- Short Review. At this point in the course we have two signal representations and how they propogate through LTI systems.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \boxed{h} \longrightarrow \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \longrightarrow \boxed{H} \longrightarrow H(s)X(s)$$

Best for causal signals.

- Recall the frequency response gives us a very easy way to determine the response to sinusoidal inputs.

$$e^{j\omega t} \longrightarrow \boxed{H(\omega)} \longrightarrow |H(\omega)|\, e^{j(\omega t + \angle H(\omega))}$$

AND by Linearity, if we apply a weighted sum of sinusoids.

$$D_1 e^{j\omega_1 t} + D_2 e^{j\omega_2 t} \longrightarrow \boxed{H(\omega)} \longrightarrow |H(\omega_1)|\, e^{j(\omega_1 t + \angle H(\omega_1))} + |H(\omega_2)|\, e^{j(\omega_2 t + \angle H(\omega_2))}$$

Thus if we can represent a signal as a linear combination of sinusoids, the response is easy to determine.

$$\sum_{n=-\infty}^{\infty} D_n e^{j\omega_n t} \longrightarrow \boxed{H(\omega)} \longrightarrow \sum_{n=-\infty}^{\infty} D_n |H(\omega_n)|\, e^{j(\omega_n t + \angle H(\omega_n))}$$

- The question: what signals can be represented this way?
  - Answer provided by Jean Baptiste Joseph Fourier in early 1800's. <u>Periodic Signals</u>.

AND a linear combination of sinusoids are called

Fourier Series: Countably infinite combination

Fourier Transform: uncountably infinite combination.

- We will focus on the Fourier Series first.

- Recall the properties of periodic functions.

$x(t) = x(t+T) \ \forall t$ — smallest such $T$ is the <u>fundamental Period $T_0$</u>.

$f_0 = \frac{1}{T_0}$ is fundamental frequency in Hz.

$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ is the fundamental frequency rad/s.

- Any integer multiple of $\omega_0$ is called a <u>harmonic</u> or <u>mode</u>.
- Example: have you ever walked by a transformer and heard a humming sound?

In the U.S. power frequency is 60Hz $\omega_0 = 120\pi$

$n\omega_0$ are the harmonics. How may could we hear?

Humans have a best case hearing range of 12 - 20kHz (in practice more like 31 - 18kHz)

so $|60n| > 12$ an $|60n| < 20\times10^3$ or $1 \le n \le 333$
so $\approx 334$ harmonics at best, In practice may of these are attenuated.

$\boxed{\text{DEMO}}$

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- Recall we can consider the Laplace transform as a change of basis. The Fourier Series is similar.

Inner product $\langle x(t), e^{-jn\omega_0 t}\rangle = \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} dt$

does not converge for periodic $x(t)$

the average over a period does converge.

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$D_n$: coefficient of Fourier Series in exponential form.

- Using Euler's identities gives us the trig form

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

– where
$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$, the averge (DC) value.

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \qquad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

– or using trig identity. the compact trig form
$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$
where
$$C_0 = a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \qquad C_n = (a_n^2 + b_n^2)^{1/2}$$
$$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

– plots of $C_n$ are called the <u>amplitude Sepetrum</u>.

– plots of $\theta_n$ are called the <u>phase Spectrum</u>

– Togethr they are the <u>frequency spectrum</u> for the periodic $x(t)$.

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Note, that since a signal $x(t)$ can be divided into an even and odd component

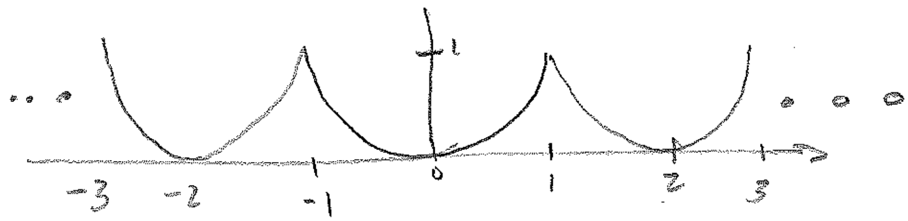
$$x(t) = x_e(t) + x_o(t) \qquad x_e(t) = \frac{1}{2}(x(t) + x(-t))$$
$$x_o(t) = \frac{1}{2}(x(t) - x(-t))$$

- For $x_e(t)$ the trig form F.S. has $b_n = 0$
- For $x_o(t)$ the trig form F.S. has $a_n = 0$

This can be a usefull check for purely even/odd signals.

– Example: $x(t) = x_p(t)$ extended with period $T_0 = 2$

$$x_p(t) = t^2 \quad -1 < t < 1$$



$f_0 = \frac{1}{2}$

$\omega_0 = \frac{2\pi}{T_0} = \pi$

Note $x(t)$ is even so we expect $b_n = 0$.

$$b_n = \frac{2}{2} \int_{-1}^{1} t^2 \sin(n\pi t)\, dt$$

From Table.

$$= \left[ \frac{(-2 + t^2 n^2 \pi^2)\cos(n\pi t)}{n^3\pi^3} + \frac{2t \sin(n\pi t)}{n^2\pi^2} \right]_{-1}^{1}$$

$$= \left[ \frac{(-2 + n^2\pi^2)\cos(n\pi)}{n^3\pi^3} + \frac{2 \overbrace{\sin(n\pi)}^{0}}{n^2\pi^2} \right]$$

$$- \left[ \frac{(-2 + n^2\pi^2)\cos(-n\pi)}{n^3\pi^3} + \frac{2 \overbrace{\sin(-n\pi)}^{0}}{n^2\pi^2} \right]$$

$$= \frac{(-2 + n^2\pi^2)}{n^3\pi^3} \underbrace{\left( \cos(n\pi) - \cos(-n\pi) \right)}_{0} = 0.$$

$$a_n = \frac{2}{2} \int_{-1}^{1} t^2 \cos(n\pi t)\, dt$$

From table

$$= \left[ \frac{2t \cos(n\pi t)}{n^2\pi^2} + \frac{(-2 + t^2 n^2 \pi^2)\sin(n\pi t)}{n^3\pi^3} \right]_{-1}^{1}$$

$$= \frac{2 \overbrace{\sin(n\pi)}^{0}}{n\pi} + \frac{4\cos(n\pi)}{n^2\pi^2} - \frac{4 \overbrace{\sin(n\pi)}^{0}}{n^3\omega_0^3}$$

$$= \frac{4}{n^2\pi^2}(-1)^n$$