* Bilateral Laplace

- Recall the definition of the Bilateral Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt$$

- for causal signals $x(t) = 0$ for $t < 0$ and this becomes the one-sided Laplace transform

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} \, dt \quad \text{for causal } x(t).$$

- we can split any signal $x(t)$ into its causal and anticausal components as.

$$x(t) = x(t)u(-t) + x(t)u(t) = x_a(t) + x_c(t)$$

and thus the Bilateral transform becomes

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} \, dt = \int_{-\infty}^{0^-} x_a(t) e^{-st} \, dt + \int_{0^+}^{\infty} x_c(t) e^{-st} \, dt$$

so that

$$X(s) = X_a(s) + X_c(s)$$

- The transform $X_c(s)$ we obtain the same way we have in the past, using the definition or one-sided table.

- To obtain $X_a(s)$ we note

$$X_a(s) = \int_{-\infty}^{0^-} x_a(t) e^{-st} \, dt = \int_{0^+}^{\infty} x_a(-t) e^{-s(-t)} \, dt$$

and

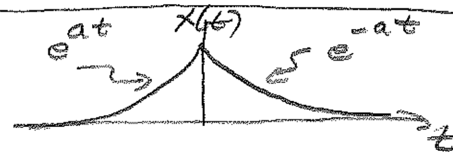
$$X_a(-s) = \int_{0^+}^{\infty} x_a(-t) e^{+s(-t)} \, dt = \underbrace{\int_{0^+}^{\infty} x_a(-t) e^{-st} \, dt}_{\text{one-sided Laplace of } x_a(-t).}$$

- Thus the Laplace transform of the anti-causal component is the one-sided Laplace of $x_a(-t)$ when s is replaced by $-s$.

- Summary of Bilateral-Laplace

1. Split into causal and anticausal components, $x_a(t)$ and $x_c(t)$
2. Compute $X_c(s)$ using one-sided transform as before (with associated ROC)
3. Compute $X_a(s)$ using one-sided transform of $x_a(-t)$ replacing s by $-s$ (including the ROC).

- Example: $x(t) = e^{-a|t|} \quad a > 0$



First, lets use the definition:

$$X(s) = \int_{-\infty}^{0^-} e^{at} e^{-st}\, dt + \int_{0^+}^{\infty} e^{-at} e^{-st}\, dt$$

$$= \int_{-\infty}^{0^-} e^{-(s-a)t}\, dt + \int_{0^+}^{\infty} e^{-(s+a)t}\, dt$$

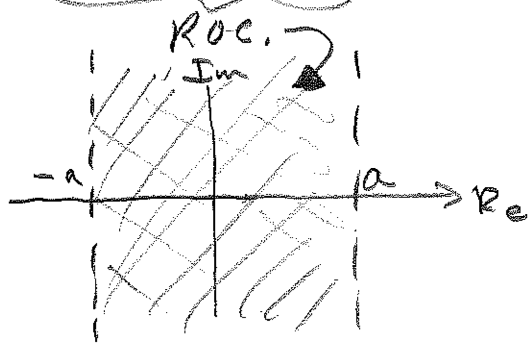
$$= \frac{-1}{s-a} e^{-(s-a)t} \Big|_{-\infty}^{0^-} + \frac{-1}{s+a} e^{-(s+a)t} \Big|_{0^+}^{\infty}$$

$$= \frac{-1}{s-a}\Big[\underbrace{e^{-(s-a)0^-}}_{1} - \underbrace{e^{-(s-a)(-\infty)}}_{=0 \text{ if } Re\{s-a\}<0 \text{ or } Re\{s\}<a}\Big]$$

$$+ \frac{-1}{s+a}\Big[\underbrace{e^{-(s+a)(\infty)}}_{0 \text{ if } Re\{s+a\}>0 \text{ or } Re\{s\}>-a} - \underbrace{e^{-(s+a)(0^+)}}_{1}\Big]$$

$$= \frac{-1}{s-a} + \frac{1}{s+a} \qquad \underbrace{-a < Re\{s\} < a}_{ROC}$$

$$= \frac{-2a}{(s-a)(s+a)}$$



- Now, lets repeat using the procedure.

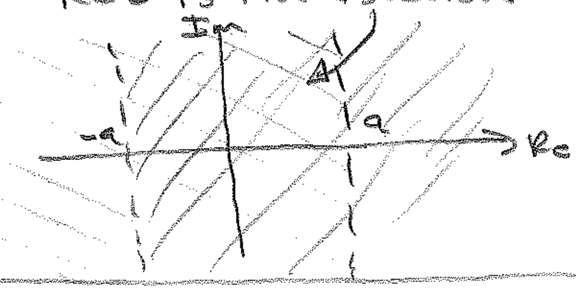
$$x(t) = x_a(t) + x_c(t) = e^{at}u(-t) + e^{-at}u(t)$$

$X_c(s) = \frac{1}{s+a}$ $\quad Re\{s\} > -a$ $\quad$ From table.

$X_a(-s) = \mathcal{L}\{x_a(-t)\} = \mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$ $\quad Re\{s\} > -a$
using table.

$X_a(s) = \frac{1}{-s+a} = \frac{-1}{s-a}$ $\quad Re\{s\} < a$

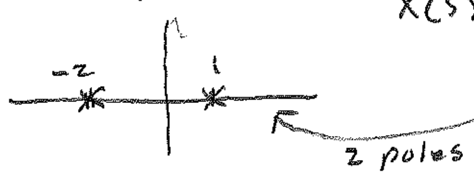
and $X(s) = X_a(s) + X_c(s)$ $\quad$ ROC is intersection



- So how does the inverse Laplace change
  - In the case of one-sided transforms we largely ignored the ROC when doing the inverse.

    Now, we have to use it since it determines the anti-causal v/s causal parts.

  - Example 4.28 $\quad X(s) = \frac{-3}{(s+2)(s-1)}$



ROC
a) $-2 < Re\{s\} < 1$
b) $Re\{s\} > 1$
c) $Re\{s\} < -2$.

- case a)
  causal ROC: $Re\{s\} > -2$
  anticausal ROC: $Re\{s\} < 1$

$$\frac{-3}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{1}{s+2} + \frac{-1}{s-1}$$

$$= X_c(s) + X_a(s)$$

$x_c(t) = \underline{e^{-2t}u(t)}$, $\quad X_a(-s) = \frac{1}{s+1}$ $\quad x_a(-t) = e^{-t}u(t)$

$x_a(t) = \underline{e^{t}u(-t)}$

- case b) $Re\{s\} > 1$ purely causal.

$$X(s) = \frac{1}{s+2} + \frac{-1}{s-1}$$

$$x(t) = (e^{-2t} - e^{t})\, u(t)$$

- case c) $Re\{s\} < -2$ purely anticausal.

$$X(s) = \frac{1}{s+2} + \frac{-1}{s-1}$$

$$X(-s) = \frac{-1}{s-2} + \frac{1}{s+1}$$

$$x(-t) = (-e^{2t} + e^{-t})\, u(t)$$

$$x(t) = (-e^{-2t} + e^{t})\, u(-t)$$

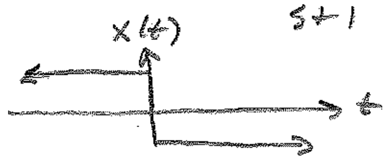
Summary: in each case we get a different inverse.

- Note that for causal systems, with TF $H(s)$

$$Y(s) = \underbrace{H(s) X_a(s)}_{\text{causal + anti-causal component}} + \underbrace{H(s) X_c(s)}_{\text{causal only.}}$$

Example: $h(t) = e^{-t} u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s+1} \quad Re\{s\} > -1$

$$x(t) = u(-t) - u(t)$$



$$X_a(s) = \frac{-1}{s} \quad Re\{s\} < 0$$

$$X_c(s) = \frac{-1}{s} \quad Re\{s\} > 0$$

$$Y(s) = H(s) X_a(s) + H(s) X_c(s)$$