★ Fourier Series Example, visualization, application.

– Consider the exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

- Expand it to just 5 terms: n=-2, n=-1, n=0, n=1, n=2

$$D_{-2} e^{-j2\omega_0 t} + D_{-1} e^{-j\omega_0 t} + D_0 + D_1 e^{j\omega_0 t} + D_2 e^{j2\omega_0 t}$$

- Now, use Euler's identity to expand each exponential

$$D_{-2}\cos(-2\omega_0 t) + jD_{-2}\sin(-2\omega_0 t) +$$

$$D_{-1}\cos(-\omega_0 t) + jD_{-1}\sin(-\omega_0 t) +$$

$$D_0 +$$

$$D_1\cos(\omega_0 t) + jD_1\sin(\omega_0 t) +$$

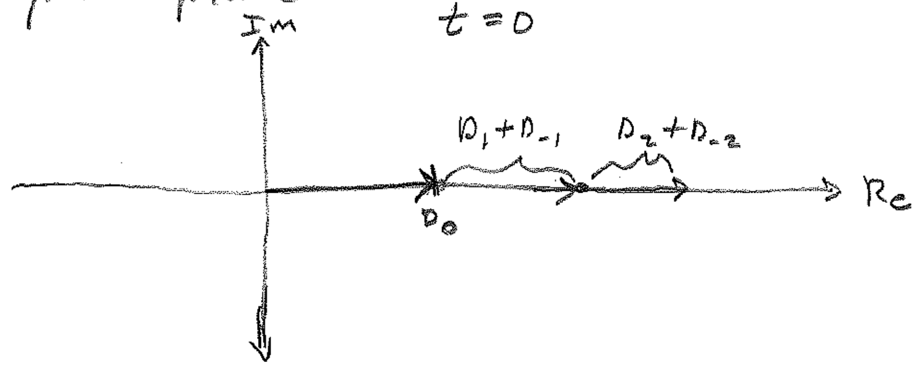
$$D_2\cos(2\omega_0 t) + jD_2\sin(2\omega_0 t)$$

- Collect real terms, Note $\cos(-\theta) = \cos(\theta)$
  $\sin(-\theta) = -\sin(\theta)$

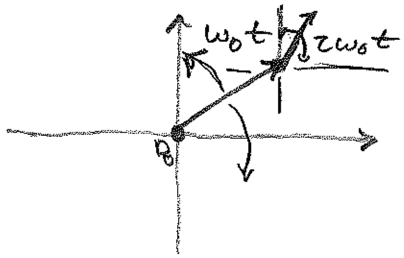
$$Re(t) = D_0 + (D_1 + D_{-1})\cos(\omega_0 t) + (D_2 + D_{-2})\cos(2\omega_0 t)$$

$$Im(t) = (D_1 - D_{-1})\sin(\omega_0 t) + (D_2 - D_{-2})\sin(2\omega_0 t)$$

- Now, for fixed time t, draw these as vectors in the complex plane



As t changes, these vectors rotate at ω_0 + $2\omega_0$



This gives us an interesting way to visualize the Fourier Series.

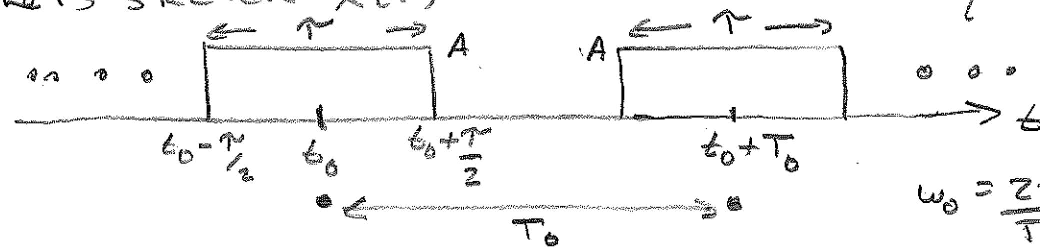
$\boxed{\text{See DEMO}}$

- Example: Pulse train.

$$x(t) = \sum_{m=-\infty}^{\infty} A \,\Pi\left(\frac{t - t_0 - mT_0}{\tau}\right) \quad \text{for } \tau < T_0$$

where Π is the "gate" function $\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else.} \end{cases}$

- Lets sketch $x(t)$



$$\omega_0 = \frac{2\pi}{T_0}$$

- what is it's Fourier Series?

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

$$D_n = \frac{1}{T_0}\int_{t_0 - \tau/2}^{t_0 + \tau/2} A e^{-jn\omega_0 t}\, dt = \frac{-A}{jn\omega_0 T_0} e^{-jn\omega_0 t} \Big|_{t_0 - \tau/2}^{t_0 + \tau/2}$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0}\left[\frac{e^{jn\omega_0 \tau/2} - e^{-jn\omega_0 \tau/2}}{2j}\right] \quad n \neq 0$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0} \sin\left(\frac{n\omega_0 \tau}{2}\right) \quad n \neq 0$$

if $n = 0$ $\quad D_0 = \text{Avg value} = \frac{A\tau}{T_0}$

$$D_n = \begin{cases} \frac{A\tau}{2} & n = 0 \\ \frac{2A}{n\omega_0 T_0} e^{jn\omega_0 t_0} \sin\left(\frac{n\omega_0 \tau}{2}\right) & \text{else} \end{cases}$$

- Since D_n is complex we plot the spectrum as

$\boxed{\text{See DEMO}}$

- System Response due to periodic inputs.
  - Recall the Frequency Response tells us how a system behaves given sinusoidal input.

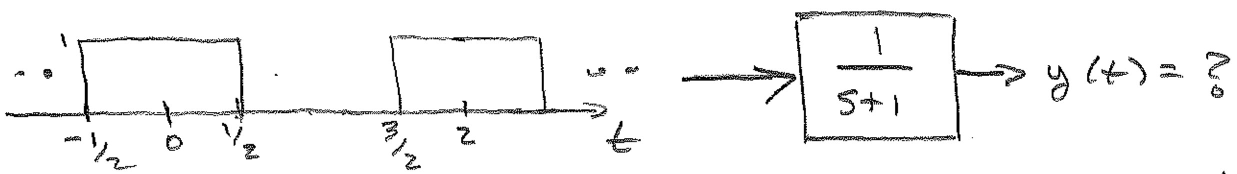
$$e^{j\omega t} \longrightarrow \boxed{H(\omega)} \longrightarrow |H(\omega)|\, e^{j(\omega t + \angle H(\omega))}$$

Applying Linearity.

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow \sum_{n=-\infty}^{\infty} |H(n\omega_0)|\, e^{j(n\omega_0 t + \angle H(n\omega_0))}$$

- Using our previous example $\tau = 1$, $T_0 = 2$, $A = 1$, $t_0 = 0$

$$D_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right) \end{cases}$$



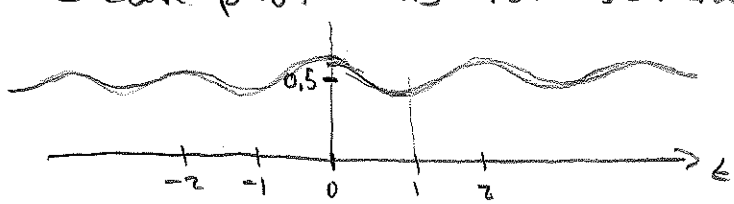
$$H(\omega) = \frac{1}{1+j\omega} \qquad |H(\omega)| = \frac{1}{(1+\omega^2)^{1/2}} \qquad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{1}\right)$$

Evaluate at $\omega = n\omega_0 = \pi n$

$$|H(n)| = \frac{1}{(1+n^2\pi^2)^{1/2}} \qquad \angle H(n) = -\tan^{-1}\left(\frac{n\pi}{1}\right)$$

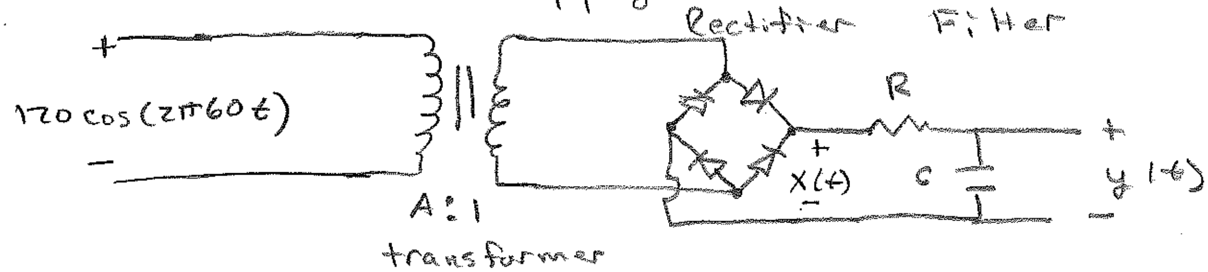
$$\text{Thus } y(t) = \sum_{n=-\infty}^{\infty} \frac{D_n}{(1+n^2\pi^2)^{1/2}}\, e^{j(n\pi t - \tan^{-1}(\frac{n\pi}{1}))}$$

We can plot this for several terms $n = -N$, to N



– A practical Application of F.S.

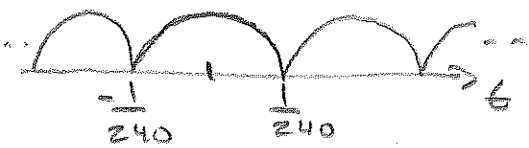
Consider a Power Supply (DC)



– We would like to know for what values of R, C does y look like a constant DC source,

- The output of the transformer and Full wave Rectifier can be modeled as

$$x(t) = \left| \frac{120}{A} \cos(120\pi t) \right|$$



- this passes through an R-C filter!

$$H(s) = \frac{1}{1+RCs}$$

- To proceed we write x(t) using the F.S.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) \qquad T_0 = \frac{1}{120} \quad \omega_0 = 240\pi$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = 120 \int_{-1/240}^{1/240} \frac{120}{A}\cos(120\pi t)\,dt = \frac{240}{A\pi}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$$

$$= 240 \int_{-1/240}^{1/240} \frac{120}{A} \cos(120\pi t)\cos(240\pi n t)\,dt$$

$$= \frac{480}{A} \frac{\cos(n\pi)}{\pi(1-4n^2)}$$

Then

$$y(t) = \frac{240}{A\pi} + \frac{480}{3A\pi} \cdot \left| \frac{1}{1+jRC240\pi n} \right| \cos(240\pi t + \angle) + \cdots$$