Ohm’s Law with Series and Parallel Combinations

Impedance and Admittance
Objective of Lecture

- Derive the equations for equivalent impedance and equivalent admittance for a series combination of components.
- Derive the equations for equivalent impedance and equivalent admittance for a parallel combination of components.

Chapter 6.3 Basic Engineering Circuit Analysis by Irwin and Nelms
Ohm’s Law in Phasor Notation

\[ V = I \frac{Z}{Y} \]
\[ I = \frac{V}{Z} \]
\[ V = I \frac{1}{Y} \]
\[ I = \frac{V}{Y} \]
Series Connections

Using Kirchhoff’s Voltage Law:
\[ V_1 + V_2 - V_s = 0 \]

Since \(Z_1, Z_2,\) and \(V_s\) are in series, the current flowing through each component is the same.

Using Ohm’s Law:
\[ V_1 = I Z_1 \quad \text{and} \quad V_2 = I Z_2 \]

Substituting into the equation from KVL:
\[ I Z_1 + I Z_2 - V_s = 0V \]
\[ I (Z_1 + Z_2) = V_s \]
Equivalent Impedance: Series Connections

We can replace the two impedances in series with one equivalent impedance, $Z_{eq}$, which is equal to the sum of the impedances in series.

$$Z_{eq} = Z_1 + Z_2$$

$$V_s = Z_{eq} I$$
Parallel Connections

Using Kirchoff’s Current Law,
\[ I_1 + I_2 - I_S = 0 \]

Since \( Z_1 \) and \( Z_1 \) are in parallel, the voltage across each component, \( V \), is the same.

Using Ohm’s Law:
\[ V = I_1 Z_1 \]
\[ V = I_2 Z_2 \]

\[ \frac{V}{Z_1} + \frac{V}{Z_2} = I_S \]
\[ I_S \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1} = V \]
Equivalent Impedance: Parallel Connections

We can replace the two impedances in series with one equivalent impedance, $Z_{eq}$, where $1/Z_{eq}$ is equal to the sum of the inverse of each of the impedances in parallel.

$$1/Z_{eq} = 1/Z_1 + 1/Z_2$$

Simplifying (only for 2 impedances in parallel)

$$Z_{eq} = Z_1Z_2 / (Z_1 + Z_2)$$
Shorthand for Parallel Connections

- An abbreviated means to show that $Z_1$ is in parallel with $Z_2$ is to write $Z_1 \parallel Z_2$. 
If you used Y instead of Z

- In series:
  The reciprocal of the equivalent admittance is equal to the sum of the reciprocal of each of the admittances in series

In this example
\[ 1/Y_{eq} = 1/Y_1 + 1/Y_2 \]

Simplifying
(only for 2 admittances in series)
\[ Y_{eq} = Y_1 Y_2 / (Y_1 + Y_2) \]
If you used Y instead of Z

- In parallel:
  The equivalent admittance is equal to the sum of all of the admittance in parallel

In this example:

\[ Y_{eq} = Y_1 + Y_2 \]
Example 1

\[ V_1 = 12V \sin(100t + 30^\circ) \]

\[ L_1 = 10\text{mH} \]

\[ R_1 = 10 \]
Impedance

\[ Z_R = 10 \, \Omega \]
\[ Z_L = j\omega L = j(100)(10\text{mH}) = 1j \, \Omega \]
\[ Z_{eq} = Z_R + Z_L = 10 +1j \, \Omega \, \text{(rectangular coordinates)} \]

In Phasor notation:

\[ Z_{eq} = (Z_R^2 + Z_L^2)^{\frac{1}{2}} \angle \tan^{-1}(\text{Im}/\text{Re}) \]
\[ Z_{eq} = (100 + 1)^{\frac{1}{2}} \angle \tan^{-1}(1/10) = 10.05 \angle 5.7^\circ \, \Omega \]
\[ Z_{eq} = 10.1 \angle 5.7^\circ \, \Omega \]

Impedances are easier than admittances to use when combining components in series.
Solve for Current

Express voltage into cosine and then convert a phasor.

\[ V_1 = 12V \cos (100t + 30^\circ - 90^\circ) = 12V \cos (100t - 60^\circ) \]

\[ V_1 = 12 \angle -60^\circ \, V \]
Solve for Current

\[ I = \frac{V}{Z_{eq}} = \frac{(12 \angle -60^\circ \text{ V})}{(10.1 \angle 5.7^\circ \Omega)} \]

\[ V = 12 \angle -60^\circ \text{ V} = 12 \text{ V} e^{-j60^\circ} \text{ (exponential form)} \]

\[ Z_{eq} = 10.1 \angle 5.7^\circ \Omega = 10.1 \Omega e^{j5.7} \text{ (exponential form)} \]

\[ I = \frac{V}{Z_{eq}} = \frac{12 \text{ V} e^{-j60^\circ}}{(10.1 e^{j5.7})} = 1.19 \text{ A} e^{-j65.7} \]

\[ I = 1.19 \text{ A} \angle -65.7^\circ \]

\[ I = \frac{V_m}{Z_m} \angle (\theta_V - \theta_Z) \]
Leading/Lagging

- $I = 1.19\, \text{A} \, e^{-j65.7^\circ} = 1.19 \angle -65.7^\circ \, \text{A}$
- $V = 12\, \text{V} \, e^{-j60^\circ} = 12 \angle -60^\circ \, \text{V}$

The voltage has a more positive angle, voltage leads the current.
Example 2
Admittance

\[ Y_R = \frac{1}{R} = 1 \ \Omega^{-1} \]

\[ Y_L = -\frac{j}{(\omega L)} = -j/[(300)(1H)] = -j \ 3.33 \ m\Omega^{-1} \]

\[ Y_C = j\omega C = j(300)(1mF) = 0.3j \ \Omega^{-1} \]

\[ Y_{eq} = Y_R + Y_L + Y_C = 1 + 0.297j \ \Omega^{-1} \]

Admittances are easier than impedances to use when combining components in parallel.
Admittances

In Phasor notation:

\[ Y_{eq} = (Y_{Re}^2 + Y_{Im}^2)^{\frac{1}{2}} \angle \tan^{-1}(\text{Im/Re}) \]

\[ Y_{eq} = (1^2 + (.297)^2)^{\frac{1}{2}} \angle \tan^{-1}(.297/1) \]

\[ Y_{eq} = 1.04 \angle 16.5^0 \Omega^{-1} \]

It is relatively easy to calculate the equivalent impedance of the components in parallel at this point as \( Z_{eq} = Y_{eq}^{-1} \).

\[ Z_{eq} = Y_{eq}^{-1} = 1/1.04 \angle 0-16.5^0 \Omega = 0.959 \angle -16.5^0 \Omega \]
Solve for Voltage

Convert a phasor since it is already expressed as a cosine.

\[ I = 4A \cos(300t - 10^\circ) \]
\[ I = 4 \angle -10^\circ \text{ A} \]
Solve for Voltage

\[ V = \frac{I}{Y_{eq}} \]

\[ V = \frac{I_m}{Y_m} \angle (\theta_I - \theta_Y) \]

\[ V = \frac{4 \angle -10^\circ \text{ A}}{(1.04 \angle 16.5^\circ \Omega^{-1})} \]

\[ V = 3.84 \angle -26.5^\circ \]

\[ V = IZ_{eq} \]

\[ V = \frac{I_m Z_m}{\angle (\theta_I + \theta_Z)} \]

\[ V = \frac{4 \angle -10^\circ \text{ A}}{(0.959 \angle -16.5^\circ \Omega^{-1})} \]

\[ V = 3.84 \angle -26.5^\circ \]
Leading/Lagging

\[ I = 4 \angle -10^\circ \text{ A} \]

\[ V = 3.84V \angle -26.5^\circ \]

Current has a more positive angle than voltage so current leads the voltage.
<table>
<thead>
<tr>
<th>Equivalent Impedances</th>
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<tbody>
<tr>
<td><strong>In Series:</strong></td>
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</tr>
<tr>
<td>$Z_{eq} = Z_1 + Z_2 + Z_3 \ldots + Z_n$</td>
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Summary

- The equations for equivalent impedance are similar in form to those used to calculate equivalent resistance and the equations for equivalent admittance are similar to the equations for equivalent conductance.
- The equations for the equivalent impedance for components in series and the equations for the equivalent admittance of components in parallel tend to be easier to use.
- The equivalent impedance is the inverse of the equivalent admittance.