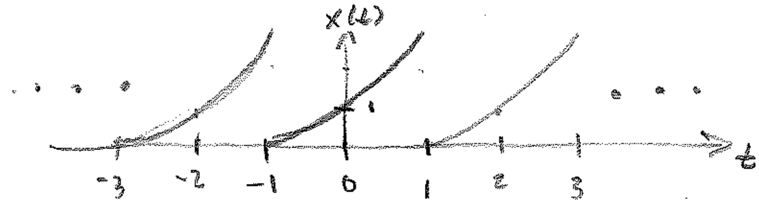
1



$T_0 = 2$

$\omega_0 = \frac{2\pi}{T_0} = \pi$

a) $a_0 = \frac{1}{2}\int_{-1}^{1}(t+1)^2\,dt = \frac{1}{2}\left[\frac{1}{3}t^3 + t^2 + t\right]\Big|_{-1}^{1}$

$= \frac{1}{2}\left[1/3 + 1 + 1\right] - \frac{1}{2}\left[-1/3 + 1 - 1\right]$

$a_0 \boxed{= \frac{4}{3}}$

$a_n = \frac{2}{2}\int_{-1}^{1}(t+1)^2\cos(n\pi t)\,dt$ From $\int$ table.

$= \frac{2\pi n\,(t+1)\cos(n\pi t) + ((t+1)^2 n^2\pi^2 - 2)\sin(n\pi t)}{n^3\pi^3}\Big|_{-1}^{1}$

$= \left[\frac{4\pi n\,\overset{(-1)^n}{\cos(n\pi)} + (4n^2\pi^2 - 2)\cancelto{0}{\sin(n\pi)}}{n^3\pi^3}\right] - \left[\frac{2\cancelto{0}{\sin(n\pi)}}{n^3\pi^3}\right]$

$a_n = \boxed{\frac{4(-1)^n}{n^2\pi^2}}$

$b_n = \frac{2}{2}\int_{-1}^{1}(t+1)^2\sin(n\pi t)\,dt$ From $\int$ table

$= \frac{2\pi n\,(t+1)\sin(n\pi t) - (n^2\pi^2(t+1)^2 - 2)\cos(n\pi t)}{n^3\pi^3}\Big|_{-1}^{1}$

$= \left[\frac{4\pi n\cancelto{0}{\sin(n\pi)} - (4n^2\pi^2 - 2)\cancelto{(-1)^n}{\cos(\pi n)}}{n^3\pi^3}\right] - \left[\frac{2\cancelto{(-1)^n}{\cos(\pi n)}}{n^3\pi^3}\right]$

$= \frac{(-1)^n}{n^3\pi^3}(-4n^2\pi^2) = \boxed{\frac{-4(-1)^n}{n\pi}} = b_n.$

① cont. b) $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

$$\underline{C_0 = a_0 = \frac{4}{3}} \qquad C_n = (a_n^2 + b_n^2)^{1/2}$$

$$= \left[\frac{4^2 \cancel{(-1)^{2n}}^{\,1}}{n^4 \pi^4} + \frac{4^2 \cancel{(-1)^{2n}}^{\,1}}{n^2 \pi^2} \right]^{1/2}$$

$$= \left[\frac{16}{n^4 \pi^4} + \frac{16}{n^2 \pi^2} \right]^{1/2}$$

$$\underline{= \frac{4}{n^2 \pi^2} \left(1 + n^2 \pi^2 \right)^{1/2}}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n} \right)$$

note, cannot cancel $(-1)^n$ terms!

$$= \tan^{-1}\left[\frac{\frac{4(-1)^n}{n\pi}}{\frac{4(-1)^n}{n^2\pi^2}} \right] = \tan^{-1}\left[\frac{(-1)^n n\pi}{(-1)^n} \right]$$

Parts c, d, see next page.

2) a) $\cos\left(\frac{3\pi}{2}t\right) = \frac{1}{2}e^{j\frac{3\pi}{2}t} + \frac{1}{2}e^{-j\frac{3\pi}{2}t}$

$$\omega_0 = \frac{3\pi}{2} \qquad T_0 = \frac{2\pi}{\omega_0} = \frac{4}{3} \qquad n = 1$$

$$D_n = \begin{cases} \frac{1}{2} & n = \pm 1 \\ 0 & \text{else} \end{cases}$$

b) $\sin(2t) = \frac{1}{2j}e^{j2t} - \frac{1}{2j}e^{-j2t}$

$$\omega_0 = 2 \qquad T_0 = \frac{2\pi}{\omega_0} = \pi \qquad n = 1$$

$$D_n = \begin{cases} -\frac{1}{2j} & n = -1 \\ \frac{1}{2j} & n = 1 \\ 0 & \text{else} \end{cases} \quad \text{or} \quad D_n = \begin{cases} -\frac{1}{2}e^{-j\pi/2} & n = -1 \\ \frac{1}{2}e^{-j\pi/2} & n = 1 \\ 0 & \text{else.} \end{cases}$$

c) $\cos(5\pi t) + \sin(7\pi t + \pi/9)$

$$\omega_1 = 5\pi \qquad \omega_2 = 7\pi$$

$$T_1 = \frac{2}{5} \qquad T_2 = \frac{2}{7}$$

GCD (14, 10)

$$\frac{2}{5}n = \frac{2}{7}m \longrightarrow \frac{14}{10} = \frac{m}{n} = \frac{7}{5} \longrightarrow T_0 = 2$$

$$\underline{\omega_0 = \frac{2\pi}{2} = \pi}$$

$$\cos(5\pi t) = \frac{1}{2}\underbrace{e^{j5\pi t}}_{n=5} + \frac{1}{2}\underbrace{e^{-j5\pi t}}_{n=-5}$$

$$\sin(7\pi t + \pi/9) = \frac{1}{2j}e^{j(7\pi t + \pi/9)} - \frac{1}{2j}e^{-j(7\pi t + \pi/9)}$$

$$= \frac{1}{2j}e^{j\pi/9}e^{j7\pi t} - \frac{1}{2j}e^{-j\pi/9}e^{-j7\pi t}$$

$$= \frac{1}{2}e^{-j\frac{7\pi}{18}}\underbrace{e^{j7\pi t}}_{n=7} - \frac{1}{2}e^{-j\frac{11\pi}{18}}\underbrace{e^{-j7\pi t}}_{n=-7}$$

$$\boxed{D_n = \begin{cases} -\frac{1}{2}e^{-j\frac{11\pi}{18}} & n = -7 \\ \frac{1}{2} & n = -5 \\ \frac{1}{2} & n = 5 \\ \frac{1}{2}e^{-j\frac{7\pi}{18}} & n = 7 \\ 0 & \text{else} \end{cases}}$$

Note: $\frac{1}{j} = e^{-j\pi/2}$

② d) $\sin(10\pi t) + 2\cos(10t)$

$\omega_1 = 10\pi \qquad \omega_2 = 10$

$T_1 = \frac{1}{5} \qquad T_2 = \frac{\pi}{5}$

$\frac{1}{5} n \neq \frac{\pi}{5} m$ for any $n, m \in \mathbb{Z}$

The signal is not periodic, No F.S. exists.

Note however the Fourier Transform does exist.

③ From the plot $T_0 = 2 \quad \omega_0 = \pi$

a) using exponential form

$$D_n = \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{0} (t+1)\, e^{-jn\pi t}\, dt + \int_{0}^{1} (-t+1)\, e^{-jn\pi t}\, dt\right]$$

$$= \underline{\frac{1-(-1)^n}{n^2\pi^2}}$$ From $\int$ table or CAS.

$D_0 = \text{Avg} = \frac{1}{2}\text{ Area } = \frac{1}{2}\left(\frac{1}{2} 2 \times 1\right) = \underline{\frac{1}{2}}$.

using trig form $b_n = 0$ since $x(t)$ is even.

$a_0 = D_0 = 1/2$

$$a_n = \frac{2}{2}\left[\int_{-1}^{0} (t+1)\cos(n\pi t)\, dt + \int_{0}^{1} (-t+1)\cos(n\pi t)\, dt\right]$$

$$= \frac{2(1-(-1)^n}{n^2\pi^2}$$ from $\int$ table or CAS.

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$

3 b)

$$H(s) = \frac{(R_1R_2C_1C_2)^{-1}}{s^2 + \frac{R_1C_1 + R_2C_2 + R_1C_2}{R_1R_2C_1C_2}s + \frac{1}{R_1R_2C_1C_2}}$$

$$= \frac{10000}{s^2 + 300s + 10000}$$ substitute values,

→ System is stable
poles at -261.8, -38.2.

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

$$= \frac{10000}{(10000-\omega^2) + j300\omega}$$

$$H(n\omega_0) = H(n\pi) = \frac{10000}{10000 - n^2\pi^2 + j300\pi n}$$

$$y(t) = \underbrace{\frac{1}{2}|H(0)|}_{1} + \sum_{n=1}^{\infty} a_n |H(n\pi)| \cos(\pi n t + \angle H(n\pi))$$

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c) See plot next page.

Observation: There is a small attenuation and phase shift, but largely the signal is unchanged.

Comment: • This makes sense given the fundamental frequency $\omega_0 = \pi$ (marked on Bode Plot) shows small attenuation & phase shift.

• If you plot for larger values of N, and zoom in near $t=0$ you can see the smoothing better.

(4) a)

$$x(t) = \lim_{a \to 0^+} \begin{cases} 2t & 0<t<1 \\ 4-2t & 1<t<2 \\ -e^{-a(t-4)} & t>4 \\ 0 & \text{else.} \end{cases} \qquad a \in \mathbb{R}^+$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_0^1 2t e^{-j\omega t}\, dt + \int_1^2 (4-2t) e^{-j\omega t}\, dt$$

$$- \lim_{a \to 0^+} \int_4^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{2\left(e^{-j\omega}(1+j\omega)-1\right)}{\omega^2} + \frac{2e^{-j2\omega}\left(e^{j\omega}(1-j\omega)-1\right)}{\omega^2}$$

$$- \lim_{a \to 0^+} e^{-j4\omega} \frac{1}{a+j\omega}$$

$$= \frac{-2(e^{j\omega}-1)^2 e^{-j2\omega}}{\omega^2} - e^{-j4\omega} \underbrace{\left[\lim_{a \to 0} \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}\right]}_{\pi\delta(\omega) + \frac{1}{j\omega}}$$

$$= \frac{-2(e^{j\omega}-1)^2 e^{-j2\omega}}{\omega^2} - e^{j4\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

b) The same approach does not work here because $r(t) = tu(t)$ is not in the table. The $\mathcal{F}\{tu(t)\}$ does exist, it can be found by taking $\lim_{a \to 0^+} \mathcal{F}\{te^{-at}u(t)\}$.

5 a) $x_1(t) = e^{-t^2}$ Table 7.1 Row 22 $\sigma = \frac{1}{\sqrt{2}}$

$$X_1(\omega) = \frac{1}{\sqrt{2}} \sqrt{2\pi}\, e^{-\sqrt{2}\,\omega^2/2} = \underline{\sqrt{\pi}\, e^{-\frac{1}{\sqrt{2}}\omega^2}}$$

b) $x_2(t) = x_1(t-2) + x_1(t+2)$

Using the shifting property.

$$X_2(\omega) = X_1(\omega)e^{-j2\omega} + X_1(\omega)e^{j2\omega}$$

$$= 2X_1(\omega)\left[\frac{1}{2}e^{-j2\omega} + \frac{1}{2}e^{j2\omega}\right]$$

Euler's formula

$$= 2X_1(\omega)\cos(2\omega)$$

$$= \underline{2\sqrt{\pi}\, e^{-\frac{1}{\sqrt{2}}\omega^2}\cos(2\omega)}$$

c) $x(t) = x_2'(t)$ Using time derivative property.

$$X(\omega) = j\omega X_2(\omega)$$

$$= \underline{j2\sqrt{\pi}\,\omega e^{-\frac{1}{\sqrt{2}}\omega^2}\cos(2\omega)}$$

d) $x(t) = x_1(t)\cos(100t)$ Using modulation (Freq shift) property.

$$x(t) = \frac{1}{2}x_1(t)e^{j100t} + \frac{1}{2}x_1(t)e^{-j100t}$$

$$X(\omega) = \frac{1}{2}X_1(\omega+100) + \frac{1}{2}X_1(\omega-100)$$

$$= \underline{\frac{\sqrt{\pi}}{2}e^{-\frac{1}{\sqrt{2}}(\omega+100)^2} + \frac{\sqrt{\pi}}{2}e^{-\frac{1}{\sqrt{2}}(\omega-100)^2}}$$

(6) This is similar to 5d, in reverse.

$$X(\omega) = X_1(\omega-5) + X_1(\omega+5)$$

$$X_1(\omega) = e^{-\omega^2}$$

Using Table 7.1 Row 22 $\quad -\frac{\sigma^2}{2} = -1 \quad \sigma = \sqrt{2}$

$$x_1(t) = \frac{1}{2\sqrt{\pi}} e^{-t^2/4}$$

Then by Frequency shift property.

$$x(t) = x_1(t)e^{j5t} + x_2(t)e^{-j5t}$$

$$= \frac{2}{2\sqrt{\pi}} x_1(t)\cos(5t) = \frac{1}{\sqrt{\pi}} e^{-t^2/4}\cos(5t)$$

(7) a) $\quad x(t) = x_1(t)\, x_2(t)$

$X_1(\omega) = 2\tau \operatorname{sinc}(\omega\tau)$ Table 7.1 Row 17

$X_2(\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ Row 9

Multiplication in time domain is convolution in the frequency domain

$$X(\omega) = \frac{1}{2\pi} X_1(\omega) X_2(\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega')X_2(\omega-\omega')\,d\omega'$$

$$= \tau\int_{-\infty}^{\infty} \operatorname{sinc}(\omega'\tau)\,\delta(\omega'-\omega_0)\,d\omega'$$

$$+ \tau\int_{-\infty}^{\infty} \operatorname{sinc}(\omega'\tau)\,\delta(\omega'+\omega_0)\,d\omega'$$

by sifting theorem

$$= \tau\operatorname{sinc}(\tau(\omega-\omega_0)) + \tau\operatorname{sinc}(\tau(\omega+\omega_0))$$

(7) b) If the system is stable it has a freq response $H(\omega)$.

If $\tau \to \infty$ then the response is

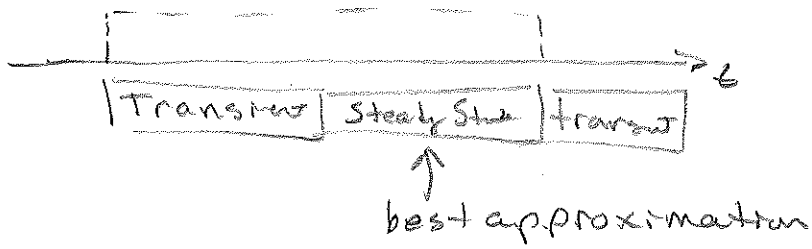
$$|H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$

c) for large τ relative to the period $\frac{2\pi}{\omega_0}$

so that

$$\operatorname{sinc}(\tau(\omega \mp \omega_0)) \approx \delta(\omega \pm \omega_0)$$

d) After transients have died away



(8) a)

$$H(\omega) = H(s)\Big|_{s=j\omega} \quad \text{since system is stable (poles at } -4, -1)$$

$$= \frac{4}{4-\omega^2+j5\omega}$$

$$X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \qquad \text{Row 11 Table 7.1}$$

$$Y(\omega) = H(\omega)X(\omega) = \frac{4}{(j\omega)(4-\omega^2+j5\omega)} + \pi\delta(\omega)$$

b)

$$Y(\omega) = \frac{\cancel{A}^{\,1}}{j\omega} + \frac{\cancel{B}^{\,-4/3}}{1+j\omega} + \frac{\cancel{C}^{\,1/3}}{4+j\omega} + \pi\delta(\omega)$$

$$y(t) = u(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{-4t}u(t)$$

8 c)

$$Y(\omega) = H(\omega) X(\omega) \qquad X(\omega) = \text{sinc}\left(\tfrac{\omega}{2}\right)$$

$$= \frac{4\,\text{sinc}(\omega/2)}{(4-\omega^2) + j5\omega}$$

d) $$y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t}\, d\omega$$

Note: $$\text{sinc}(\omega/2) = \frac{\sin(\omega/2)}{\omega/2} = \frac{1}{j\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

thus $$Y(\omega) = \frac{1}{(j\omega)(4-\omega^2+j5\omega)} e^{j\omega/2} - \frac{1}{(j\omega)(4-\omega^2+j5\omega)} e^{-j\omega/2}$$

time shift by $-1/2$ time shift by $1/2$

using PFE $$\frac{1}{(j\omega)(4-\omega^2+j5\omega)} = \frac{A}{j\omega} + \frac{B}{1+j\omega} + \frac{C}{4+j\omega}$$

$A = 1 \quad B = -4/3 \quad C = 1/3$

The inverse Fourier of this is

$$\frac{1}{2}\,\text{sgn}(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{-4t} u(t)$$

Applying the shifts

$$y(t) = \frac{1}{2}\,\text{sgn}(t+\tfrac{1}{2}) - \frac{4}{3} e^{-(t+1/2)} u(t+\tfrac{1}{2}) + \frac{1}{3} e^{-4(t+1/2)} u(t+1/2)$$
$$- \frac{1}{2}\,\text{sgn}(t-1/2) + \frac{4}{3} e^{-(t-1/2)} u(t-1/2) - \frac{1}{3} e^{-4(t-1/2)} u(t-1/2)$$

(8) e) When $x(t) = u(t)$ the Laplace version is straightforward

$$Y(s) = \frac{4}{s(s^2+5s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$y(t) = (A + Be^{-t} + Ce^{-4t})\, u(t)$$

same as 8 b.

When $x(t) = \sqcap(t)$ we could take 2 approaches

1. use Bilateral Laplace (General approach)
2. <u>since $x(t) = 0$</u> prior to $t = -1/2$ we can first shift $x(t)$ by $1/2$ so that it is causal, solve using laplace to get the response, then shift back by $1/2$ (we can do this because of <u>time invariance</u>).

Lets take the 2nd (simpler) approach

Define $x_1(t)$ as $x(t - 1/2)$ Thus

$$x_1(t) = u(t) - u(t-1)$$

$$Y_1(s) = H(s)X_1(s) = \frac{4}{s(s+1)(s+4)} - \frac{4}{s(s+1)(s+4)}e^{-s}$$

$$Y_1(s) = \left(\frac{1}{s} + \frac{-4/3}{s+1} + \frac{1/3}{s+4}\right) - \left(\frac{1}{s} + \frac{-4/3}{s+1} + \frac{1/3}{s+4}\right)e^{-s}$$

$$y_1(t) = \left(1 - 4/3\, e^{-t} + 1/3\, e^{-4t}\right)u(t)$$
$$- \left(1 - 4/3\, e^{-t} + 1/3\, e^{-4t}\right)u(t-1)$$

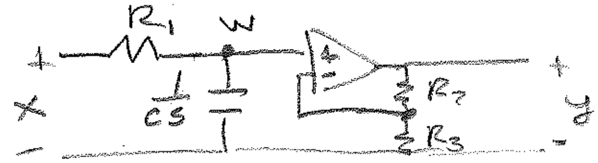
Now $y(t) = y_1(t + 1/2)$

$$y(t) = \left(1 - \tfrac{4}{3}e^{-t} + 1/3\, e^{-4t}\right)u(t - 1/2)$$
$$- \left(1 - 4/3\, e^{-t} + 1/3\, e^{-4t}\right)u(t + 1/2)$$

same as 8 c.

Note $\frac{1}{2}\mathrm{sgn}(t+\frac{1}{2}) - \frac{1}{2}\mathrm{sgn}(t - 1/2) = u(t - 1/2) - u(t + 1/2)$

(9) a)



KCL @ w: $\frac{x-w}{R_1} = Csw$ KVL @ output: $\frac{y-w}{R_2} = \frac{w}{R_3}$

$x - w = R_1 C s w$ $y - w = \frac{R_2}{R_3} w$

$x = (1 + R_1 C s) w$ $y = \left(1 + \frac{R_2}{R_3}\right) w$

$$H(s) = \frac{y}{w} \cdot \frac{w}{x} = \left(1 + \frac{R_2}{R_3}\right)\left(\frac{1}{1+R_1 C s}\right) = \left(1 + \frac{R_2}{R_3}\right)\frac{(R_1 C)^{-1}}{s + (R_1 C)^{-1}}$$

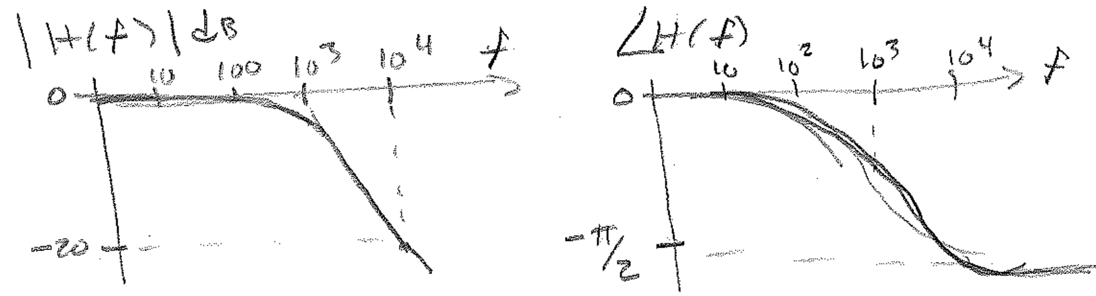
$$= \frac{K}{s+a}$$

$a = 1000 \rightarrow R_1 C = 1000$
Let $C = 1\mu F$ $R_1 = 1K$

$K = 0$ @ DC $\rightarrow H(0) = 1$
requires $R_2 = 0$ $R_3 = \infty$ (open).

Thus $R_1 = 1K$ $C = 1\mu F$, $R_2 = 0$, $R_3 = \infty$

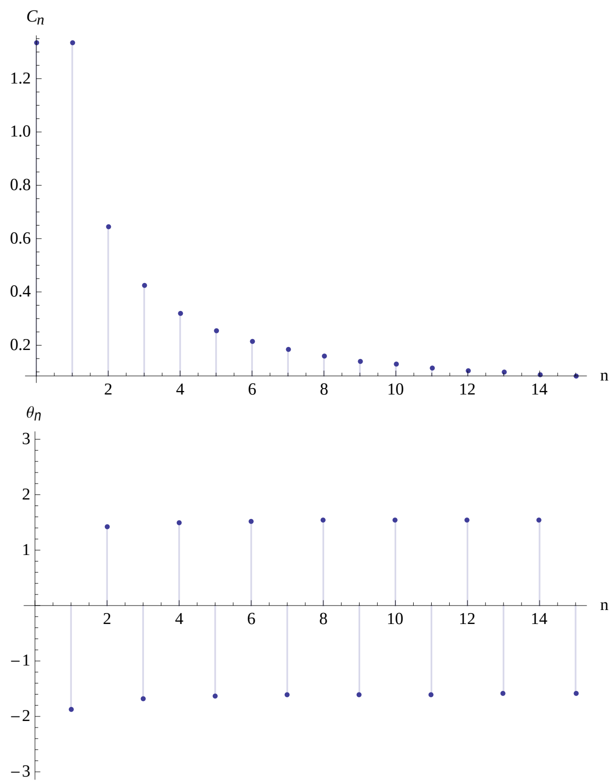
b)



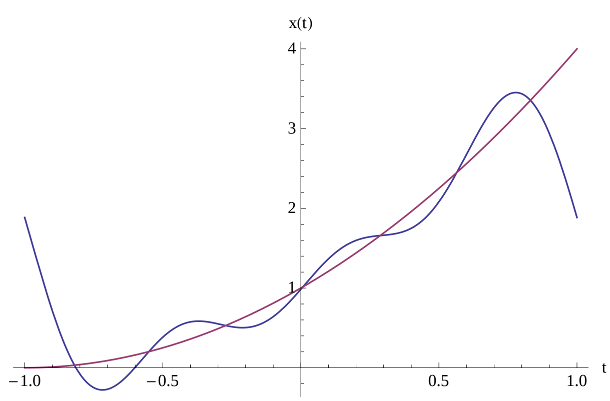
c) When you did this it should have been
d) impossible to read ~~[illegible]~~ $|H(f)|$ $\angle H(f)$
for $f > 1$ kHz

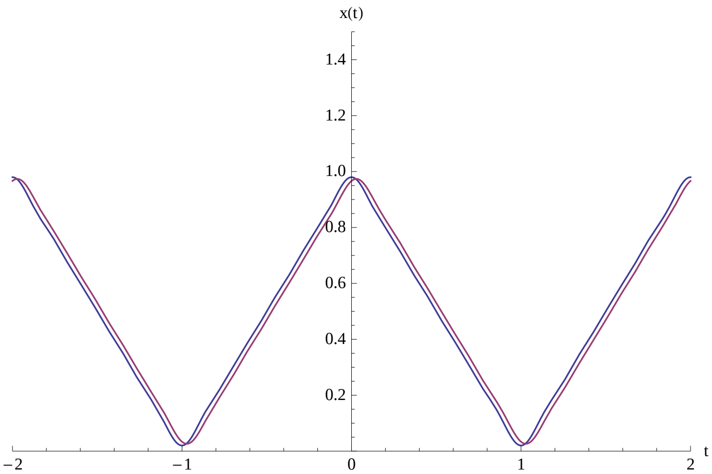
otherwise is should match up reasonably well.

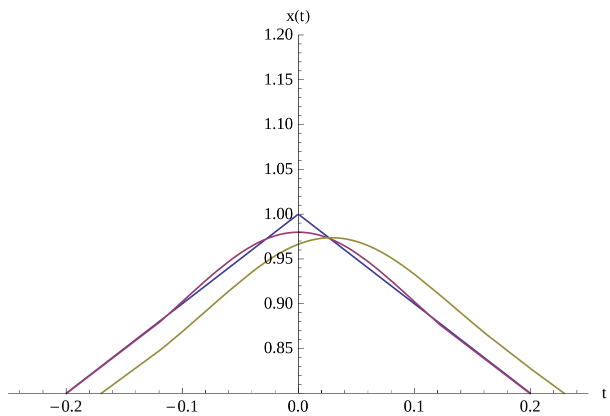
c)

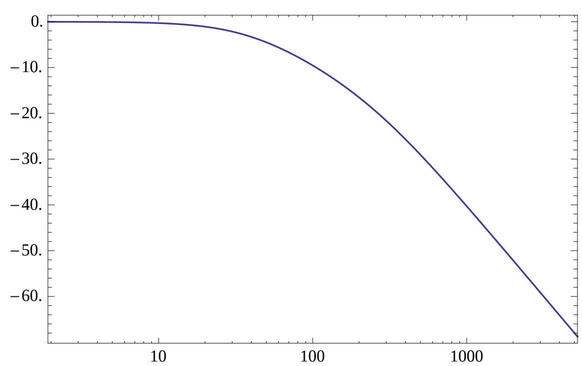

C_n
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
12
14
n
θ_n
3
2
1
−1
−2
−3

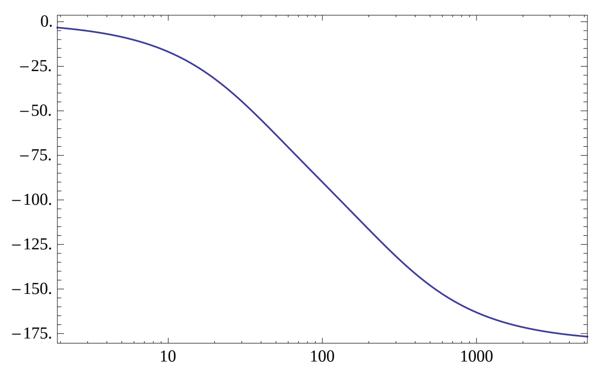
d)


x(t)
4
3
2
1
−1.0
−0.5
0.5
1.0
t


x(t)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
t
−2
−1
0
1
2


x(t)
1.20
1.15
1.10
1.05
1.00
0.95
0.90
0.85
t
−0.2
−0.1
0.0
0.1
0.2


0.
−10.
−20.
−30.
−40.
−50.
−60.
10
100
1000


0.
−25.
−50.
−75.
−100.
−125.
−150.
−175.
10
100
1000