★ Introduction to Fourier Transform

- Quick Review of Signal Representations, thus far.

Laplace: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$

Fourier Series: $D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$ for periodic $x(t)$

Both are frequency domain representations.

- Today we will add another, the Fourier Transform.

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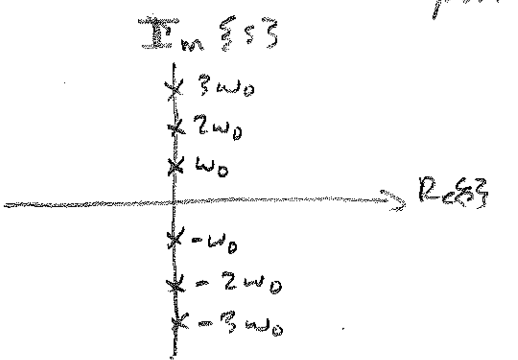
- Let's start by taking the Laplace transform of the Fourier Series.

$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ $\qquad X(s) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} e^{-st}\, dt$

$X(s) = \sum_{n=-\infty}^{\infty} D_n \int_{-\infty}^{\infty} e^{jn\omega_0 t} e^{-st}\, dt$

Table: $e^{\lambda t} \Longleftrightarrow \frac{1}{s-\lambda}$

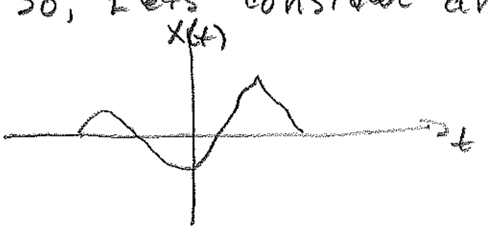
$$= \sum_{n=-\infty} D_n \Big( \underbrace{\frac{-1}{s - jn\omega_0}}_{\text{anticausal part}} + \underbrace{\frac{1}{s + jn\omega_0}}_{\text{causal part}} \Big)$$



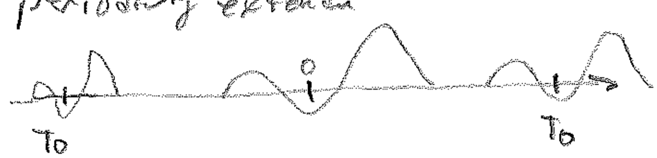

⇐ This is a sampling of the $j\omega$-axis.

- The "density" of the sampling is determined by $\omega_0 = \frac{2\pi}{T_0}$
- as $T_0 \to \infty$, $\omega_0 \to 0$

- So, lets consider an arbitrary signal $x(t)$



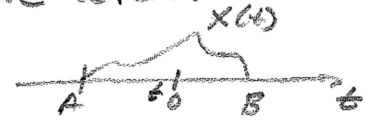

periodically extend




as $T_0 \to \infty$.

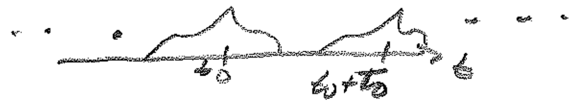
– Consider an a-periodic signal of finite width

$x(t) = 0$ for $t < A$ and $t > B$.




and its periodic extension

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$




– The Fourier Series of $x_p(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ $\quad \omega_0 = \frac{2\pi}{T_0}$

$$D_n = \frac{1}{T_0} \int_{t_0 - T_0/2}^{t_0 + T_0/2} x(t) e^{-jn\omega_0 t} \, dt$$

since $x(t) = 0$ outside $t \in [A, B]$

$$= \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} \, dt.$$

– Define $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$ as the <u>Fourier Transform</u> of $x$.

Then,
$$D_n = \frac{1}{T_0} X(\omega) \Big|_{\omega = n\omega_0}$$

AND $$x_p(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} X(n\omega_0) e^{jn\omega_0 t}.$$

– Now, let $T_0 \to \infty$, $x_p(t) \to x(t)$, $n\omega_0 \to \omega d\omega$ and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

$n \frac{2\pi}{T_0} \to \frac{1}{2\pi} \omega d\omega$

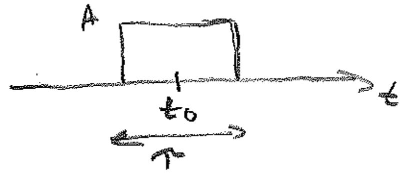
– This gives the Fourier transform Pair.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

| Forward Transform | Inverse Transform |
| --- | --- |
| $\mathcal{F}\{x(t)\}$ | $\mathcal{F}^{-1}\{X(\omega)\}$ |

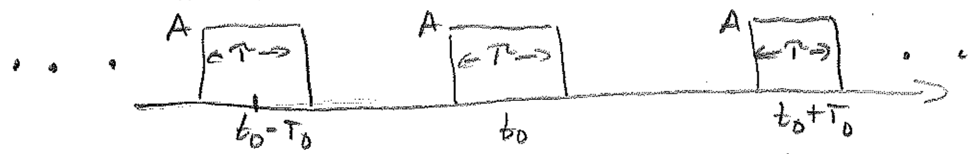
- Lets repeat that derivation using an example.
- Consider the pulse train from last lecture.

$$x(t) = A\,\Pi\left(\frac{t-t_0}{\tau}\right) \quad \text{where } \Pi(t) = \begin{cases} 1 & |t|<\frac{1}{2} \\ 0 & \text{else.} \end{cases}$$




copies at, $t_0, \pm t_0 + mT_0$

$$x_p(t) = \sum_{m=-\infty}^{\infty} A\,\Pi\left(\frac{t-t_0-mT_0}{\tau}\right) \qquad \tau < T_0$$




From last time $D_n = \dfrac{A}{\pi n} e^{-j\frac{2\pi n t_0}{T_0}} \sin\left(\dfrac{\pi n \tau}{T_0}\right) \quad n \neq 0$

$$D_n = \frac{A\tau}{T_0} \quad n = 0.$$

We can combine this using the sinc function

$$\text{sinc}(z) \equiv \frac{\sin(\pi z)}{\pi z}$$

Thus, $D_n = \dfrac{AT_0}{\tau} \,\text{sinc}\left(\dfrac{n\tau}{T_0}\right) e^{-j\frac{2\pi t_0}{T_0}}$

Note if $t_0 = 0$ then the signal is even and $D_n$ real.

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Compare this to the Fourier Transform of $x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt = \int_{t_0-\tau/2}^{t_0+\tau/2} A e^{-j\omega t} = \frac{A}{-j\omega} e^{-j\omega t}\Big|_{t_0-\tau/2}^{t_0+\tau/2}$$

$$= \frac{A}{-j\omega} e^{-j\omega t_0}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right)$$

$$= \frac{2A}{\omega} e^{-j\omega t_0}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right) \qquad f = \frac{\omega}{2\pi}$$

$$= 2A e^{-j\omega t_0} \frac{\sin(\omega\tau/2)}{\omega} = A\,\text{sinc}(ft)$$

- $\boxed{\text{Demo}}$ Plot F.S. $D_n$ for ⊓ and Let $T_0 \to \infty$

  Samples approach $X(\omega)$

- Another view on this is the Laplace transform when $s = j\omega$.

  Given $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ with ROC that includes the $j$-$\omega$ axis.

  $$X(s)\Big|_{s=j\omega} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \quad \text{the } \mathcal{F}\text{ Transform}$$

  The inverse Laplace

  $$x(t) = \frac{1}{2\pi} \oint_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds \qquad s \in \mathbb{C}$$

  complex region becomes a line $\int$.

  Becomes $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\underline{\omega} \qquad \omega \in \mathbb{R}.$

- As we shall see $Y(\omega) = H(\omega) X(\omega)$ for LTI systems,

  So, why use Laplace?

  - The Fourier transform only exists for Energy signals. (or in limit for some others).

    e.g. $e^{t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-1}$ $\qquad e^{t}u(t) \overset{\mathcal{F}}{\longleftrightarrow}$ does not exist.

    $\underline{\underline{Re\{s\} > 1}} \longrightarrow$ does not include $j\omega$-axis

  - Fourier methods only apply to <u>Stable</u> systems.

Otherwise Fourier methods can be used Similarly to Laplace.

E.g. $\frac{1}{cs}$ becomes $\frac{1}{j\omega C}$

$Ls$ becomes $j\omega L$

in Linear Circuits.