- Convergence of the Fourier Series.

- The Fourier Series is, strictly speaking, an approximation of $x(t)$

$$x(t) \approx \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$
$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$x(t) \approx a_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t)$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$
$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n \omega_0 t) dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n \omega_0 t) dt$$

$$x(t) \approx C_0 + \sum_{n=1}^{\infty} c_n \cos(n \omega_0 t + \Theta_n)$$

$$c_n = (a_n^2 + b_n^2)^{1/2}$$
$$\Theta_n = \arctan\left(-\frac{b_n}{a_n}\right)$$

- Why is it just an approximation? To answer this we need to talk about convergence and existence.

- Existence: When do the integrals for $D_n, a_n, b_n$ converge to finite values?

  When $\int_{T_0} |x(t)| dt < \infty$ absolutely integral.

- Convergence: When does $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$ exactly?

  Let $x_N(t) = \sum_{n=-N}^{N} D_n e^{jn\omega_0 t}$ be the truncated Fourier Series.

  We can define the error $E(N, t) = x(t) - x_N(t)$.

There are two notions of convergence

$$\int_{T_0} |E(N, t)| dt$$
$$\int_{T_0} |E(M, t)|^2 dt$$ (mean-square)

if $\lim_{N \to \infty}$ of these measures is zero, the series converges.
Example, Square Wave.

\[ x(t) = \begin{cases} A & 0 < t < \frac{T_0}{2} \\ -A & \frac{T_0}{2} < t < T_0 \end{cases} \] extended periodically.

\[ \omega_0 = \frac{2\pi}{T_0} \]

Let's use the trig form for \( T_0/2 \)

\[ a_0 = \frac{1}{T_0} \int_0^{T_0/2} x(t) \, dt = \frac{1}{T_0} \int_0^{T_0/2} A \, dt + \frac{1}{T_0} \int_0^{T_0/2} -A \, dt = 0 \]

\[ a_n = 0 \text{ because signal is odd.} \]

\[ b_n = \frac{2}{T_0} \int_0^{T_0/2} x(t) \sin(n \omega_0 t) \, dt \]

\[ = \frac{2}{T_0} \left[ \int_0^{T_0/2} A \sin(n \omega_0 t) \, dt + \int_0^{T_0/2} -A \sin(n \omega_0 t) \, dt \right] \]

\[ = \frac{2A}{T_0} \left[ \frac{-\cos(n \omega_0 t_0/2)}{n \omega_0} + \frac{\cos(n \omega_0 t_0)}{n \omega_0} \right] \]

\[ = \frac{2A}{T_0} \left[ \frac{-\cos(n \omega_0 T_0/2)}{n \omega_0} + \frac{1}{n \omega_0} + \frac{\cos(n \omega_0 T_0)}{n \omega_0} - \frac{\cos(n \omega_0 T_0/2)}{n \omega_0} \right] \]

Note \( \omega_0 T_0 = 2\pi \) \( \omega_0 \frac{T_0}{2} = \pi \)

\[ = \frac{2A}{T_0} \left[ -\frac{\cos(\pi n)}{n \omega_0} + \frac{1}{n \omega_0} + \frac{1}{n \omega_0} + \frac{\cos(\pi n)}{n \omega_0} \right] \]

\[ = \frac{2A}{\pi n} \left[ 2 - 2\cos(\pi n) \right] \]

\[ = \frac{2A}{\pi n} \left[ 1 - \cos(\pi n) \right] = \begin{cases} 0 & \text{even} \\ \frac{4A}{\pi n} & \text{odd} \end{cases} \]

\[ x(t) = \frac{4A}{\pi} \left[ \sin(\omega_0 t) + \frac{1}{3} \sin(3 \omega_0 t) + \frac{1}{5} \sin(5 \omega_0 t) + \cdots \right] \]

**Demo**
The Fourier Series converges in mean-square sense if:

1. The signal has a finite number of discontinuities per period $T_0$.

2. The signal has a finite number of max/min over $T_0$.

3. The signal is bounded $\int_{-T_0}^{T_0} |x(t)| dt < \infty$ (exists).

*This keeps the signal from being "pathological".*

**Example:** $x(t) = \frac{1}{|t|}$ for $-T < t < T$ periodically extended.

\[ x(t) \]

\[ \int_{-T}^{T} \frac{1}{|t|} dt = \infty \quad \text{no F.S. exists.} \]

**Example:** Impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$.

*Does it meet the conditions above?*

*Let's use the exponential form*

\[ x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0 t} \]

\[ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jnw_0 t} dt \]

\[ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t-nT_0) e^{-jnw_0 t} dt = \frac{1}{T_0} \]

by shifting property.

\[ x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jnw_0 t} \]

**Spectrum**

\[ |D_n| = \frac{1}{T_0} \quad \angle D_n = 0 \]

\[ D_n \]

\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \cdots \quad n \]
- Usually about this point students wonder what the P.S. is good for...

- Consider the circuit

\[ x(t) \xrightarrow{\text{System}} y(t) \]

Diode:

\[ y_d = I_S (e^{v_d/|V_d|} - 1) \]

This diode circuit "clips" the \( \cos \),

\[ x(t) = \left\{ \begin{array}{lcr} \cos(\omega_0 t) & -\frac{\pi}{\omega_0} < t < \frac{\pi}{\omega_0} \\ 0 & \frac{\pi}{\omega_0} < t < \frac{2\pi}{\omega_0} \end{array} \right. \]

Suppose this was the input to an LTI system. How could we determine the response?

- Time Domain
  \[ y(t) = x(t) * h(t) \]
  \[ = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \]

- Laplace
  \[ X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt = \int_{-\infty}^{0} x(t) e^{-st} \, dt + \int_{0}^{\infty} x(t) e^{-st} \, dt \]
  - Anticausal
  - Causal

- Use the Fourier Series
  \[ x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j n \omega_0 t} \]
  \[ y(t) = \sum_{n=-\infty}^{\infty} D_n |H(n \omega)| e^{j (n \omega_0 t + \angle H(n \omega))} \]

where \( H(\tau) \) is the Frequency Response of the System.