★ Fourier transform Examples

- Recall the Fourier Transform Pair

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\underbrace{e^{-j\omega t}}_{\text{Forward Kernel}}\,dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\underbrace{e^{j\omega t}}_{\text{Inverse Kernel}}\,dt$$

$\mathcal{F}\{x(t)\}$ $\qquad\qquad$ $\mathcal{F}^{-1}\{X(\omega)\}$

- Today we will see several examples of $\mathcal{F}$.

- Example #1, $x(t) = e^{-at}u(t)$ $a > 0$

$$X(\omega) = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$= \underbrace{\left[\lim_{t\to\infty} \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\right]}_{\substack{\text{because } a>0 \\ \text{converges to zero}}} + \frac{1}{a+j\omega}$$

$$\mathcal{F}\{e^{-at}u(t)\} = \boxed{\frac{1}{a+j\omega}}$$

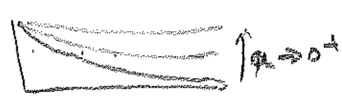
Remarks:
- Notice NO ROC as in Laplace
- compare to $\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$

- Example #2 $x(t) = u(t)$

$$X(\omega) = \int_{0}^{\infty} e^{-j\omega t}\,dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{0}^{\infty}$$

$$= \underbrace{\left[\lim_{t\to\infty} \frac{-1}{j\omega} e^{-j\omega t}\right]}_{?\ \text{convergence.}} + \frac{1}{j\omega}$$

Note: $u(t) = \lim_{a\to 0^+} e^{-at}u(t)$



— Example #2 cont.

$$\lim_{a\to 0^+} e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \lim_{a\to 0^+} \frac{1}{a+j\omega}$$

• Lets Rewrite this as Real + Imaginary

$$\frac{1}{a+j\omega}\cdot\frac{a-j\omega}{a-j\omega} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}$$

• Then take Limit

$$\underbrace{\lim_{a\to 0^+} \frac{a}{a^2+\omega^2}}_{\pi\delta(\omega)} - j\underbrace{\frac{\omega}{a^2+\omega^2}}_{\to 0 \text{ as } a\to 0^+}$$

DEMO

Thus $\mathcal{F}\{u(t)\} = \boxed{\pi\delta(\omega) + \frac{1}{j\omega}}$

Remark:
• Compare to $\mathcal{L}\{u(t)\} = \frac{1}{s}$

— Example #3 $x(t) = e^{-at}\cos(\omega_0 t)u(t)$ $a > 0$

• Lets rewrite $\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$

• Then $e^{-at}\cos(\omega_0 t) = \frac{1}{2}\left[e^{-(a-j\omega_0)t} + e^{-(a+j\omega_0)t}\right]$

So:

$$X(\omega) = \int_0^\infty \frac{1}{2}\left[e^{-(a-j\omega_0)t} + e^{-(a+j\omega_0)t}\right]e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(a-j\omega_0)t}e^{-j\omega t}\,dt + \int_0^\infty e^{-(a+j\omega_0)t}e^{-j\omega t}\,dt\right]$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(a+j(\omega-\omega_0))t}\,dt + \int_0^\infty e^{-(a+j(\omega+\omega_0))t}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{-1}{a+j(\omega-\omega_0)}e^{-(a+j(\omega-\omega_0))t}\Big|_0^\infty\right.$$

$$\left. + \frac{-1}{a+j(\omega+\omega_0)}e^{-(a+j(\omega+\omega_0))t}\Big|_0^\infty\right.$$

- Example #3 cont.

$$\mathbf{X}(\omega) = \frac{1}{2}\left[0 + \frac{1}{a+j(\omega-\omega_0)} + 0 + \frac{1}{a+j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2}\left\{\frac{a+j(\omega+\omega_0) + a+j(\omega-\omega_0)}{a^2 + aj(\omega-\omega_0) + aj(\omega+\omega_0) - (\omega-\omega_0)(\omega+\omega_0)}\right\}$$

$$aj\omega - \cancel{aj\omega_0} + aj\omega + \cancel{aj\omega_0}$$

$$= \frac{1}{\cancel{2}}\left[\frac{2a + j2\omega}{a^2 + 2ja\omega + \omega^2 + \omega_0^2}\right]$$

$$\boxed{= \frac{a+j\omega}{(a+j\omega)^2 + \omega_0^2}}$$

Compare to:

$$\mathcal{L}\{e^{-at}\cos(\omega_0 t)\} = \frac{s+a}{(s+a)^2+\omega_0^2}$$

- Example #4 $x(t) = \cos(\omega_0 t)$

• Lets write

$$x(t) = \lim_{a\to 0^+} e^{-at}\cos(\omega_0 t)u(t) + e^{at}\cos(\omega_0 t)u(-t)$$

Since $\int_a^b f(t)\,dt = -\int_b^a f(-t)\,dt$

Then $\mathcal{F}\{x(t)\} =$

$$\lim_{a\to 0} \frac{a+j\omega}{(a+j\omega)^2+\omega_0^2} - \frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$$

- Lets focus on the first term and write as Real + Imaginary

$$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2} = \underbrace{\frac{a^3+a\omega_0^2+a\omega^2}{4a^2\omega^2+(a^2+\omega_0^2-\omega^2)^2}} + j\underbrace{\frac{a^2\omega+\omega_0^2\omega-\omega^3}{4a^2\omega^2+(a^2+\omega_0^2-\omega^2)^2}}_{\lim_{a\to 0^+} = \frac{j\omega}{\omega_0^2-\omega^2}}$$

$\boxed{\text{DEMO}}$ $\lim_{a\to 0} = \frac{\pi}{2}\delta(\omega-\omega_0) + \frac{\pi}{2}\delta(\omega+\omega_0)$

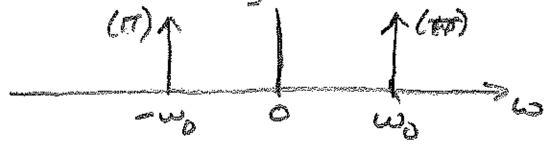
- For the second term we get

$$\frac{\pi}{2}\delta(\omega-\omega_0) + \frac{\pi}{2}\delta(\omega+\omega_0) - \frac{j\omega}{\omega_0^2-\omega^2}$$

- Example #4 cont.

- Putting the two terms together we get

$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

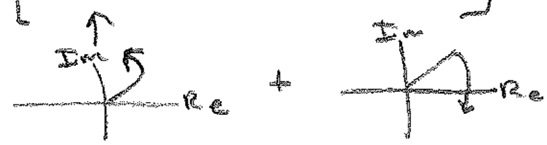


- This makes intuitive sense, there is only one frequency in $\cos(\omega_0 t)$, ω_0.
- What about $-\omega_0$? Lets consider the inverse, $\mathcal{F}^{-1}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]e^{j\omega t}\,dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,dt + \int_{-\infty}^{\infty}\delta(\omega+\omega_0)e^{j\omega t}\,dt\right]$$

$$= \frac{1}{2}\left[e^{j\omega_0 t} + e^{j(-\omega_0)t}\right] = \cos(\omega_0 t)$$



both ω_0 and $-\omega_0$ needed to get cos.

- Now that we have $\mathcal{F}\{\cos(\omega_0 t)\}$ we can take the Fourier transform of any periodic signal with a F.S. representation.

$$\mathcal{F}\left\{\sum_{n=0}^{\infty} C_n\cos(n\omega_0 + \theta_n)\right\}$$

See Properties next lecture.

- We see a clear relationship between Laplace and Fourier.

<u>If</u> ROC includes Imaginary Axis then

$$X(\omega) = X(s)\Big|_{s=j\omega}.$$

- See the Table of Fourier Transforms, Table 7.1 Lathi pg. 699.