Filter Design Basics.

- Recall the frequency response determines the output due to an input sinusoid at a given frequency \( f_0 \).

- It is common to design a transfer function, and thus a frequency response, to affect sinusoids at different frequencies differently.

\[
\text{Amplify: } \cos(\omega t) \rightarrow H(s) \rightarrow A \cos(\omega t + B) \\
A > 1 \quad B \approx 0 \text{ (ideally)}
\]

\[
\text{Suppress: } \cos(\omega t) \rightarrow H(s) \rightarrow A \cos(\omega t + B) \\
A < 0 \quad B \approx 0 \text{ (ideally)}
\]

- This is done by placing poles (roots of \( \phi(s) \)) and/or zeros (roots of \( \Phi(s) \)). We design polynomials.

Effects of Poles and Zeros.

- Simple example: \( \cos(\omega_0 t) u(t) \rightarrow H(s) \rightarrow \sin(\omega_0 t) u(t) \).

In Laplace Domain: \[
\frac{s}{s^2 + \omega_0^2} \rightarrow H(s) \rightarrow \frac{\omega_0}{s^2 + \omega_0^2}
\]

\[H(s) = \frac{Y(s)}{X(s)} = \frac{\omega_0}{s} \text{, an integrator.}
\]

- We can also view this as a pole-zero cancellation.

This gives us some intuition about how to amplify or suppress.

Suppose \( x(t) = \cos(2\pi f_0 t) u(t) + \cos(4\pi f_0 t) u(t) \).

We want to e.g. Amplify \( \cos(2\pi t) \) but

\[
\text{Amplify: } \cos(2\pi t) \rightarrow H(s) \rightarrow A \cos(2\pi t + B) \quad A > 1
\]

\[
\text{Select: } \cos(4\pi t) \rightarrow H(s) \rightarrow C \cos(4\pi t + D) \quad B, D \text{ as small as possible,}
\]

\[A = \left| H(2\pi) \right| \quad \text{and} \quad C = \left| H(4\pi) \right|
\]

Goal: Design \( H(s) \) such that \( |H(2\pi)| \text{ large while } |H(4\pi)| \text{ small.} \)
- Effect of poles on \( \cos(2\pi t)u(t) \)

\[
H(s) = \frac{K}{(s+a+jb)(s+a-jb)} = \frac{K}{s^2+2as+a^2+b^2}
\]

\[
X(s) = \frac{s}{s^2+(2\pi)^2}
\]

Then \( Y(s) = H(s)X(s) = \frac{KS}{(s^2+(2\pi)^2)(s^2+2as+a^2+b^2)} \)

**Demo:** Let \( a > 0 \), \( b = 2\pi \)
Then let \( a < 0 \), unstable
Then let \( a = 0 \), \( b = \pi \)
Then let \( a \to 0, b \to 2\pi \), resonance.

- Effect of zeros on \( \cos(2\pi t)u(t) \)

\[
H(s) = \frac{1}{(s+a+jb)(s+a-jb)} = \frac{1}{s^2+2as+a^2+b^2}
\]

\[
Y(s) = \frac{s}{s^2+2as+a^2+b^2}
\]

**Demo:** Let \( a > 0 \), \( b = 2\pi \)
Then let \( a \to 0 \)
Then switch to \( a = 0 \), \( b = \pi \) Let \( b \to 2\pi \)

- In summary, poles can be used to amplify, and zeros (with poles) can be used to attenuate.

- Ideal Filters. (All phases are ideally \( \pm 90^\circ \)).

\[
\text{Low Pass} \quad H(w) = \frac{1}{1+\omega_0^2}
\]

\[
\text{High Pass} \quad H(w) = \frac{1}{1-\omega_0^2}
\]

\[
\text{Notch (Band Stop)} \quad H(w) = \frac{1}{1+\omega_0^2}
\]

\[
\text{Band Pass} \quad H(w) = \frac{1}{1-\omega_0^2}
\]
- Ideal filters cannot be realized.

Example: low-pass $|H(w)|$

- Practical filters, $|H(w)|$

Low-Pass

$\Rightarrow$

- Number of poles is the order of the filter.

**DEMO**
- place poles
- note response shape
- note phase and DC gain.

- To determine the order and exact shape we relax the ideal filter.

Low-Pass (Non-Ideal)

$|H(w)|$

$G_s = \text{Stop-band gain at } w_s$

$G_p = \text{Pass-band gain at } w_p$

For a Butterworth filter the "wall of poles" is the polynomial,

$|H(w)|^2 = \frac{G_p}{1 + j\left(\frac{w}{w_p}\right)^{2n}}$

$n = \text{order of filter, } G_p = \text{DC gain}$

The corresponding Transfer function is

$H(s) = \frac{G_p}{\prod_{k=1}^{n} \left(\frac{w_p}{s - S_k}\right)}$

$S_k = w_p e^{i \frac{2(k+1)\pi}{2n}}, k=1, 2, ..., n$

The location of poles: the transition between frequencies.
- To reduce circuit complexity and sensitivity to R, C values, filters are implemented in stages.

- **Two basic stages**

  \[ n=1 \text{ } (1^{\text{st}} \text{ord}) \text{ Butterworth} \quad R \quad \frac{1}{s} \]

- **2nd order Butterworth using Sallen-Key Topology:**

- For other filter types:
  
  **High Pass**
  
  Rarely used.

  \[ \text{"Wall of Zeros"} \]

  \[ \text{pole.} \]

  **Notch**

  \[ \text{poles} \]

  **Band Pass**

  \[ \text{poles} \]