RC and RL Circuits

First Order Circuits
Objective of Lecture

- Explain the transient response of a RC circuit
  - as the capacitor releases energy when there is:
    - a transition in a unit step function source \([1-u(t-t_o)]\)
    - or a voltage or current source is switched out of the circuit.

- Explain the transient response of a RL circuit
  - as the inductor releases energy when there is:
    - a transition in a unit step function source \([1-u(t-t_o)]\)
    - or a voltage or current source is switched out of the circuit.
Natural Response

- The behavior of the circuit with no external sources of excitation.
  - There was a transition in the source in the circuit where the unit step function changed from 1 to 0 at $t = 0$ s.
  - There is stored energy in the capacitor or inductor at time $t = 0$ s.
  - For $t > 0$ s, the stored energy is released
    - Current flows through the circuit and voltages exist across components in the circuit as the stored energy is released.
    - The stored energy will decay to zero as time approaches infinite, at which point the currents and voltages in the circuit become zero.
Suppose there is some charge on a capacitor at time $t = 0$ s. This charge could have been stored because a voltage or current source had been in the circuit at $t < 0$ s, but was switched off at $t = 0$ s.

We can use the equations relating voltage and current to determine how the charge on the capacitor is removed as a function of time.

The charge flows from one plate of the capacitor through the resistor $R$ to the other plate to neutralize the charge on the opposite plate of the capacitor.
Equations for RC Circuit

\[-v_C + v_R = 0\]

\[i_C = -i_R\]

\[i_C = C \frac{dv_C}{dt}\]

\[i_R = \frac{v_R}{R}\]
\[ C \frac{dV_C}{dt} + \frac{V_R}{R} = 0 \]

\[ V_R = V_C \]

\[ \frac{dV_C}{dt} + \frac{V_C}{RC} = 0 \]

\[ \frac{1}{V_C} \frac{dV_C}{dt} + \frac{1}{RC} = 0 \]

\[ \frac{dV_C}{V_C} = -\frac{1}{RC} \, dt \]

\[ \ln(V_C) = -\frac{t}{RC} + \ln(V_C|_{t=0s}) \]
If \( V_o = V_C \big|_{t=0s} \) and \( \tau = RC \),

\[
p_R(t) = V_R I_R = \frac{V_o^2}{R} e^{-\frac{2t}{\tau}}
\]

\[
V_C(t) = V_o e^{-\frac{t}{\tau}} \quad \text{when} \ t \geq 0s
\]

\[
I_R(t) = -I_C(t) = \frac{V_o}{R} e^{-\frac{t}{\tau}}
\]

Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the capacitor.
RL Circuits

\[ V_L + V_R = 0 \]
\[ I_L = I_R \]

\[ V_L = L \frac{dI_L}{dt} \]
\[ I_R = \frac{V_R}{R} \]
\[ L \frac{dI_L}{dt} + RI_R = 0 \]

\[ \frac{dI_L}{dt} + \frac{RI_L}{L} = 0 \]

\[ \frac{1}{I_L} \frac{dI_L}{dt} + \frac{R}{L} = 0 \]

\[ \frac{dI_L}{I_L} = -\frac{R}{L} \, dt \]

\[ \ln(I_L) = -\frac{R}{L} t + \ln\left(I_L \bigg|_{t=0s}\right) \]
If \( I_o = I_L \big|_{t=0s} \) and \( \tau = \frac{L}{R} \),

\[ I_L(t) = I_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s \]

\[ V_R(t) = -V_L(t) = -RI_o e^{-\frac{t}{\tau}} \]

Since the currents are equal and the voltage have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the inductor.

\[ p_R(t) = V_R I_R = RI_o^2 e^{-\frac{2t}{\tau}} \]

\[ w(t) = \int_{0s}^{t} p_R(t) dt = \frac{LI_o^2}{2} \left[ 1 - e^{-\frac{2t}{\tau}} \right] \]
Initial Condition

- Can be obtained by inserting a d.c. source to the circuit for a time much longer than $\tau$ (at least $t = -5\tau$).
  - **Capacitor**
    - $V_0$ is the voltage calculated by replacing the capacitor with a resistor with infinite resistance (an open circuit) after the voltage across the capacitor has reached a constant value (steady state).
  - **Inductor**
    - $I_0$ is the current flowing through the inductor calculated by replacing the inductor with a resistor with zero resistance (a short circuit) after the current flowing through the inductor has reached a constant value (steady-state).
You can set the initial condition on a capacitor or inductor by doubling clicking on the part symbol. Then, enter a value for IC in the pop-up window that opens.
Time constant, $\tau$

- The time required for the voltage across the capacitor or current in the inductor to decay by a factor of $1/e$ or 36.8% of its initial value.
Example #1

\[ V(t) = 6V \left[ 1 - u(t) \right] \]
Example #1 (con’t)
Example #1 (con’t)

Find the initial condition.

Therefore,

\[ I_o = 2 \text{mA} \]
Example #1 (con’t)
Example #1 (con’t)

\[ t > 0s \]

\[ \tau = \frac{L}{R} = \frac{10 \text{mH}}{3\text{k}\Omega} = 3.33 \mu\text{s} \]

\[ I_L = I_R = I_0 e^{-t/\tau} = 2\text{mA} \ e^{-(t/3.33\mu\text{s})} \]

\[ V_R = 3\text{k}\Omega \ I_R = 6\text{V} \ e^{-(t/3.33\mu\text{s})} \]

\[ V_L = L \frac{dI_L}{dt} = -6\text{V} \ e^{-(t/3.33\mu\text{s})} \]

Note \( V_R + V_L = 0 \text{ V} \)
Example #2
Example #2 (con’t)

\[ \text{t < 2ms} \]

\[ R1 \quad 1k \]

\[ R2 \quad 3k \]

\[ R3 \quad 12k \]

\[ + \quad V_c \quad - \]

\[ C \quad 2\mu F \]
Example #2 (con’t)

- Calculate the initial condition - the voltage on the capacitor. Replace the capacitor with an open circuit and find the voltage across the two terminals.

- Note that in this circuit, current will flow through R3 so there will be a voltage across C, but it will not be equal to the magnitude of the voltage source in the circuit.
Example #2 (con’t)

- The voltage across the capacitor is equal to the voltage across the 12kΩ resistor.

\[ V_C = V_o = \left[\frac{12k\Omega}{15k\Omega}\right] 5V = 4V \]
Example #2 (con’t)

\[ V_C \]

\[ R_2 \quad 3k \]

\[ R_1 \quad 1k \quad R_3 \quad 12k \]

\[ C \quad 2uF \]

\[ 5V \]

\[ t > 2ms \]
Example #2 (con’t)

- Further simplification of the circuit

\[ R_{eq} = \left( \frac{1k\Omega + 3k\Omega}{12k\Omega} \right) = 3k\Omega \]
Example #2 (con’t)

\( \tau = R_{eq} C = 3k\Omega (2\mu F) = 6\text{ms} \)

when \( t > 2\text{ms} \)

\[ V_C = V_C(t) \bigg|_{t=2\text{ms}} e^{-(t-2\text{ms})/\tau} \]

\[ V_C(t) = 4Ve^{-\frac{(t-2\text{ms})}{6\text{ms}}} \]

\[ V_C(t) = V_{R_{eq}}(t) \]

\[ I_C(t) = C \frac{dV_C(t)}{dt} = 2\mu F \left( -4V / 6\text{ms} \right)e^{-\frac{(t-2\text{ms})}{6\text{ms}}} \]

\[ I_C(t) = -1.33\text{mA} \ e^{-(t-2\text{ms})/6\text{ms}} \]

\[ I_{R_{eq}}(t) = -I_C(t) = 1.33\text{mA} \ e^{-(t-2\text{ms})/6\text{ms}} \]

\[ I_{R_{eq}}(t) + I_C(t) = 0 \]
Summary

- The initial condition for:
  - the capacitor voltage \( V_o \) is determined by replacing the capacitor with an open circuit and then calculating the voltage across the terminals.
  - The inductor current \( I_o \) is determined by replacing the inductor with a short circuit and then calculating the current flowing through the short.
- The time constant for:
  - an RC circuit is \( \tau = RC \)
  - an RL circuit is \( \tau = L/R \)
- The general equations for the natural response of:
  - the voltage across a capacitor is \( V_C(t) = V_o e^{-(t-t_o)/\tau} \)
  - the current through an inductor is \( I_L(t) = I_o e^{-(t-t_o)/\tau} \)