* Review of Fourier Methods.

Summary:

- Periodic functions can (in most cases) be represented by the Fourier Series.

Exponential Form: $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ $\omega_0$ = fundamental frequency. $= \frac{2\pi}{T_0}$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\,dt$$

Trig Form: $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

Compact Trig Form:

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$C_0 = a_0 = D_0 \quad C_n = (a_n^2 + b_n^2)^{1/2} \quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

- The Fourier transform Pair is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\,d\omega$$

exists for only some signals.

- Convolution Property: $Y(\omega) = H(\omega)\, X(\omega)$

- When ROC includes $j\omega$-Axis $X(s)\big|_{s=j\omega} = X(\omega)$

In particular when $H(s)$ is Stable

$H(\omega) = H(s)\big|_{s=j\omega}$ the Frequency Response.

– Example 1.

$$x(t) = 2\cos(10\pi t) + \sin(19\pi t)$$

a) What is F.S. in exponential form.

$\omega_0 = \pi \qquad 10\omega_0 = 10\pi \qquad 19\omega_0 = 19\pi$

$$x(t) = 2\left(\tfrac{1}{2}e^{j10\pi t} + \tfrac{1}{2}e^{-j10\pi t}\right) + \frac{1}{2j}\left(e^{j19\pi t} - e^{-j19\pi t}\right)$$

Compare to $\displaystyle x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$

$$D_n = \begin{cases} 1 & n = \pm 10 \\ \frac{1}{2j} & n = 19 \\ \frac{-1}{2j} & n = -19 \\ 0 & \text{else} \end{cases}$$

b) What is the system response when $H(s) = \dfrac{s}{s^2+3s+2}$
$(s+2)(s+1)$

poles at $-1, -2$ $\therefore$ Stable

$$H(\omega) = \frac{j\omega}{-\omega^2+2+j3\omega}$$

$$|H(\omega)| = \frac{|j\omega|}{|2-\omega^2+j3\omega|} = \frac{\omega}{((2-\omega^2)^2+9\omega^2)^{1/2}}$$

$$\angle H(\omega) = \underbrace{\tan^{-1}\left(\frac{\omega}{0}\right)}_{\pm\pi/2} - \tan^{-1}\left(\frac{3\omega}{2-\omega^2}\right)$$

$$y(t) = 2|H(10\pi)|\cos(10\pi t + \angle H(10\pi)) + |H(19\pi)|\sin(19\pi t + \angle H(19\pi))$$

or equivalently

$$y(t) = \sum_{n=-\infty}^{\infty} D_n |H(n\pi)| e^{jn\pi t + \angle H(n\pi)}$$

— Example #2.

$$\mathcal{F}\{(1+te^{-t})\cos(2\pi t)\}$$

$$\mathcal{F}\{\cos(2\pi t) + te^{-t}\cos(2\pi t)\}$$

$$\mathcal{F}\{\cos(2\pi t)\} + \underbrace{\mathcal{F}\{te^{-t}\cos(2\pi t)\}}_{\text{does not exist}}$$

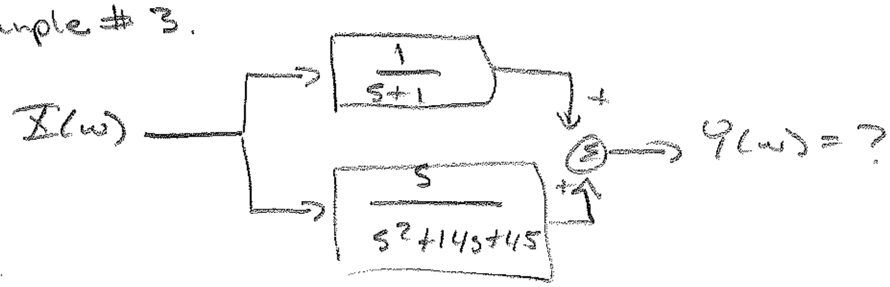
Note: What if it was $(1+te^{-t})\cos(2\pi t)u(t)$

Then From table Row 13 + 16.

$$X(\omega) = \frac{\pi}{2}\left[\delta(\omega-2\pi)+\delta(\omega+2\pi)\right] + \frac{j\omega}{4\pi^2-\omega^2} +$$

$$\frac{1+j\omega}{(1+j\omega)^2+4\pi^2}$$

— Example #3.




$$H(s) = \frac{1}{s+1} + \frac{s}{s^2+14s+45} = \frac{s^2+14s+45+s^2+s}{(s+1)(s^2+14s+45)}$$

$$= \frac{2s^2+15s+45}{(s+1)(s^2+14s+45)}$$

Poles at $-1, -5, -9$

$\therefore$ Stable.

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{45-2\omega^2+j15\omega}{(1+j\omega)(45-\omega^2+j14\omega)}$$

$$x(t) = \sin(2\pi t) = \frac{2\pi}{\pi}\cdot\left(\frac{\pi}{2\pi}\right)\operatorname{sinc}(2\pi t)$$

$$X(\omega) = \frac{1}{2}\,\Pi\left(\frac{\omega}{4\pi}\right)$$

$$Y(\omega) = H(\omega)\,X(\omega) = \frac{1}{2}\,\Pi\left(\frac{\omega}{4\pi}\right)\frac{45-2\omega^2+j15\omega}{(1+j\omega)(45-\omega^2+j14\omega)}$$

Example # 4. Given Bode Plot what is response due to

a) $x(t) = 1 = \cos(0t)$ aka DC.

$|H(0)| \approx 12\,dB$ $\qquad \angle H(0) \approx 0$ rad

$10^{\frac{12}{20}} \approx 4$

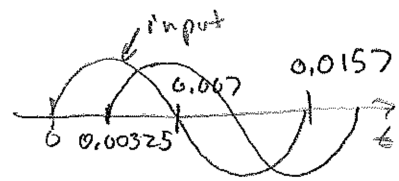
$y(t) \approx 4$.

b) $x(t) = \sin(400t)$

$|H(400)| \approx 0\,dB$ $\qquad \angle H(400) \approx -1.3$ rad

$10^{0/20} = 1$

$y(t) \approx \sin(400t - 1.3)$




$\sin(400t - 400t_0)$ $\qquad 400t_0 = 1.3$

$t_0 \approx 0.00325$

$T_0 = \frac{2\pi}{400} = 0.0157$

c) $x(t) = \cos(10{,}000t)$

$|H(10{,}000)| \approx -27\,dB$ $\qquad \angle H(10{,}000) \approx -1.5$

$10^{-27/20} \approx 0.044$

$y(t) = 0.044\cos(10{,}000t - 1.5)$