Impedance and Admittance

Ohm’s Law with Phasors
Objective of Lecture

- Describe the mathematical relationships between ac voltage and ac current for a resistor, capacitor, and inductor.
  - Discuss the phase relationship between the ac voltage and current.
- Explain how Ohm’s Law can be adapted for inductors and capacitors when an ac signal is applied to these components.
  - Derive the mathematical formulas for the impedance and admittance of a resistor, inductor, and capacitor.

Chapter 8.5 in Basic Engineering Circuit Analysis by Irwin and Nelms
Resistors

Ohm’s Law

\[
\text{if } i(t) = I_m \cos(\omega t + \theta) \\
\text{then } v(t) = Ri(t) = R I_m \cos(\omega t + \theta)
\]

\[V = RI_m \angle \theta = RI \text{ where } \theta = \phi\]

The voltage and current through a resistor are in phase as there is no change in the phase angle between them.
Capacitors

\[ i(t) = C \frac{dv(t)}{dt} \text{ where } v(t) = V_m \cos(\omega t) \]
\[ i(t) = -C\omega V_m \sin(\omega t) \]
\[ i(t) = \omega CV_m \sin(\omega t + 180^\circ) \]
\[ i(t) = \omega CV_m \cos(\omega t + 180^\circ - 90^\circ) \]
\[ i(t) = \omega CV_m \cos(\omega t + 90^\circ) \]
Capacitors

\[ V = V_m \angle 0^\circ \]

\[ I = \omega CV_m \cos(\omega t + 90^\circ) \]

\[ V_m \cos(\omega t + 90^\circ) = V e^{j90^\circ} = V \angle 90^\circ = jV \]

\[ I = j\omega CV = \omega CV \angle 90^\circ \]

or

\[ V = (1/j\omega C) I = - (j/\omega C) I = (1/\omega C) I \angle -90^\circ \]
Capacitors

- $90^\circ$ phase difference between the voltage and current through a capacitor.
  - Current needs to flow first to place charge on the electrodes of a capacitor, which then induce a voltage across the capacitor.
  - Current leads the voltage (or the voltage lags the current) in a capacitor.
Inductors

\[ v(t) = L \frac{d}{dt} i(t) \quad \text{where} \quad i(t) = I_m \cos(\omega t) \]

\[ v(t) = -L \omega I_m \sin(\omega t) = \omega L I_m \cos(\omega t + 90^\circ) \]

\[ V = \omega L I_m \angle 90^\circ \]

\[ I = I_m \cos(\omega t) \]

\[ I_m \cos(\omega t + 90^\circ) = I e^{j90^\circ} = I \angle 90^\circ = jI \]

\[ V = j\omega L I = \omega L I \angle 90^\circ \]

or

\[ I = \frac{1}{j\omega L} V = -\frac{j}{\omega L} V = \frac{1}{\omega L} V \angle -90^\circ \]
Inductors

- $90^\circ$ phase difference between the voltage and current through an inductor.
- The voltage leads the current (or the current lags the voltage).
Impedance

If we try to force all components to following Ohm's Law, \( V = Z I \), where \( Z \) is the impedance of the component.

Resistor:
\[ Z_R = R \quad R \angle 0^\circ \]

Capacitor:
\[ Z_C = - \frac{j}{\omega C} \quad 1/\omega C \angle -90^\circ \]

Inductor:
\[ Z_L = j\omega L \quad \omega L \angle 90^\circ \]
Admittance

If we rewrite Ohm’s Law:

\[ I = Y V \quad (Y = \frac{1}{Z}) \]

where \( Y \) is admittance of the component

- Resistor:
  \[ Y_R = \frac{1}{R} = G \quad G \angle 0^\circ \]
- Capacitor:
  \[ Y_C = j\omega C \quad \omega C \angle 90^\circ \]
- Inductor:
  \[ Y_L = -\frac{j}{\omega L} \quad \frac{1}{\omega L} \angle -90^\circ \]
<table>
<thead>
<tr>
<th>Impedances</th>
<th>Value at $\omega =$</th>
<th>Admittance $Y$</th>
<th>Value at $\omega =$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0 \text{ rad/s}$</td>
<td>$\infty \text{ rad/s}$</td>
<td>$0 \text{ rad/s}$</td>
</tr>
<tr>
<td>$Z_R = R = 1/G$</td>
<td>$R$</td>
<td>$R$</td>
<td>$Y_R = 1/R = G$</td>
</tr>
<tr>
<td>$Z_L = j\omega L$</td>
<td>$\infty \Omega$</td>
<td>$\infty \Omega$</td>
<td>$Y_L = -j/(\omega L)$</td>
</tr>
<tr>
<td>$Z_C = -j/(\omega C)$</td>
<td>$\infty \Omega$</td>
<td>$0 \Omega$</td>
<td>$Y_C = j\omega C$</td>
</tr>
</tbody>
</table>

Inductors act like short circuits under d.c. conditions and like open circuits at very high frequencies.

Capacitors act like open circuits under d.c. conditions and like short circuits at very high frequencies.
Impedance

Generic component that represents a resistor, inductor, or capacitor.

\[ Z = |Z| \angle \phi \]
\[ Z = R + jX \]
\[ |Z| = \sqrt{R^2 + X^2} \]
\[ \phi = \tan^{-1}(X/R) \]

\[ R = |Z| \cos(\phi) \]
\[ X = |Z| \sin(\phi) \]
Admittance

\[ Y = 1/Z = \frac{1}{R + jX} \]

\[ G = \frac{R}{R^2 + X^2} \]

\[ B = \frac{-X}{R^2 + X^2} \]

\[ Y = |Y| \angle \gamma \]

\[ Y = G + jB \]

\[ |Y| = \sqrt{G^2 + B^2} \]

\[ \gamma = \tan^{-1}(B/G) \]

\[ G = |Y| \cos(\gamma) \]

\[ B = |Y| \sin(\gamma) \]
Summary

- Ohm’s Law can be used to determine the ac voltages and currents in a circuit when impedance or admittance are used.
  - A resistor’s voltage and current are in phase.
  - Voltage leads current through an inductor by $90^\circ$.
  - Current leads voltage through a capacitor by $90^\circ$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Impedance</th>
<th>Admittance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$Z_R$</td>
<td>$R$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$Z_C$</td>
<td>$-j/\omega C$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$Z_L$</td>
<td>$j\omega L$</td>
</tr>
</tbody>
</table>