§4 Wheatstone Bridge

(Hambley sect 2.8)
This is a circuit which can be used in principle to find the value of an unknown resistor very accurately.
Often used for Platinum resistance thermometer, or strain gauge to measure temperature or weight.

In operation the resistance of three of the resistors is known accurately, and the 4th is the unknown. A sensitive galvanometer G (ammeter) is used to measure when the bridge is balanced, that is when no current flows from A to B.
At this point the voltages $V_A$ and $V_B$ are the same and we have $V_A = V_B$ or $V_{AB} = 0$

Because, for this condition, the same current $I_1$ flows through the resistors $R_1$ and $R_2$ we may use the potential divider rule and write.

$$
\frac{V_A}{V_S} = \frac{I_1 R_2}{I_1 (R_1 + R_2)} = \frac{R_2}{R_1 + R_2}
$$

Similarly

$$
\frac{V_B}{V_S} = \frac{I_2 R_4}{I_2 (R_3 + R_4)} = \frac{R_4}{R_3 + R_4}
$$

Note again: this is only true if no current flows A to B!

In this case

$$
\frac{R_2}{R_1 + R_2} = \frac{R_4}{R_3 + R_4}
$$

So

$$
R_2(R_3 + R_4) = R_4(R_1 + R_2)
$$

$$
R_2 R_3 + R_2 R_4 = R_1 R_4 + R_2 R_4
$$

$$
R_2 R_3 = R_1 R_4
$$

$$
\frac{R_3}{R_4} = \frac{R_1}{R_2}
$$

That is: the ratio of the resistances in both branches is the same, if the bridge is balanced.
This makes sense: the fraction of the voltage dropped in the first resistor in each branch must be the same.
The power of this method is that, since no current flows in the Ammeter, its physical properties (such as internal resistance) do not matter. It is effectively not part of the circuit.
But now suppose the bridge is out of balance: how much current would flow in the meter? This will depend on the meter properties as we will see.
Look at the Thevenin equivalent of the circuit. Divide the circuit up into the component under study, The ammeter (Galvanometer) G, and the rest of the circuit.
First calculate the Thevenin resistance $R_T$. Short circuit the voltage source - this means that nodes C and D are connected together.

\[
R_T = \frac{R_1R_2}{R_1 + R_2} + \frac{R_3R_4}{R_3 + R_4}
\]
Now for the Thevenin voltage
If we open circuit the load (meter) between A and B we would get

\[ V_T = V_{AB} = V_A - V_B = V_s \frac{R_2}{R_1 + R_2} - V_s \frac{R_4}{R_3 + R_4} \]

Suppose the set up we had was

\[ R_T = \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} = \frac{60 \times 30}{60 + 30} + \frac{60 \times 40}{60 + 40} = 20\Omega + 24\Omega = 44\Omega \]

So

\[ V_T = V_s \left( \frac{30}{30 + 60} - \frac{40}{40 + 60} \right) = 10V \left( \frac{1}{3} - \frac{2}{5} \right) = -\frac{2}{3}V \]
Thévenin equivalent circuit plus load is therefore:

Current flowing is $V/R = 2/3 \times 1/54 = 12.4 \text{ mA}$

Voltage across $10\Omega$ resistor is $10\Omega \times 12.35\text{ mA} = 0.124\text{ V}$

In this example we could not have reduced the circuit to resistors in series or in parallel until the whole circuit was represented by a single equivalent resistor. This worked here, but we require a general technique which could be used in all circumstances.

One such general technique is Nodal Analysis. Briefly, we write a set of equations, describing the current into key Nodes, and solve them simultaneously. The hard part is making sure all the signs are right!
Nodal analysis for the Wheatstone bridge.

Because Ohm's law only tells us about potential differences, we have to assume one of the potentials and work relative to that. Here we will assume, as is conventional, that the lowest potential is 0V. That is: $V_D = 0V$

Node C is then at +10V, i.e. $V_C = +10V$

Now apply Kirchhoff’s current law at the remaining nodes. Watch the signs! We must always be consistent. One method is to imagine standing at the node and looking at potentials from that node to each adjacent node. Use this to work out currents.
Here we go: Node A first

Current flowing from A to C is \( \frac{V_A - V_C}{60} \)
Current flowing from A to B is \( \frac{V_A - V_B}{10} \)
Current flowing from A to D is \( \frac{V_A - V_D}{30} \)

Since the total current into the node must equal the total current out of the node, the sum of these three currents must be zero.

\[
\sum I_{\text{node}} = \sum I_{\text{in}} + \sum I_{\text{out}} = 0
\]

So

\[
\frac{V_A - V_C}{60} + \frac{V_A - V_B}{10} + \frac{V_A - V_D}{30} = 0
\]

We can do this since we are using a consistent sign convention: all the voltages are with respect to \( V_A \). All the currents are written as currents flowing out of the node.

Setting \( V_C = +10\text{V} \) and \( V_D = 0\text{V} \) we have:

\[
\frac{V_A - 10}{60} + \frac{V_A - V_B}{10} + \frac{V_A}{30} = 0
\]

\[
V_A \left( \frac{1}{60} + \frac{1}{10} + \frac{1}{30} \right) + V_B \left( - \frac{1}{10} \right) - \frac{10}{60} = 0
\]
\[ V_A \left(1 + \frac{6 + 2}{60}\right) + V_B \left(-\frac{1}{10}\right) = \frac{10}{60} \]
\[ V_A \left(\frac{9}{60}\right) + V_B \left(-\frac{1}{10}\right) = \frac{10}{60} \]

\[ 9V_A - 6V_B = 10 \quad \text{Equation 1} \]

Node B

Similarly we take the algebraic sum of the currents flowing from node B we have:

\[ \frac{V_B - V_A}{10} + \frac{V_B - V_C}{60} + \frac{V_B - V_D}{40} = 0 \]

Setting \( V_C = +10\text{V} \) and \( V_D = 0\text{V} \) we have:

\[ \frac{V_B - V_A}{10} + \frac{V_B - 10}{60} + \frac{V_B}{40} = 0 \]
\[ V_A \left(-\frac{1}{10}\right) + V_B \left(\frac{1}{60} + \frac{1}{10} + \frac{1}{40}\right) - \frac{10}{60} = 0 \]
\[ V_A \left(-\frac{1}{10}\right) + V_B \left(\frac{12 + 2 + 3}{120}\right) - \frac{10}{60} = 0 \]
\[ V_A \left(-\frac{1}{10}\right) + V_B \left(\frac{17}{120}\right) = \frac{10}{60} \]

\[ -12V_A + 17V_B = 20 \quad \text{Equation 2} \]

Previously we had

\[ 9V_A - 6V_B = 10 \quad \text{Equation 1} \]
4 times this is
\(36V_A - 24V_B = 40\)

3 times Equation 2 is
\[-36V_A + 51V_B = 60\]

Adding gives
\[0V_A + (51 - 24)V_B = 40 + 60\]

\[100\]
\[27\]

so \(V_B = \frac{100}{27} = 3.704\) V

From Equation 1 \(9V_A - 6V_B = 10\)

\[V_A = \frac{6V_B + 10}{9} = \frac{6 \times 3.704 + 10}{9} = 3.580\) V

\(V_{BA} = 3.704 - 3.580 = 0.124\) V

Current flow from B to A is
\[\frac{0.124}{10} = 0.0124\) A = 12.4 mA.

Just as before!
Another quick example:

\[
\begin{align*}
\text{Node 2:} & \quad \frac{V_2 - 60}{40} + \frac{V_2 - V_3}{10} + \frac{V_2}{60} = 0 \\
\text{Multiply by 120 gives:} & \quad 3V_2 - 180 + 12V_2 - 12V_3 + 2V_2 = 0 \text{ or } 17V_2 - 12V_3 = 180 \quad \text{Eq1} \\
\text{Node 3:} & \quad \frac{V_3 - 0}{20} + \frac{V_3 - V_2}{10} + \frac{V_3 - 60}{80} = 0 \\
\text{Multiply by 80 gives:} & \quad 4V_3 + 8V_3 - 8V_2 + V_3 - 60 = 0 \text{ or } 13V_3 - 8V_2 = 60 \quad \text{Eq2} \\
\text{Multiply Eq 1 by 13 and Eq 2 by 12 gives} & \quad 221V_2 - 156V_3 = 2340 \quad \text{Eq1a} \\
& \quad 156V_3 - 96V_2 = 720 \quad \text{Eq2} \\
\text{Adding} & \quad 125V_2 = 3060 \text{ so } V_2 = 24.48V \\
\text{Substituting in either Eq 1 or Eq 2 gives} & \quad V_3 = 19.68V
\end{align*}
\]
Nodal analysis: a more general example.

Writing an equation for node 1 in the form $\frac{V_A - V_B}{R_{AB}}$ we have:

$$\frac{V_1 - 0}{R_{10}} + \frac{V_1 - V_2}{R_{12}} + \frac{V_1 - V_3}{R_{13}} = I_1$$

Look at this carefully. $(V_1 - V_X)/R_{1X}$ is the current leaving node 1 in the direction of node X over resistance $R_{1X}$. The current leaving the node on these three paths must equal the current coming into the node on the other path which is $I_j$. One could just as easily have written the same equation entirely in terms of currents leaving the node. Signs!
\[ \frac{V_1 - 0}{R_{10}} + \frac{V_1 - V_2}{R_{12}} + \frac{V_1 - V_3}{R_{13}} - I_1 = 0 \]

Writing equations for the other nodes we have:

Node 2: \[ \frac{V_2 - 0}{R_{20}} + \frac{V_2 - V_1}{R_{12}} + \frac{V_2 - V_3}{R_{23}} = 0 \]

Node 3: \[ \frac{V_3 - 0}{R_{30}} + \frac{V_3 - V_1}{R_{13}} + \frac{V_3 - V_2}{R_{23}} = 0 \]

Rewriting these equations gives

\[ V_1 \left( \frac{1}{R_{10}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} \right) - V_2 \frac{1}{R_{12}} - V_3 \frac{1}{R_{13}} = I_1 \]

\[ -V_1 \frac{1}{R_{12}} + V_2 \left( \frac{1}{R_{20}} + \frac{1}{R_{12}} + \frac{1}{R_{23}} \right) - V_3 \frac{1}{R_{23}} = 0 \]

\[ -V_1 \frac{1}{R_{13}} - V_2 \frac{1}{R_{23}} + V_3 \left( \frac{1}{R_{30}} + \frac{1}{R_{13}} + \frac{1}{R_{23}} \right) = 0 \]

Note that, while the order of current and voltage labels matter, the order of resistance labels does not

\[ V_1 \left( \frac{1}{R_{10}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} \right) - V_2 \frac{1}{R_{12}} - V_3 \frac{1}{R_{13}} = I_1 \]

\[ -V_1 \frac{1}{R_{21}} + V_2 \left( \frac{1}{R_{20}} + \frac{1}{R_{21}} + \frac{1}{R_{23}} \right) - V_3 \frac{1}{R_{23}} = 0 \]
\[-V_1 \frac{1}{R_{31}} - V_2 \frac{1}{R_{32}} + V_3 \left( \frac{1}{R_{30}} + \frac{1}{R_{31}} + \frac{1}{R_{32}} \right) = 0 \]

This can be written in matrix form as:

\[
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \left( \frac{1}{R_{10}} + \frac{1}{R_{12}} + \frac{1}{R_{13}} \right) & -\frac{1}{R_{12}} & -\frac{1}{R_{13}} \\ -\frac{1}{R_{21}} & \left( \frac{1}{R_{20}} + \frac{1}{R_{21}} + \frac{1}{R_{23}} \right) & -\frac{1}{R_{23}} \\ -\frac{1}{R_{31}} & -\frac{1}{R_{32}} & \left( \frac{1}{R_{30}} + \frac{1}{R_{31}} + \frac{1}{R_{32}} \right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\]

This means that we can work on these equations, and solve them, using standard matrix techniques. For big complicated circuits this proves to be (by far!) the easiest technique. For such purposes it is often more convenient to work in terms of the inverse resistance, called the conductance, \( G \).

For a simple resistance, \( G = \frac{1}{R} \) so we can rewrite Ohm's law as either \( V = IR \) or \( I = GV \)

Here we have a matrix of inverse resistances, that is an conductance matrix.

\[
\overline{G} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}
\]
For the full equation we can write, simply.

\[ \bar{I} = \bar{G} \times \bar{V} \]

Where \( \bar{G} \) is the admittance matrix, \( \bar{V} \) is the voltage vector and \( \bar{I} \) is the current vector. This means that, for example \( G_{jk} \) is the conductance connected between nodes \( j \) and \( k \) and \( V_j \) is the voltage at node \( j \).

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix} =
\begin{bmatrix}
I_1 \\
0 \\
0
\end{bmatrix}
\]

where \( I_1 \) is the current injected by a source at node 1. Since there are no sources connected to nodes 2 and 3, \( I_2 = I_3 = 0 \). Notice that we have a circuit with four nodes, but we end up with the equations. This is because we choose one node to be at 0V.

Remember: we only know about potential differences. It is always the voltage with respect to something, and not the absolute voltage. We choose the analysis reference voltage for our convenience.
As an aside, it is unusual to see $G$ used in this way very often in real analysis.

This is because in more realistic circuit analysis we often deal with a.c. signals and the resistance $R$ and conductance $G$ are replaced by terms which are more general, the impedance $Z$ and the admittance $Y$.

It is therefore more usual to see $\bar{Y} \times \bar{V} = \bar{I}$ as the matrix equation, where,

$$
\bar{Y} = \begin{bmatrix}
y_{11} & y_{12} & y_{13} \\
y_{21} & y_{22} & y_{23} \\
y_{31} & y_{32} & y_{33}
\end{bmatrix}
$$