* LTI System Response in Fourier (real frequency) domain

- The convolution property of Fourier Transforms gives us the ability to find the response to a given input.

Time Domain:

$$\begin{array}{ccccc} x(t) & \longrightarrow & \boxed{h(t)} & \longrightarrow & y(t) = (x*h)(t) \\ \mathcal{F}\downarrow & & \mathcal{F}\downarrow \text{ (if stable)} & & \uparrow \mathcal{F}^{-1} \\ X(\omega) & \longrightarrow & \boxed{H(\omega)} & \longrightarrow & Y(\omega) = H(\omega)X(\omega) \\ & & \uparrow s = j\omega & & \\ & & \boxed{H(s)} & & \end{array}$$

- An advantage of Fourier is the domain is one-dimensional $\omega \in \mathbb{R}$ vs $s \in \mathbb{C}$

  - We can express Fourier Transforms in polar form.

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)}$$

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

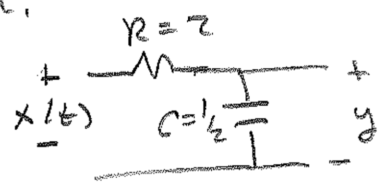
$$Y(\omega) = |Y(\omega)|\, e^{j\angle Y(\omega)}$$

So that,

$$Y(\omega) = |Y(\omega)|\, e^{j\angle Y(\omega)} = |H(\omega)|\,|X(\omega)|\, e^{j\left(\angle H(\omega) + \angle X(\omega)\right)}$$

  - This makes it easy to see how $H(\omega)$ "shapes" the input spectrum to produce the output spectrum.

- Example



Let $x(t) = e^{-2t}u(t)$ Input

$$X(\omega) = \frac{1}{2+j\omega}$$

$$|X(\omega)| = \frac{1}{(4+\omega^2)^{1/2}} \qquad \angle X(\omega) = \tan^{-1}\left(\frac{-\omega}{2}\right)$$

System

$$H(s) = \frac{1}{s+1}$$

$$H(\omega) = \frac{1}{1+j\omega}$$

$$|H(\omega)| = \frac{1}{(1+\omega^2)^{1/2}}$$

$$\angle H(\omega) = \tan^{-1}\left(\frac{-\omega}{1}\right)$$

Output: $Y(\omega) = H(\omega)X(\omega) = \dfrac{1}{(2+j\omega)(1+j\omega)} = \dfrac{\cancel{A}\ {-1}}{2+j\omega} + \dfrac{\cancel{B}\ 1}{1+j\omega}$

$$y(t) = -e^{-2t}u(t) + e^{-t}u(t)$$

$\boxed{\text{See Plots}}$

- Experimental Measurement of the Frequency Response.

$$x(t) \longrightarrow \boxed{?} \longrightarrow y(t)$$

- if the unknown system is LTI, consider the response to inputs.

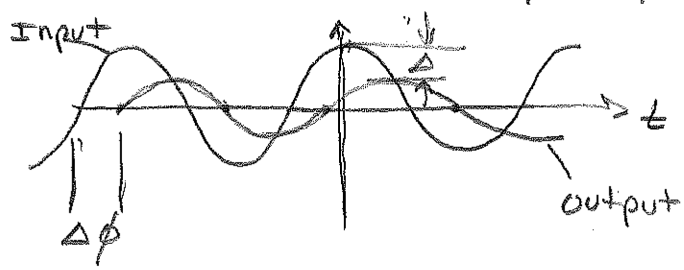
"Frequency Sweep"

$$x(t) = \cos(\omega_0 t) \quad \omega_0 = 0, 1, 10, 100, 1000, \text{etc.}$$

$$x(t) = \cos(0t) = 1 \rightarrow \boxed{?} \rightarrow |H(0)|$$

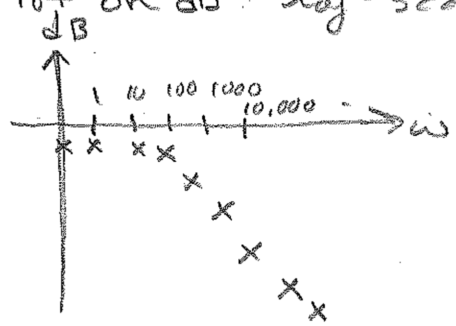
$$x(t) = \cos(t) \rightarrow \boxed{?} \rightarrow |H(1)|\cos(t + \angle H(1))$$

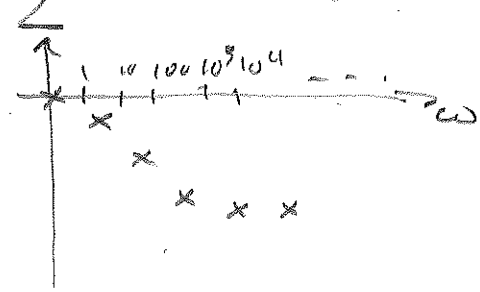
$$\cos(10t) \rightarrow \boxed{?} \rightarrow |H(10)|\cos(10t + \angle H(10))$$



- at a given frequency, <u>measure</u> Δ, $\Delta\phi$

Plot on dB - log-scale. & radians - log-scale





☆ This is the experimental equivalent of the Bode Plot of $H(\omega)$. ☆

$\boxed{\text{DEMO}}$

* Numerically computing the Fourier Transform
The Discrete Fourier transform (DFT).

- For many real-world signals we don't have an analytic form for $x(t)$.

  e.g. $x(t) =$ microphone output

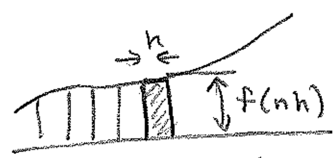
- How can we compute the Fourier Transform? AND it's inverse?

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad \text{Note } e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$= \underbrace{\int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt}_{\substack{X_R(\omega) \\ \text{Real Part of } X(\omega)}} + j \underbrace{\int_{-\infty}^{\infty} -x(t)\sin(\omega t)\, dt}_{\substack{X_I(\omega) \\ \text{Imaginary Part of } X(\omega)}}$$

- Now we approximate the integral.

$$\int_{-\infty}^{\infty} f(t)\, dt = \lim_{h \to 0} \sum_{n=-\infty}^{\infty} h\, f(nh)$$



Let $h = \Delta t$ a small increment.

$$\approx \sum_{n=-\infty}^{\infty} \Delta t\, f(n\Delta t)$$

- Now suppose $f(t) \approx 0$ for $L < t < U$

$$\int_{-\infty}^{\infty} f(t)\, dt \approx \int_{L}^{U} f(t)\, dt \qquad n\Delta t = L, \quad n = \left\lfloor \frac{L}{\Delta t} \right\rfloor$$

$$\text{Thus} \quad \int_{-\infty}^{\infty} f(t)\, dt \approx \sum_{n=L}^{U} \Delta t\, f(n\Delta t) \qquad n\Delta t = U, \quad n = \left\lceil \frac{U}{\Delta t} \right\rceil$$

- For the Fourier transform.

$$X_R(\omega) \approx \sum_{n=\lfloor L/\Delta t \rfloor}^{\lceil U/\Delta t \rceil} \Delta t \, x(n\Delta t) \cos(\omega n \Delta t)$$

$$X_I(\omega) \approx \sum_{n=\lfloor L/\Delta t \rfloor}^{\lceil U/\Delta t \rceil} -\Delta t \, x(n\Delta t) \sin(\omega n \Delta t)$$

This gives an algorithm.
Input: $x(t)$, L, U, Δt, ω
output $X_r(\omega)$, $X_I(\omega)$

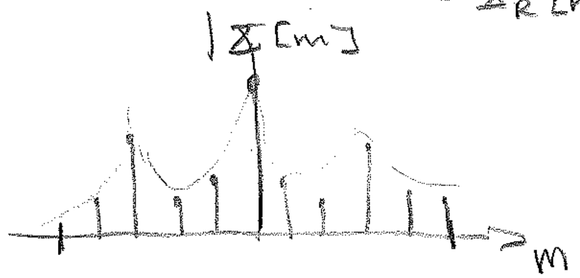
- If we sample the time axis. $x(n\Delta t) = x[n]$ $\quad n = 1 \dots N$
- We can also sample the ω-Axis $\omega = m\Delta\omega$ ↓ frequency sample rate,
  to get $X(m\Delta\omega) = X[m]$

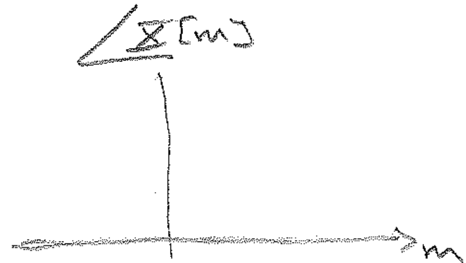
- The resulting algorithm is.

```
For m = -M to M
    X_R[m] = 0; X_I[m] = 0
    for n = 1 to N
        X_R[m] += Δt x[n] cos(m n Δω Δt)
        X_I[m] += -Δt x[n] sin(m n Δω Δt)
    end for
end for.
```

- Need to choose Δt and $\Delta\omega$.
- $|X[m]| = \left(X_R[m]^2 + X_I[m]^2\right)^{1/2}$

$$\angle X[m] = \tan^{-1}\left(\frac{X_I[m]}{X_R[m]}\right)$$





$\boxed{\text{See DEMO}}$