Source-Free RLC Circuit

Parallel RLC Network
Objective of Lecture

- Derive the equations that relate the voltages across and the currents flowing through a resistor, an inductor, and a capacitor in parallel as:
  - the unit step function associated with voltage or current source changes from 1 to 0 or
  - a switch disconnects a voltage or current source in the circuit.
- Describe the solution to the 2\textsuperscript{nd} order equations when the condition is:
  - Overdamped
  - Critically Damped
  - Underdamped
RLC Network

- A parallel RLC network where the current source is switched out of the circuit at $t = t_o$. 

![RLC Network Diagram](image-url)
Boundary Conditions

- You must determine the initial condition of the inductor and capacitor at $t < t_o$ and then find the final conditions at $t = \infty s$. Replace the capacitor with an open circuit and the inductor with a short circuit.
  - Since the current source has a magnitude of $I_s$ at $t < t_o$
    - $i_L(t_o^-) = I_s$ and $v(t_o^-) = v_C(t_o^-) = 0V$
    - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$
  - Once the steady state is reached after the current source been removed from the circuit at $t > t_o$ and the stored energy has dissipated through R.
    - $i_L(\infty s) = 0A$ and $v(\infty s) = v_C(\infty s) = 0V$
    - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$
Selection of Parameter

- **Initial Conditions**
  - $i_L(t_o^-) = I_s$ and $v(t_o^-) = v_C(t_o^-) = 0V$
  - $v_L(t_o^-) = 0V$ and $i_C(t_o^-) = 0A$

- **Final Conditions**
  - $i_L(\infty s) = 0A$ and $v(\infty s) = v_C(\infty s) = 0V$
  - $v_L(\infty s) = 0V$ and $i_C(\infty s) = 0A$

- Since the current through the inductor is the only parameter that has a non-zero boundary condition, the first set of solutions will be for $i_L(t)$. 
Kirchoff’s Current Law

\[ i_R(t) + i_L(t) + i_C(t) = 0 \]

\[ v(t) = v_R(t) = v_L(t) = v_C(t) \]

\[ \frac{v_R(t)}{R} + i_L(t) + C \frac{dv_C(t)}{dt} = 0 \]

\[ v_L(t) = v(t) = L \frac{di_L(t)}{dt} \]

\[ L C \frac{d^2 i_L(t)}{dt^2} + \frac{L}{R} \frac{di_L(t)}{dt} + i_L(t) = 0 \]

\[ \frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC} \frac{di_L(t)}{dt} + \frac{i_L(t)}{LC} = 0 \]
General Solution

\[ s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \]

\[ s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]

\[ s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \]
\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \]
\[ \alpha = \frac{1}{2RC} \]
\[ \omega_o = \frac{1}{\sqrt{LC}} \]

\[ s^2 + 2\alpha s + \omega_o^2 = 0 \]

Note that the equation for the natural frequency of the RLC circuit is the same whether the components are in series or in parallel.
Overdamped Case

- $\alpha > \omega_0$
  - implies that $L > 4R^2C$

$s_1$ and $s_2$ are negative and real numbers

\[ i_{L1}(t) = A_1e^{s_1\Delta t} \]
\[ i_{L2}(t) = A_2e^{s_2\Delta t} \]

$\Delta t = t - t_o$

\[ i_{L}(t) = i_{L1}(t) + i_{L2}(t) = A_1e^{s_1\Delta t} + A_2e^{s_2\Delta t} \]
Critically Damped Case

- \( \alpha = \omega_0 \)
  - implies that \( L = 4R^2C \)
    
    \[ s_1 = s_2 = -\alpha = -\frac{1}{2}RC \]

\[
 i_L(t) = A_1 e^{-\alpha \Delta t} + A_2 \Delta t e^{-\alpha \Delta t}
\]
Underdamped Case

- $\alpha < \omega_o$
- implies that $L < 4R^2C$

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} = -\alpha + j \omega_d \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} = -\alpha - j \omega_d \]
\[ \omega_d = \sqrt{\omega_o^2 - \alpha^2} \]
\[ i_L(t) = e^{-\alpha \Delta t} [A_1 \cos \omega_d \Delta t + A_2 \sin \omega_d \Delta t] \]
Solve for Coefficients $A_1$ and $A_2$

- Use the boundary conditions at $t_o^-$ and $t = \infty s$ to solve for $A_1$ and $A_2$.

\[ i_L(t_o^-) = I_S \]

- Since the current through an inductor must be a continuous function of time.

\[ i_L(t_o^-) = i_L(t_o^+) = i_{L1}(t_o^+) + i_{L2}(t_o^+) = I_S \]

\[ A_1 e^{s_1(0s)} + A_2 e^{s_2(0s)} = A_1 + A_2 = I_S \]

- Also know that

\[ v_L(t_o) = L \frac{di_L(t_o)}{dt} = \frac{d}{dt} [i_{L1}(t_o) + i_{L2}(t_o)] = 0 \]

\[ s_1 A_1 e^{s_1(0s)} + s_2 A_2 e^{s_2(0s)} = s_1 A_1 + s_2 A_2 = 0 \]
Other Voltages and Currents

- Once current through the inductor is known:

\[
v_L(t) = L \frac{di_L(t)}{dt}
\]

\[
v_L(t) = v_C(t) = v_R(t)
\]

\[
i_C(t) = C \frac{dv_C(t)}{dt}
\]

\[
i_R(t) = v_R(t) / R
\]
Summary

• The set of solutions when \( t > t_0 \) for the current through the inductor in a RLC network in parallel was obtained.
  • Selection of equations is determined by comparing the natural frequency \( \omega_0 \) to \( \alpha \).
  • Coefficients are found by evaluating the equation and its first derivation at \( t = t_0^- \).
  • The current through the inductor is equal to the initial condition when \( t < t_0 \).
• Using the relationships between current and voltage, the voltage across the inductor and the voltages and currents for the capacitor and resistor can be calculated.