★ Properties of the Fourier transform.

- Last time we computed the forward Fourier transform for some common signals using the definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \underline{\text{see Table 7.1}}$$

- Note the inverse transform is similar.

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- so that if $x(t) \xrightarrow{\mathcal{F}} X(\omega)$ $\quad X(t) \longrightarrow 2\pi x(-\omega)$

Example: $\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} = 1$

$$\mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

- This (near) symmetry means that the properties work similarly in both domains as well.

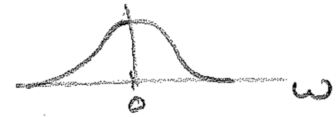
- Linearity: if $x_1(t) \xrightarrow{\mathcal{F}} X_1(\omega)$ and $x_2(t) \xrightarrow{\mathcal{F}} X_2(\omega)$

then $x(t) = x_1(t) + x_2(t) \xrightarrow{\mathcal{F}} X_1(\omega) + X_2(\omega)$

- Conjugate Symmetry: if $x(t) \xrightarrow{\mathcal{F}} X(\omega)$
then $x^*(t) \xrightarrow{\mathcal{F}} X^*(-\omega)$

- this implies if $x(t)$ is real valued
$x(t) = x^*(t)$ then $X^*(-\omega) = X(\omega)$
and $|X(\omega)|$ is an <u>even</u> function.

Example $e^{-a|t|} \xrightarrow{\mathcal{F}} \dfrac{2a}{a^2+\omega^2}$
$a > 0$

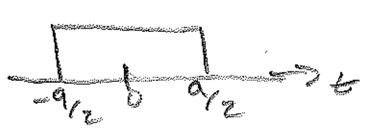


- Scaling: if $x(t) \xrightarrow{\mathcal{F}} X(\omega)$ then $x(at) \xrightarrow{\mathcal{F}} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

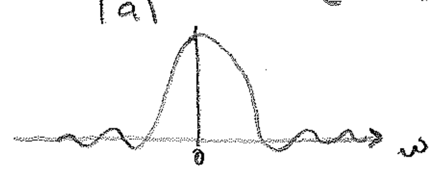
Example: Pulse $\Pi(t) = \begin{cases} 0 & |t| > 1/2 \\ 1 & \text{else.} \end{cases}$

$\Pi(t) \xrightarrow{\mathcal{F}} \text{sinc}\left(\frac{\omega}{2}\right)$

$x(at) = \begin{cases} 0 & |at| > 1/2 \\ 1 & \text{else} \end{cases} \xrightarrow{\mathcal{F}} \dfrac{1}{|a|} \text{sinc}\left(\dfrac{\omega}{2a}\right)$



DEMO



– Note in particular, $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-\omega)$

– time shift: if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$x(t-\tau) \longleftrightarrow X(\omega)e^{-j\omega\tau}$$ compare to Laplace.

Example: $x_1(t) = \cos(\omega_0 t)$

$$X_1(\omega) = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

$$x_2(t) = x_1(t-\tau) = \cos(\omega_0(t-\tau))$$

$$= \cos(\omega_0 t \underbrace{- \omega_0\tau}_{\text{phase shift}})$$

Then $X_2(\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] e^{-j\omega\omega_0\tau}$

In particlar $\mathcal{F}\{\sin(\omega_0 t)\}$

$$\sin(\omega_0 t) = \cos(\omega_0 t - \tfrac{\pi}{2}) = \cos(\omega_0(t - \tfrac{\pi}{2\omega_0}))$$

AND

$$\mathcal{F}\{\sin(\omega_0 t)\} = \pi\,\delta(\omega-\omega_0)\underbrace{e^{-j\frac{\omega}{\omega_0}\frac{\pi}{2}}}_{\substack{\omega=\omega_0 \\ e^{-j\pi/2} = -j}} + \pi\,\delta(\omega+\omega_0)\underbrace{e^{-j\frac{\omega}{\omega_0}\frac{\pi}{2}}}_{\substack{\omega=-\omega_0 \\ e^{j\pi/2} = j}}$$

$$= j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

– Frequency shift property (dual of time shift)

if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then $x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$

- A usefull application of this property is modulation

$$y(t) = x(t)\cos(\omega_0 t) = \frac{1}{2}x(t)e^{j\omega_0 t} + \frac{1}{2}x(t)e^{-j\omega_0 t}$$

$$Y(\omega) = \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega-\omega_0)$$

- This allows us to multiplex two signals over one channel
- Also allows us to transmit signals at radio freq.

Together these give AM Radio.

- Another Important Property is Convolution.

  if $x_1(t) \underset{\mathcal{F}}{\Longleftrightarrow} X_1(\omega)$ and $x_2(t) \underset{\mathcal{F}}{\Longleftrightarrow} X_2(\omega)$

  then $x_1(t) * x_2(t) \underset{\mathcal{F}}{\Longleftrightarrow} X_1(\omega) \cdot X_2(\omega)$.

- In particular if one of the signals is the impulse response of a causal system

$$y(t) = (h * x)(t) \Longleftrightarrow Y(\omega) = H(\omega)\, X(\omega)$$

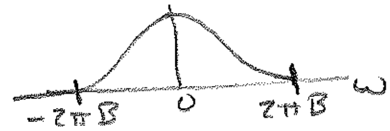
[more about this next time] — Frequency response.

- There are other important examples, of the dual.

  - Another View of Modulation. Suppose $x_1(t) \Longleftrightarrow X_1(\omega)$

    Example: Natural Voice $B \approx 4\,KHz$

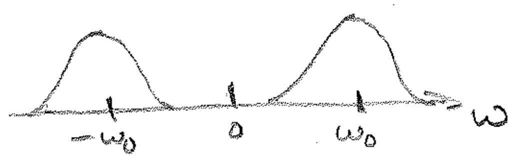
    Music $B \approx 20\,KHz$.



Bandwidth B.

Let $X_2(\omega) = \pi\,[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$

Then $X_1(\omega) * X_2(\omega)$

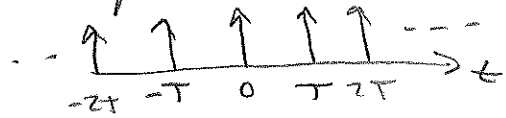


In time domain

$$x_1(t)\, x_2(t) = x_1(t) \cos(\omega_0 t)$$

- Sampling Theorem: $x_1(t) \overset{\mathcal{F}}{\Longleftrightarrow} X_1(\omega)$

$$x_2(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$, the impulse train



Multiplying $x_1(t)$ by $x_2(t)$ samples x_1 at nT.

$$x_1(t)\, x_2(t) = \sum_{n=-\infty}^{\infty} x_1(t)\, \delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)$$

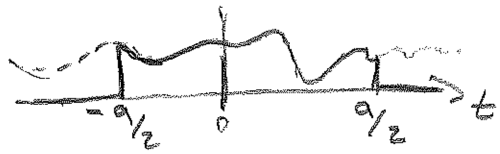
$= x[n]$ a discrete time signal

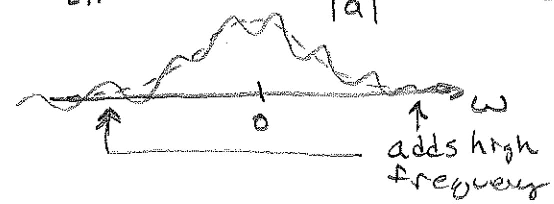
see lecture # 41 & 42 and ECE 3704

— The convolution property also helps give intuition about Fourier transforms.

- Truncation in time domain.

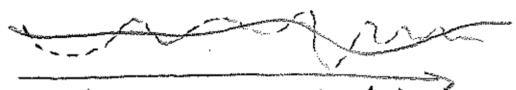
$$x(t)\,\Pi(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * \frac{1}{|a|}\,\mathrm{sinc}\left(\frac{\omega}{2a}\right)$$



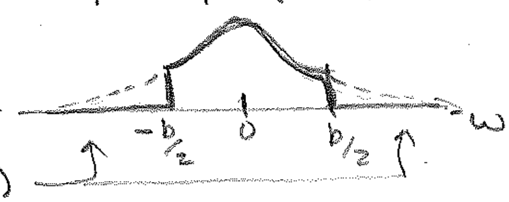


- Truncation in Frequency domain.

$$x_1(t) * \frac{1}{2\pi b}\,\mathrm{sinc}\left(\frac{t}{2b}\right) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)\,\Pi(b\omega)$$



Smoothes $x_1(t)$
(reduces frequency content)



— time derivative. $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

$$\frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)$$

compare to Laplace.

Example. $x(t) = \cos(\omega_0 t) \longleftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$$\frac{dx}{dt} = -\omega_0 \sin(\omega_0 t) \longleftrightarrow j\omega X(\omega)$$

$$= j\omega\pi\,\delta(\omega-\omega_0) + j\omega\pi\,\delta(\omega+\omega_0)$$

$$= j\omega_0\pi\,\delta(\omega-\omega_0) - j\omega_0\pi\,\delta(\omega+\omega_0)$$

$$= -\omega_0\left[j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)\right]$$

$$\downarrow \mathcal{F}^{-1}$$

$$= -\omega_0 \sin(\omega_0 t).$$

— time integral: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ then

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(\omega) + \pi X(0)\,\delta(\omega)$$

compare to Laplace.

Q: Could we use these properties to solve LCCDE?
A. In general NO, only if we knew it was stable....