Today we will discuss the z-transform, the discrete time analog of Laplace in continuous time. After today you should be able to:

1. determine the z-transform and the associated region-of-convergence using the definition
2. define the transfer function of a discrete time system using the z-transform of the impulse response
3. use the table of z-transforms and properties to determine the z-transform and its inverse

Transfer Functions in Discrete Time Systems

Let $x[n] = z^n$ for $z \in \mathbb{C}$. Then the zero-state response is

$$y[n] = h[n] * (z^n) = \sum_{m=-\infty}^{\infty} h[m]z^{n-m} = z^n \sum_{m=-\infty}^{\infty} h[m]z^{-m} = z^n H(z)$$

The factor $H(z)$ is the transfer function of the system since it specifies how to scale the input to get the output for each value of $z$. We define it to be the (bilateral) z-transform of the impulse response. For an arbitrary signal, $x[n]$, this transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Thus, substituting $x[n] = h[n]$ gives the definition of the transfer function above.

Another Perspective

We can derive the z-transform from another perspective by focusing on a causal continuous time signal, $x(t)$, sampled with period $T$ by the impulse train to give $x_d(t)$

$$x_d(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT)$$
Now let’s take the (one-sided) Laplace transform

\[ X_d(s) = \int_0^\infty \sum_{n=0}^{\infty} x(nT)\delta(t-nT)e^{-st} \, dt \]

\[ = \sum_{n=0}^{\infty} x(nT) \int_0^\infty \delta(t-nT)e^{-st} \, dt \]

\[ = \sum_{n=0}^{\infty} x(nT)e^{-snT} \]

If we define \( z = e^{sT} \) then this becomes a function of \( z \)

\[ X_d(z) = \sum_{n=0}^{\infty} x(nT)z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n} \]

This is the one-sided z-transform and is equivalent to the Laplace transform of the sampled continuous time signal. We will return to this idea a little later in the context of selecting the sampling rate.

**Region of Convergence**

As with the Laplace transform, in order for the z-transform to exist there are constraints on the allowed values of \( z \), i.e. those for which the sum converges to a finite value. This is the region of convergence or ROC.

As with Laplace the ROC is needed to fully specify the transform due to overlap in the expressions for causal and anti-causal signals. When the one-sided transform is used the signals must be causal and so the ROC need not be specified.

**Inverse z-Transform**

As you might suspect, the inverse z-transform is very similar to the inverse Laplace, and is defined using complex integration. This is beyond the scope of this course so we will use the same polynomial expansion techniques used in 2704 with a table of z-transforms and properties. This will enable use to invert transforms in a wide range of useful cases.

1. See Lathi Tables 5.1 and 5.2 on pages 498 and 514.