Recall that a signal \( x(t) \) can be divided into an even and odd component

\[
X(t) = X_e(t) + X_o(t) \quad X_e(t) = \frac{1}{2} \left( x(t) + x(-t) \right) \\
X_o(t) = \frac{1}{2} \left( x(t) - x(-t) \right)
\]

For \( X_e(t) \) the trigonometric F.S. has \( b_n = 0 \)
For \( X_o(t) \) the trigonometric F.S. has \( a_n = 0 \) (e.g., square)
This is a useful check if you know that \( x(t) \) is even/odd.

**Example:** \( x(t) = x_e(t) \) extended \( x_e(t) = t^2 \quad -1 < t < 1 \)

\[
X(t) = X_e(t) + X_o(t) = \begin{cases} 
0 & \text{if } -1 < t < 1 \\
\frac{2}{\pi} & \text{otherwise}
\end{cases}
\]

\[ T_0 = 2 \]
\[ \omega_0 = \frac{2\pi}{T_0} = \pi \]
\( x(t) \) is even so we expect \( b_n \) to be 0.

\[
b_n = \frac{2}{\pi} \int_{-1}^{1} t^2 \sin(n \pi t) \, dt
\]

\[
= \left[ \frac{(-2 + t^2 \pi^2 n^2) \cos(n \pi t)}{n^2 \pi^3} \right]_{-1}^{1} + \frac{2 \pi \sin(n \pi t)}{n^2 \pi^3}
\]

\[ = 0 \]

\[
a_n = \frac{2}{\pi} \int_{-1}^{1} t \cos(n \pi t) \, dt
\]

\[
= \left[ \frac{2 \pi \sin(n \pi t)}{n^2 \pi^3} \right]_{-1}^{1} + \left[ \frac{(-2 + t^2 \pi^2 n^2) \sin(n \pi t)}{n^2 \pi^3} \right]_{-1}^{1}
\]

\[ = \frac{4}{n^2 \pi^2} \]

\[
= \frac{(-1)^n}{n^2 \pi^2}
\]
So we have said that the F.S. is an approximation to \( x(t) \). How good an approximation is it?

Let \( x_N(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n \omega t) + b_n \sin(n \omega t) \quad N < \infty \)

define the error \( e(t) = x(t) - x_N(t) \) and the mean-square error as

\[
E = \frac{1}{T} \int_{-T/2}^{T/2} |e(t)|^2 \, dt
\]

For the F.S. as \( N \to \infty \) \( E \to 0 \).

Another Example: Impulse Train.

\[ x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \]

Let's use Exp Form.

\[ x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn \omega t} \]

\[ D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn \omega t} \, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn \omega t} \, dt = \frac{1}{T_0} \delta(-jn \omega) \quad \text{by shifting property} \]

\[ x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \delta(-jn \omega) e^{jn \omega t} \]

Spectrum

\[ D_n \]
The F.S. is guaranteed to converge in the mean-square sense if the Dirichlet conditions hold:

1. The signal has a finite # of discontinuities over T.
2. The signal has a finite # of maxima, minima over T.
3. The signal is bounded. \[ \int_{-T}^{T} |x(t)| \, dt < \infty \]

This keeps \( x(t) \) from being pathological.

E.g. \( x(t) = \frac{1}{|t|} \) for \( t \neq 0 \) extended.

[Diagram of a signal with repeated u-shaped waves]

\( x(t) \) has no F.S. but the impulse train does.

**So let's try to relate the F.S. back to Laplace.**

What is the Laplace transform of the F.S.?

\[
X(s) = \sum_{n=-\infty}^{\infty} D_n e^{s n T} \\
X(s) = \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} D_n e^{s n T} \right) e^{-s \tau} \, d\tau \\
= \sum_{n=-\infty}^{\infty} D_n \int_{-\infty}^{\infty} e^{s n T - s \tau} \, d\tau \\
= \sum_{n=-\infty}^{\infty} D_n \left[ \frac{1}{s - n s T} - \frac{1}{s + n s T} \right] \\
= \sum_{n=-\infty}^{\infty} D_n \left[ \frac{1}{s - n s T} - \frac{1}{s + n s T} \right] \\
\text{ (causal)} \\
\text{ (anticausal)}
So, given the limitations of the F.S. why do we use it?

1) Recall the response of a LTI C system to a complex sinusoid.

\[ P \rightarrow \boxed{H(f_0)} \rightarrow P \boxed{h(t)} \]

This makes it easy to get intuition to understand what is going on in filter systems.

2) In power systems the power of \( x(t) \) is related simply to the Dn's via Parseval's theorem.

\[ P_x = \sum_{n=-\infty}^{\infty} |D_n|^2 \]

3) The practical "steps" left over after approximation and bounded in magnitude and don't matter too much in real systems.

---

**Example: Diode Circuit.**

\[ x(t) = \cos(\omega t) \]

\[ v(t) = \text{Diode} \]

\[ f(t) = \text{Small resistor} \]

\[ V_d \]

\[ V_{dc} \]

\[ I = \frac{1}{2} V \]

\[ V_{in} \]

\[ V(t) \text{ vs. } t \]

\[ x(t) \text{ vs. } t \]