* Laplace Domain Techniques.

Summary:

- The Laplace transform maps time domain signals to a signal that is a function of $s \in \mathbb{C}$.

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt.$$

- Given a system description $Q(D)y(t) = P(D)x(t)$ or an impulse response $h(t)$, the system can be described by the Transfer Function

$$H(s) = \mathcal{L}\{h(t)\} \quad \text{or} \quad H(s) = \frac{P(s)}{Q(s)}$$

- The zero-state response is the product of $H(s)$ and $X(s)$

$$Y(s) = H(s)X(s)$$

- The system is BIBO stable if poles of TF are in the left hand of complex plane.

- The system is internally stable if roots of $Q(s)$ are in left hand of complex plane.

- The response to a stable system is

$$Y(s) = Y_0(s) + Y_{zs}(s)$$
$$= Y_0(s) + Y_{transient}(s) + Y_{ss}(s)$$

- Systems can be realized from $H(s)$ using multipliers, summers, and integrators.

- <u>If the system is stable</u> $H(s)$ can be converted to the frequency Response

$$H(s)\big|_{s=j\omega} = H(\omega)$$

- This can be plotted as a Bode Plot.
- This can be used to find the sinusoidal response or sinusoidal steady-state response.

Example #1

Given $y'' + 8y' + 15y = \cos(2\pi t)u(t)$
with $y(0^-) = 0$, $y'(0^-) = 1$ find total
response $y(t)$ using Laplace.

- Take Laplace Transform of Diff Eq.

$$\left[s^2Y(s) - s\cancel{y(0^-)}^{0} - \cancel{y'(0^-)}^{1}\right] + 8\left[sY(s) - \cancel{y(0^-)}^{0}\right] + 15Y(s)$$
$$= \frac{s}{s^2+4\pi^2}$$

- Solve for $Y(s)$:

$$(s^2+8s+15)Y(s) = \frac{s}{s^2+4\pi^2} + 1$$

$$Y(s) = \frac{s}{(s^2+8s+15)(s^2+4\pi^2)} + \frac{1}{s^2+8s+15}$$

- Take Inverse Laplace. $(s^2+8s+15) = (s+5)(s+3)$

$$Y(s) = \frac{A}{s+5} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4\pi^2} + \frac{E}{s+5} + \frac{F}{s+3}$$

$$A = \left.\frac{s}{(s+3)(s^2+4\pi^2)}\right|_{s=-5} = \frac{-5}{(-2)(25+4\pi^2)} = \frac{5}{2(25+4\pi^2)}$$

$$B = \left.\frac{s}{(s+5)(s^2+4\pi^2)}\right|_{s=-3} = \frac{-3}{(2)(9+4\pi^2)}$$

to find C & D: $A(s+3)(s^2+4\pi^2) +$
$B(s+5)(s^2+4\pi^2) +$
$(Cs+D)(s^2+8s+15) = s$

$A(s^3 + 4\pi^2 s + 3s^2 + 12\pi^2) +$

$B(s^3 + 4\pi^2 s + 5s^2 + 20\pi^2) +$

$Cs^3 + 8Cs^2 + 15Cs + Ds^2 + 8Ds + 15D = s$

- Equating coefficients

$$A + B + C = 0$$

$$3A + 5B + 8C + D = 0$$

$$4\pi^2 A + 4\pi^2 B + 15C + 8D = 1$$

$$12\pi^2 A + 20\pi^2 B + 15D = 0$$

or $C = -(A+B) = \dfrac{-5}{2(25+4\pi^2)} + \dfrac{3}{2(9+4\pi^2)}$

$$D = \frac{-12\pi^2 A - 20\pi^2 B}{15} = \frac{\dfrac{-12\pi^2 \cdot 5}{2(25+4\pi^2)} + \dfrac{20\pi^2 \times 3}{2(9+4\pi^2)}}{15}$$

- Now to find E, F

$$E = \left.\frac{1}{s+3}\right|_{s=-5} = \frac{-1}{2} \qquad F = \left.\frac{1}{s+5}\right|_{s=-3} = \frac{1}{2}$$

- Now

$$y(t) = Ae^{-5t}u(t) + Be^{-3t}u(t) + C\cos(2\pi t)u(t) + \frac{D}{2\pi}\sin(2\pi t)u(t) - \frac{1}{2}e^{-5t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

- Example #2 $\quad H(s) = \dfrac{1}{s^2+8s+12}$

a) Is the system stable, in what sense?

we can only determine BIBO stability.

$(s^2+8s+12) = (s+6)(s+2)$

both poles in LHP, $\therefore$ BIBO stable.

Example #2 b) What is the response in the Laplace domain to

$$x(t) = e^{-t}u(t)$$

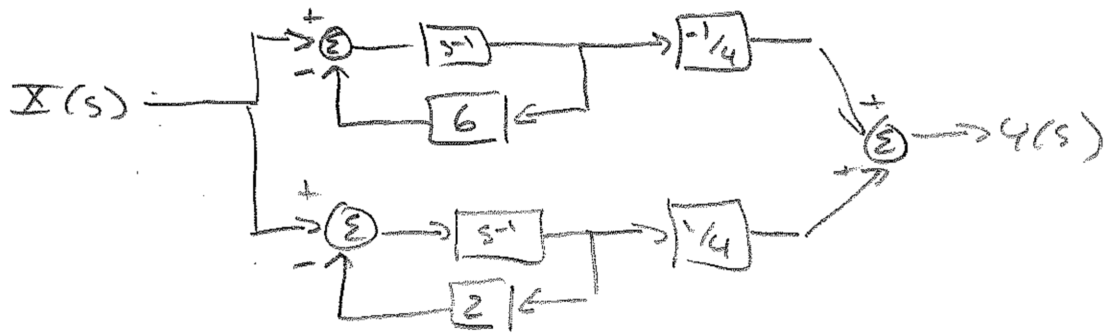
$$Y(s) = H(s)X(s) \qquad X(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s^2+8s+12)}$$

- Example #3, Realize previous $H(s)$ in parallel form if possible.

$$H(s) = \frac{A}{s+6} + \frac{B}{s+2} \qquad A = \left.\frac{1}{s+2}\right|_{s=-6} = -\frac{1}{4}$$

$$= \frac{-\frac{1}{4}s^{-1}}{1+6s^{-1}} + \frac{\frac{1}{4}s^{-1}}{1+2s^{-1}} \qquad B = \left.\frac{1}{s+6}\right|_{s=-2} = \frac{1}{4}$$



- Example #4 : $$H(s) = \frac{\frac{1}{s(s+2)}}{1+\frac{1}{s(s+2)}} = \frac{1}{s(s+2)+1} = \frac{1}{s^2+2s+1}$$

- Example #5: Suppose $H(s) = \frac{6}{s+3}$

What is the steady-state response due to.

$$x(t) = \cos(2t)\,u(t) + 2\sin(3t)\,u(t)$$

- The system is stable pole at -3
- The Frequency response $H(\omega) = \frac{6}{3+j\omega}$.

$$|H(\omega)| = \frac{6}{(9+\omega^2)^{1/2}} \qquad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{3}\right)$$

$$\underline{y_{ss}(t)} = |H(2)|\cos(2t + \angle H(2)) + 2|H(3)|\sin(3t + \angle H(3))$$

$$|H(2)| = \frac{6}{(9+4)^{1/2}} \qquad \angle H(2) = -\tan^{-1}\left(\frac{2}{3}\right)$$

$$|H(3)| = \frac{6}{(9+9)^{1/2}} \qquad \angle H(3) = -\tan^{-1}\left(\frac{3}{3}\right)$$