Today we will start our discussion of DT systems. There is a lot of material but its similarity to that in CT systems is such that I believe we can move pretty fast. After today you should be able to:

1. classify DT systems
2. describe DT systems by linear difference equations
3. solve for the zero-input response given a difference equation
4. determine the impulse response from a difference equation

Classifying DT Systems

Just as with CT systems we can classify DT systems via several attributes. In the following we represent a system producing an output $y$ from input $x$ as $x \mapsto y$.

- linearity - a DT system is linear if $x_1 \mapsto y_1, x_2 \mapsto y_2$ implies $ax_1 + bx_2 \mapsto ay_1 + by_2$ for constants $a, b$ and all $n$.

- time invariant - a system is time-invariant if $x[n] \mapsto y[n]$ implies $x[n-m] \mapsto y[n-m]$. That is delaying the input, delays the output.

- causality - a system is causal if $y[n]$ depends only on values of the input and output at indices less than the current.

- invertability - a system if invertable if $x \mapsto y \mapsto x$.

- stability - a system is stable if, initially at a fixed point, after a finite perturbation, it returns to the same fixed point. As with CT systems there are two versions internal and external stability.

- memory - a system whose output $y[n]$ depends on inputs other than $x[n]$ is said to have memory, or to be dynamic.

Difference Equations

Whereas differential equations are the canonical model of CT systems, their discrete analog, the difference equation, is used to model DT systems. The general advanced form linear, constant coefficient difference equation is given by

$$y[n+N] + a_1y[n+N-1] + \cdots + a_Ny[n] = b_{N-M}x[n+M] + b_{N-M+1}x[n+M-1] + \cdots + b_Nx[n]$$
with $N$ initial conditions $y[m + N - 1], y[m + N - 2], \ldots, y[m]$ at some starting index $m$ (often zero). Since this equation holds for all $n$, it holds for shifted versions as well. Shifting by $-N$ gives the causal form

$$y[n] + a_1 y[n - 1] + \cdots + a_N y[n - N] = b_{N-M} x[n - N + M] + b_{N-M+1} x[n - N + M - 1] + \cdots + b_N x[n - N]$$

with $N$ initial conditions $y[m - 1], y[m - 2], \ldots, y[m - N]$ at some starting index $m$ (often zero). The order of the system is $\max(N, M)$.

Difference equations lend themselves to solution via computer because the output at an index can be written as a linear combination of previously computed outputs $y$ and inputs $x$. We will also see how to solve them analytically as well. The solution of difference equations is again very similar to that of differential equations as in CT systems. Just as solving a differential equation is simplified using Laplace, we will shortly see the $z$-transform eases the solution of difference equations. Nevertheless it is instructive to see how to solve difference equations directly in the (discrete) time domain. The total solution is the sum of the zero-input and zero-state solutions.

**Zero-Input Response**

Consider the zero-input equation using the advanced form

$$y_0[n + N] + a_1 y_0[n + N - 1] + \cdots + a_N y_0[n] = 0$$

we define the advancing operator as $E^m x[n] \equiv x[n + m]$ so that now

$$E^N y_0[n] + a_1 E^{N-1} y_0[n] + \cdots + a_N y_0[n] = (E^N + a_1 E^{N-1} + \cdots + a_N) y_0[n] = 0$$

The form of the solution is $y_0[n] = c \gamma^n$, which when substituted into the above gives

$$c (\gamma^n + a_1 \gamma^{n-1} + \cdots + a_N) = 0$$

which implies the characteristic polynomial is

$$Q(\gamma) = \gamma^n + a_1 \gamma^{n-1} + \cdots + a_N$$

The roots of $Q(\gamma)$ determine the form of the solution. Assuming distinct roots, this is

$$y_0[n] = c_1 \gamma_1^n + c_2 \gamma_2^n + \cdots$$

where $\gamma_i$ is the $i$th root of $Q(\gamma)$. When there is a repeated root (of order $r$) the solution corresponding to that root is

$$y_0[n] = \cdots + (c_1 + c_2 n + \cdots c_i n^{r-1}) \gamma_i^n + \cdots$$

where the $i$th root is repeated. In both cases, the constants $c_1, c_2,$ etc are determined using the initial conditions. Note when a root is complex the solution will be complex, however it can always be written in real form using a cosine.
**Impulse Response**

To analytically solve for the zero-state solution given an arbitrary input we will need the impulse response. The difference equation with an impulse input in operator form is

\[
Q[E]h[n] = P[E]δ[n]
\]

with all initial conditions of \( h \) zero. The general form for the impulse response is

\[
h[n] = \frac{b_N}{a_N} δ[n] + c_1 γ_1^n + c_2 γ_2^n + \cdots
\]

where \( γ_i \) is the \( i^{th} \) root of the characteristic equation (the \( i^{th} \) mode of the system). The constants \( c_1, c_2 \), etc are determined using the initial conditions.