  - Since the resistance of $R$ is not usually equal to zero, some energy will be dissipated through the resistor as energy is transferred between the inductor and capacitor.
    - $\alpha$ determined the rate of the damping response.
Properties of RLC network

- Behavior of RLC network is described as damping, which is a gradual loss of the initial stored energy
  - The resistor R causes the loss
  - \( \alpha \) determined the rate of the damping response
    - If \( R = 0 \), the circuit is loss-less and energy is shifted back and forth between the inductor and capacitor forever at the natural frequency.
- Oscillatory response of a lossy RLC network is possible because the energy in the inductor and capacitor can be transferred from one component to the other.
  - Underdamped response is a damped oscillation, which is called ringing.
Properties of RLC network

- Critically damped circuits reach the final steady state in the shortest amount of time as compared to overdamped and underdamped circuits.
  - However, the initial change of an overdamped or underdamped circuit may be greater than that obtained using a critically damped circuit.
Set of Solutions when $t > t_o$

- There are three different solutions which depend on the magnitudes of the coefficients of the $\frac{dv_c(t)}{dt}$ and the $v_c(t)$ terms.
- To determine which one to use, you need to calculate the natural angular frequency of the series RLC network and the term $\alpha$.

\[
\omega_o = \frac{1}{\sqrt{LC}}
\]
\[
\alpha = \frac{R}{2L}
\]
Transient Solutions when $t > t_o$

- **Overdamped response** ($\alpha > \omega_o$)
  
  where $\Delta t = t - t_o$

  
  $v_c(t) = A_1 e^{s_1\Delta t} + A_2 e^{s_2\Delta t}$

  
  
  $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

  
  $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

- **Critically damped response** ($\alpha = \omega_o$)

  
  $v_c(t) = (A_1 + A_2\Delta t)e^{-\alpha\Delta t}$

- **Underdamped response** ($\alpha < \omega_o$)

  
  $v_c(t) = [A_1 \cos(\omega_d\Delta t) + A_2 \sin(\omega_d\Delta t)]e^{-\alpha\Delta t}$

  
  $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Find Coefficients

• After you have selected the form for the solution based upon the values of $\omega_0$ and $\alpha$
  • Solve for the coefficients in the equation by evaluating the equation and its first derivative at $t = t_o$ using the initial boundary conditions.
  • $v_C(t_o) = V_s$ and $dv_C(t_o)/dt = i_C(t_o)/C = 0\text{V/s}$
Other Voltages and Currents

Once the voltage across the capacitor is known, the following equations for the case where $t > t_0$ can be used to find:

\[
i_C(t) = C \frac{dv_C(t)}{dt}
\]

\[
i(t) = i_C(t) = i_L(t) = i_R(t)
\]

\[
v_L(t) = L \frac{di_L(t)}{dt}
\]

\[
v_R(t) = Ri_R(t)
\]
Solutions when $t < t_0$

- The initial conditions of all of the components are the solutions for all times $-\infty < t < t_0$.
  - $v_C(t) = V_s$
  - $i_C(t) = 0\text{A}$
  - $v_L(t) = 0\text{V}$
  - $i_L(t) = 0\text{A}$
  - $v_R(t) = 0\text{V}$
  - $i_R(t) = 0\text{A}$
Summary

- The set of solutions when $t > t_o$ for the voltage across the capacitor in a RLC network in series was obtained.
  - Selection of equations is determined by comparing the natural frequency $\omega_o$ to $\alpha$.
  - Coefficients are found by evaluating the equation and its first derivative at $t = t_o$.
  - The voltage across the capacitor is equal to the initial condition when $t < t_o$.
- Using the relationships between current and voltage, the current through the capacitor and the voltages and currents for the inductor and resistor can be calculated.