- Convergence of the Fourier Series.

  - The Fourier Series is, strictly speaking, an <u>approximation</u> of $x(t)$

$$x(t) \approx \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

$$x(t) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt \qquad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\, dt$$

$$x(t) \approx C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad C_n = (a_n^2 + b_n^2)^{1/2}$$

$$\theta_n = \arctan\left(\frac{-b_n}{a_n}\right)$$

  - Why is it just an approximation? To answer this we need to talk about <u>convergence</u> and <u>existance</u>.

    - Existance: When do the integrals for D_n, a_n, b_n converge to finite values?

$$\text{when } \int_{T_0} |x(t)|\, dt < \infty \qquad \text{absolutly integral.}$$

    - Convergence: when does $x(t) \underset{\text{exactly}}{\overset{\uparrow}{=}} \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$

Let $x_N(t) = \sum_{n=-N}^{N} D_n e^{jn\omega_0 t}$ be the truncated Fourier Series.

We can define the error $E(N;t) = x(t) - x_N(t)$.

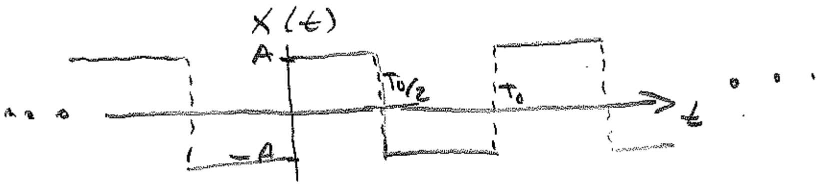
There are two notions of convergence

$$\int_{T_0} |E(N,t)|\, dt \quad \text{and} \quad \int_{T_0} |E(N,t)|^2\, dt \quad (\text{mean-square})$$

if $\lim_{N\to\infty}$ of these measures is zero, the series <u>converges</u>.

— Example, square wave.

$$x(t) = \begin{cases} A & 0 < t < T_0/2 \\ -A & \frac{T_0}{2} < t < T_0 \end{cases} \quad \text{extended periodically.}$$



$$\omega_0 = \frac{2\pi}{T_0}$$

Lets use the trig form

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t) = \frac{1}{T_0}\int_0^{T_0/2} A\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} -A\,dt = \underline{0}.$$

$a_n = 0$ because signal is odd.

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2} A\sin(n\omega_0 t)\,dt + \int_{T_0/2}^{T_0} -A\sin(n\omega_0 t)\,dt\right.$$

$$= \frac{2A}{T_0}\left[\frac{-\cos(n\omega_0 t)}{n\omega_0}\Big|_0^{T_0/2} + \frac{\cos(n\omega_0 t)}{n\omega_0}\Big|_{T_0/2}^{T_0}\right]$$

$$= \frac{2A}{T_0}\left[\frac{-\cos(n\omega_0 T_0/2)}{n\omega_0} + \frac{1}{n\omega_0} + \frac{\cos(n\omega_0 T_0)}{n\omega_0} - \frac{\cos(n\omega_0 \frac{T_0}{2})}{n\omega_0}\right]$$

Note $\omega_0 T_0 = 2\pi$ $\quad \omega_0 \frac{T_0}{2} = \pi$

$$= \frac{2A}{T_0}\left[\frac{-\cos(\pi n)}{n\omega_0} + \frac{1}{n\omega_0} + \frac{1}{n\omega_0} - \frac{\cos(\pi n)}{n\omega_0}\right]$$

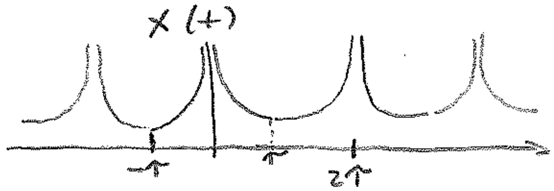
$$= \frac{2A}{2\pi n}\left[2 - 2\cos(\pi n)\right]$$

$$= \frac{2A}{\pi n}\left[1 - \cos(\pi n)\right] = \begin{cases} 0 & n \text{ even} \\ \frac{4A}{\pi n} & n \text{ odd} \end{cases}$$

$$x(t) = \frac{4A}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \cdots\right.$$

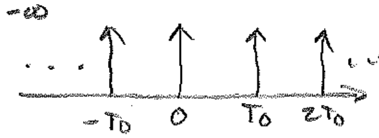
$\boxed{\text{DEMO}}$

- The Fourier Series Converges in mean-square sense if
  (1) The signal has a finite number of discontinuities per period T_0
  (2) The signal has a finite number of max/min over T_0
  (3) the signal is bounded $\int_{T_0} |x(t)| dt < \infty$ (exists).
  - This keeps the signal from being "pathological".
- Example: $x(t) = \frac{1}{|t|}$ $-\tau < t < \tau$ periodically extended.



$$\int_{-\tau}^{\tau} \frac{1}{|t|} dt = \infty \quad \therefore \text{ NO F.S. exists.}$$

- Example: Impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$



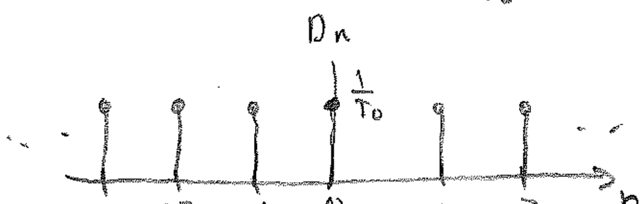
  - Does it meet the conditions above?
  - Lets use the exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \quad \text{by sifting property.}$$

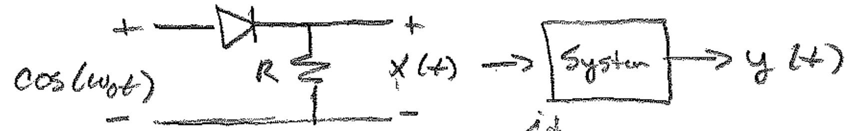
$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jn\omega_0 t}$$

Spectrum $|D_n| = \frac{1}{T_0}$ $\angle D_n = 0$

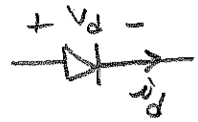


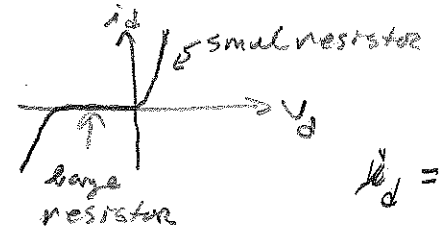
- Usually about this point students wonder what the F.S. is good for.....

• Consider the circuit



Diode :

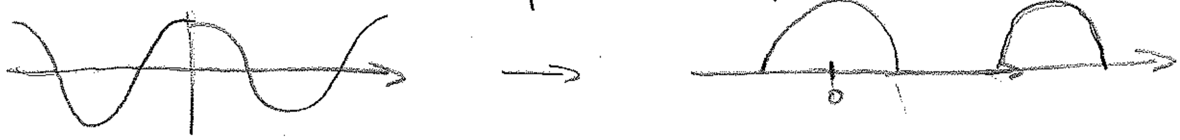




$$i_d = I_s\left(e^{\frac{V_d}{nV_T}} - 1\right)$$

a non-linear function.

This diode circuit "clips" the cos.



$$x(t) = \begin{cases} \cos(\omega_0 t) & -\frac{\pi}{\omega_0} < t < \frac{\pi}{\omega_0} \\ 0 & \frac{\pi}{\omega_0} < t < \frac{2\pi}{\omega_0} \end{cases}$$ extended periodically.

Suppose this was the input to an LTI system. How could we determine the response?

- Time Domain $\quad y(t) = (x * h)(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

- Laplace $\quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} = \underbrace{\int_{-\infty}^{0} x(t) e^{-st} dt}_{\text{anticausal}} + \underbrace{\int_{0}^{\infty} x(t) e^{-st} dt}_{\text{causal.}}$

- Use the Fourier Series.

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$y(t) = \sum_{n=-\infty}^{\infty} D_n \, |H(n\omega_0)| \, e^{j(n\omega_0 t + \angle H(n\omega_0))}$$

where $H(\,)$ is the Frequency Response of the system.