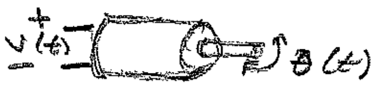
★ Control Example: Lets look at a practical example of designing a transfer function, combined with analysis in Laplace and time domain gives us powerful engineering tools.

— Consider a servo, a mechanism often used in machine design and robotics.

$x(t) \longrightarrow \boxed{\text{servo}} \longrightarrow y(t)$ the actual angle.
desired angle

• One way to make a servo is to use a DC PM motor combined with a sensor to measure the angle.

— Lets begin with a simple model of a motor.



$$V(s) \rightarrow \boxed{\frac{1}{s(s+1)}} \rightarrow \theta(s)$$

— Suppose we apply a step as the input voltage, what is the angle output θ?

$$\theta(s) = \frac{1}{s(s+1)} V(s) = \frac{1}{s(s+1)} \cdot \frac{1}{s} = \frac{1}{s^2(s+1)}$$

• To get $\theta(t)$ we need to do a Partial Fraction Exp.

$$\theta(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$ because of repeated root $s=0$.

Find B: $s^2\theta(s) = \left.\frac{1}{s+1}\right|_{s=0} = 1$

Find A: $\frac{d}{ds}\left[s^2\theta(s)\right] = \left.\frac{-1}{(s+1)^2}\right|_{s=0} = -1$

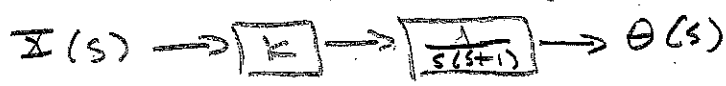
Find C: $(s+1)\theta(s) = \left.\frac{1}{s^2}\right|_{s=-1} = 1$

Thus, $\theta(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

and $\theta(t) = -u(t) + t\,u(t) + e^{-t}u(t) = (t - 1 + e^{-t})u(t)$

Does this make physical sense? $= t\,u(t) + (e^{-t} - 1)u(t)$

- So now lets see how to control the speed of the motor, relative to the input voltage.



X = speed.

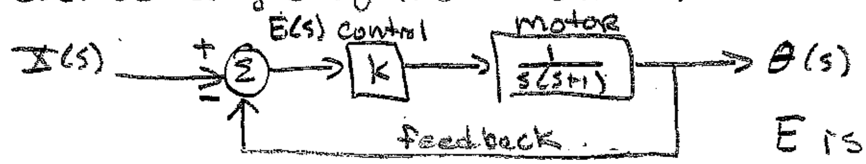
- By linearity the step response is

$$\theta(t) = K\left[t - 1 + e^{-t}\right]u(t)$$

and the gain K, controls the speed relative to input

$\boxed{\text{DEMO}}$

- Now, lets turn this into a servo. Let X be the desired angle of motor shaft.



E is the "error" signal.

- The overall transfer function is then

$$G(s) = \frac{K}{s(s+1)} \qquad H(s) = 1$$

$$\frac{\Theta(s)}{X(s)} = \frac{\frac{K}{s(s+1)}}{1 + \frac{K}{s(s+1)}} = \frac{K}{s^2+s+K} \leftarrow \text{note roots changed.}$$

- Now, if we apply a step as the input,

$$\Theta(s) = \frac{K}{s^2+s+K} \cdot \frac{1}{s}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2+s+K}$$

Clearing the fractions and Equating Coefficients,

$$\left.\begin{array}{l} A(s^2+s+K) + s(Bs+C) = K \\ (A+B)s^2 + (A+C)s + AK = K \end{array}\right\} \begin{array}{l} A+B=0 \\ A+C=0 \\ AK = K \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \\ C=-1 \end{array}$$

$$\Theta(s) = \frac{1}{s} - \frac{(s+1)}{s^2+s+K}$$

- The second term is 2nd order, I would expand it as

$$\frac{s+1}{s^2+s+k} = \frac{s+1}{(s+\frac{1}{2})^2+(k+\frac{1}{4})} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(k-\frac{1}{4})} + \frac{\frac{1}{2}}{(s+\frac{1}{2})^2+(k-\frac{1}{4})}$$

$$= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(k-\frac{1}{4})} + \frac{1}{2\sqrt{k-1/4}} \cdot \frac{\sqrt{k-1/4}}{(s+\frac{1}{2})^2+(k-\frac{1}{4})}$$

Then using the Table:

$$\theta(t) = u(t) - e^{-1/2 t}\cos\left(\sqrt{k-\tfrac{1}{4}}\, t\right)u(t) - \frac{1}{2\sqrt{k-\frac{1}{4}}}\sin\left(\sqrt{k-\tfrac{1}{4}}\, t\right)u(t)$$

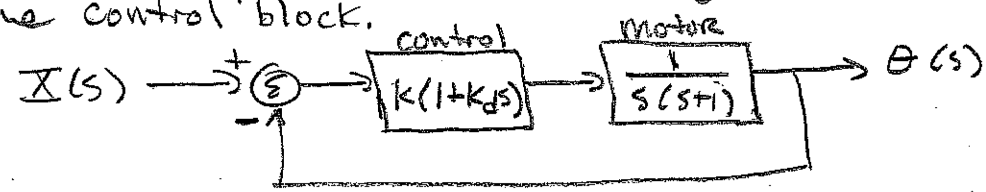
$$= \left(1 - e^{-1/2 t}\left(\cos(\omega_0 t) + \frac{1}{2\omega_0}\sin(\omega_0 t)\right)\right)u(t)$$

$$\omega_0 = \sqrt{k - 1/4}$$

- Note:
  - at steady-state (t large) $\theta \to 1$
  - the oscillation frequency and magnitude depend on the gain k.

$\boxed{\text{DEMO}}$

- We can improve the behavior of the servo by modifying the control block.



- $k_d \cdot s$ is a derivative, acting on error signal

[If you take controls you will see why this helps]

- Now the overall transfer function is

$$\frac{\theta(s)}{X(s)} = \frac{\frac{k(1+k_d s)}{s(s+1)}}{1 + \frac{k(1+k_d s)}{s(s+1)}} = \frac{k(1+k_d s)}{\underbrace{s^2+(kk_d+1)s+k}_{\text{again roots change.}}}$$

- Since the voltage is proportional to the error and the derivative of the error, this is called a proportional derivative (PD) controller.

- The step response of the PD Servo.

$$\Theta(s) = \frac{k(1+k_d s)}{s(s^2+(k\cdot k_d+1)s+k)} = \frac{1}{s} - \frac{(s+1)}{s^2+(kk_d+1)s+k}$$

Now we have to resort to Row 10d of Table, for the second term.

$$\theta(t) = u(t) - e^{-at}\left[A\cos(bt) + \frac{B-Aa}{b}\sin(bt)\right]u(t)$$

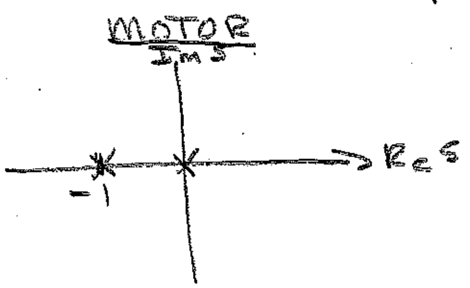
where $A=1$, $B=1$ $a = \frac{(k\cdot k_d+1)}{2}$ $b = \left(k - \frac{(k\cdot k_d+1)^2}{4}\right)^{1/2}$

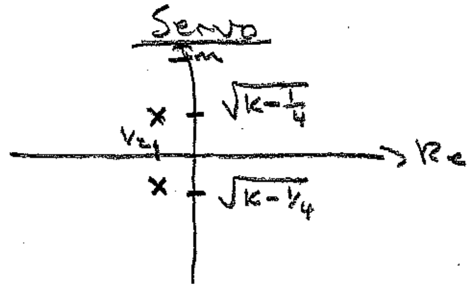
- To make the servo respond fast, but have smaller overshoot want a large, $\frac{B-Aa}{b}$ small.

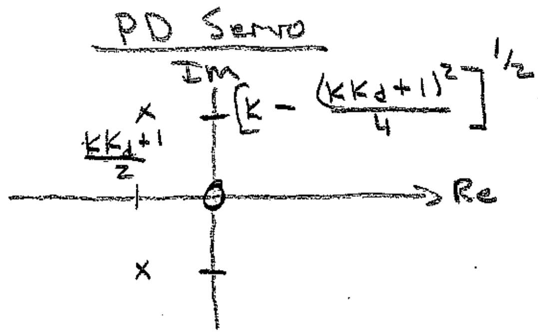
$\boxed{\text{DEMO}}$

- A controls course is largely about placing poles (and sometimes zeros) to achieve a desired behavior.

pole = x zero = 0







at each stage of the design we get <u>more</u> control of where the poles <u>are</u>.

- Thus, systems design is about placement of poles and zeros.

This will be a reocurring theme (see Filters)