☆ Stable Systems and Frequency Response.

- Given the general form for a system in the time Domain.

$$Q(D)\,y(t) = P(D)\,x(t)$$

- In the Laplace Domain under zero-state conditions this becomes

$$Q(s)\,Y(s) = P(s)\,X(s)$$

where each order of derivative in Q(D) and P(D) is replaced by a power of s in Q(s), P(s)

e.g. $y'' + 2y' + 6y = 2x' + x$

$$Q(D) = D^2 + 2D + 6 \qquad P(D) = 2D + 1$$

Taking Laplace transform

$$\underbrace{(s^2+2s+6)}_{Q(s)}\,Y(s) = \underbrace{(2s+1)}_{P(s)}\,X(s) + \underbrace{F(s)}_{\text{From I.C.}}$$

The transfer Function $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{P(s)}{Q(s)}$

- roots of P(s) are "zeros"
- roots of Q(s) are "poles"

- Recall the two definitions of stability & criteria.

BIBO Stability

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$$

Lyapunov/Internal/Asymptotic Stability

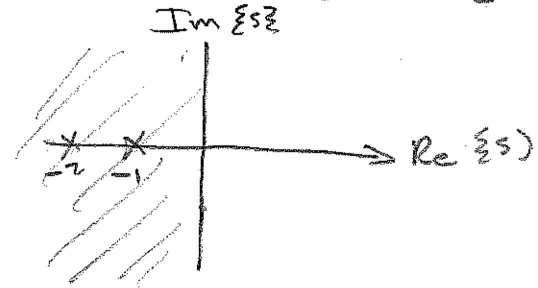
roots of Q(D) in LHP.

- In the Laplace domain roots of Q(D) = roots of Q(s) are the poles.

  - The criteria for Internal/Asymptotic stability is the same, just use roots of Q(s)

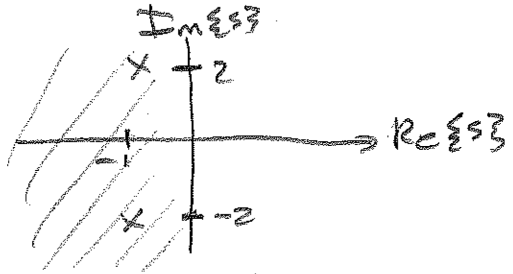
  - For BIBO stability the criteria is the poles of H(s). If any pole is in the right hand complex plane then $\int_{-\infty}^{\infty} |h(\tau)|\,d\tau$ diverges

Example: $H(s) = \dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)}$



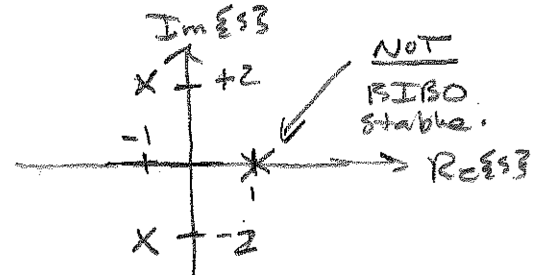
Both poles are in LHP. thus BIBO stable.

$$H(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s+1+j2)(s+1-j2)}$$



BIBO Stable.

$$H(s) = \frac{1}{(s-1)(s^2+2s+5)}$$



— A point of confusion is that the poles of $H(s)$ seem like the same as roots of $Q(s)$

$$H(s) = \frac{P(s)}{Q(s)}$$

However in some cases they are Not the same, Consider the system.

$$y'' + y' - 2y = x' - x$$

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s-1}{s^2+s-2} = \frac{\cancel{s-1}}{(\cancel{s-1})(s+2)} = \frac{1}{s+2}$$

pole-zero cancellation.

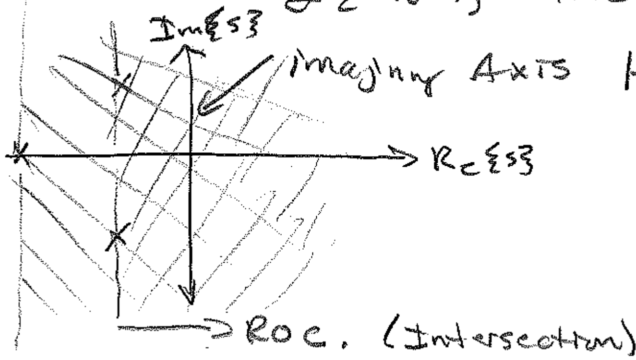
This system is BIBO stable but internally unstable.

$$Y(s) = \frac{1}{s+2}X(s) + \underbrace{\frac{F(s)}{(s-1)(s+2)}}$$

initial condition cause divergence.

- <u>If</u> a system is stable, this implies the ROC of

$H(s) = \mathcal{L}\{h(t)\}$ includes the imaginary axis.



$$H(s)\Big|_{s=j\omega} = H(j\omega) = H(\omega)$$

★ Frequency Response ★

— Recall the response in the time domain to the complex exponential.

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow h * e^{st} \equiv H(s)e^{st}$$

- when $s = \sigma + j\omega$, $\sigma = 0$, $s = j\omega$ and we have a pure sinusoid,

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)\, e^{j\omega t}$$

- taking the $Re\{e^{j\omega t}\} = \cos(\omega t)$, since it is a linear system

$$\cos(\omega t) \longrightarrow \boxed{h(t)} \longrightarrow Re\{H(j\omega)\, e^{j\omega t}\}$$

Note $H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$ thus,

$$\cos(\omega t) \longrightarrow \boxed{h(t)} \longrightarrow |H(\omega)| \cos(\omega t + \angle H(\omega))$$

magnitude scale. depends on frequency ω. change phase.

- This gives us an easy shortcut to the response of a system to a sinusoid.

- Soon we will see how to decompose any signal into a sum of such sinusoids.

$x(t)$ periodic $\quad x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \omega_0 = \frac{2\pi}{T_0}$

The Fourier Sum

$x(t)$ non periodic

$x(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} X(\omega)\, d\omega$ the Fourier transform.

- One potential point of confusion is that so far we have focused on one sided - Laplace transforms and causal systems.

- Lets modify the sinusoidal input above to be causal.

$$x(t) = e^{j\omega t} u(t) \xleftarrow{\mathcal{L}} \frac{1}{s-j\omega}$$

- Then $Y(s) = H(s)X(s) = \frac{P(s)}{Q(s)} \cdot \frac{1}{s-j\omega}$

where $Q(s)$ can be factored as $Q(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_N)$
So that.

$$Y(s) = \frac{P(s)}{(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_N)(s-j\omega)}$$

$$= \sum_{i=1}^{N} \frac{k_i}{s+\lambda_i} + \frac{A}{s-j\omega} \quad \text{using PFE}$$

To Find A $\quad (s-j\omega)Y(s)\Big|_{s=j\omega} = \sum_{i=1}^{N} \frac{k_i(s-j\omega)}{(s+\lambda_i)} + A\Bigg|_{s=j\omega}$

$$= A = H(s)\Big|_{s=j\omega} = H(j\omega)$$

- Taking $\mathcal{L}^{-1}$

$$y(t) = \underbrace{\sum_{i=1}^{N} k_i e^{\lambda_i t} u(t)}_{\text{Transient Response.}} + \underbrace{H(j\omega) e^{j\omega t} u(t)}_{\substack{\text{Steady-state response.} \\ t \gg 0}}$$

- <u>If</u> the system is stable, then $\lambda_i < 0$

and for $t \gg 0$ the transient response goes to zero and

$$y_{ss}(t) \equiv y(t)\Big|_{t \gg 0} = H(j\omega) e^{j\omega t}$$

$$y_{ss}(t) = H(j\omega)\cos(\omega t) + jH(j\omega)\sin(\omega t)$$

$$= \underset{\substack{\uparrow \\ \text{mag}}}{|H(\omega)|} \cos(\omega t + \underset{\substack{\uparrow \\ \text{phase}}}{\angle H(\omega)}) + j \underset{\substack{\uparrow \\ \text{mag}}}{|H(j\omega)|} \cos(\omega t + \frac{\pi}{2} + \underset{\substack{\uparrow \\ \text{phase.}}}{\angle H(\omega)})$$