★ Solving LCCDE's using Laplace

The time derivative property of the Laplace Transform gives us a powerful method to solve LCCDE's.

$$D^n x = \frac{d^n x}{dt^n} = s^n X(s) - \sum_{k=1}^{n} s^{n-k} D^{k-1} x(0^-)$$

Recall the generic form of the LCCDE.

$$D^N y + a_1 D^{N-1} y + a_2 D^{N-2} y + \cdots a_N y = b_{N-m} D^m x + \cdots b_N x$$

Lets consider the zero-state case (initially at rest)
Taking the Laplace transform

$$s^N Y(s) + a_1 s^{N-1} Y(s) + \cdots a_N Y(s) = b_{N-m} s^m X(s) + \cdots b_N X(s)$$

$$\underbrace{(s^N + a_1 s^{N-1} + \cdots a_N)}_{Q(s)} Y(s) = \underbrace{(b_{N-m} s^m + \cdots b_N)}_{P(s)} X(s)$$

$$Q(s) Y(s) = P(s) X(s) \implies Y(s) = \frac{P(s)}{Q(s)} X(s)$$

- we call $\frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)} = H(s)$ the <u>transfer function</u>.

- If we have non-zero initial conditions then the form we will get is

$$Y(s) = \underbrace{H(s) X(s)}_{\substack{\text{zero-state} \\ \text{response}}} + \underbrace{H_0(s)}_{\substack{\text{zero-input response} \\ \text{(due to I.C.)}}}$$

★ This is the solution, just in the Laplace Domain.

- To find the response in the time domain, we take the inverse Laplace transform.

$$y(t) = y_{zs}(t) + y_0(t)$$

where $y_{zs}(t) = \mathcal{L}^{-1}\{H(s) X(s)\}$ using integration or tables.

$$y_0(t) = \mathcal{L}^{-1}\{H_0(s)\}$$

– Example: Find the solution to

$$y' + 2y = 2x \quad \text{when} \quad y(0^-) = 1 \quad x(t) = 2u(t)$$

1. Take Laplace transform of both sides.

$$\underbrace{sY(s) - y(0^-)}_{\mathcal{L}\{y'\}} + \underbrace{2Y(s)}_{\mathcal{L}\{2y\}} = \underbrace{2X(s)}_{\mathcal{L}\{2x\}}$$

and solve for $Y(s)$

$$(s+2)Y(s) = 2X(s) + y(0^-)$$

$$Y(s) = \frac{2}{s+2}X(s) + \frac{y(0^-)}{s+2}$$

2. Take Laplace transform of input

$$\mathcal{L}\{2u(t)\} = \frac{2}{s} \quad \text{and apply } y(0^-) = 1$$

so then $$Y(s) = \frac{4}{s(s+2)} + \frac{1}{s+2}$$

3. Take inverse Laplace transform.

$$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{1}{s+2}$$

$A(s+2) + Bs = 4$
$(A+B)s + 2A = 4$
$A = 2, \; B = -2$

$$= \frac{2}{s} + \frac{-2}{s+2} + \frac{1}{s+2}$$

→ we could combine

$$y(t) = \underbrace{2u(t) - 2e^{-2t}u(t)}_{\text{zero-state response}} + \underbrace{e^{-2t}u(t)}_{\text{zero-input response.}}$$

– We can compare this to how we solved this in the time domain.

$$Q(D)y = P(D)x \qquad Q(D) = D+2, \quad P(D) = 2$$

– solve $Q(D)y_0 = 0 \quad y_0(0^-) = 1$

– Find $h(t)$ by solving $Q(D)y_n = 0$ with special I.C. and $h(t) = b_0\delta(t) + [P(D)y_n]u(t)$

– Performed convolution $y_{zs}(t) = (h * x)(t)$.

- Another Example: Solve

$$y'' + 7y' + 6y = 42x' \qquad y(0^-) = 0 \quad y' = 168$$

$$x(t) = 6u(t)$$

$$\underbrace{s^2Y(s) - sy(0^-) - y'(0^-)}_{\mathcal{L}\{y''\}} + \underbrace{7sY(s) - 7y(0^-)}_{\mathcal{L}\{7y'\}} + \underbrace{6Y(s)}_{\mathcal{L}\{6y\}}$$

$$= \underbrace{42(sX(s) - x(0^-))}_{\mathcal{L}\{42x'\}}$$

collecting terms

$$s^2Y(s) + 7sY(s) + 6Y(s) = 42s\underbrace{X(s)}_{=\frac{6}{s}} - \overbrace{x(0^-)}^{0} + 7\underbrace{y(0^-)}_{0} + s\overbrace{y(0^-)}^{0} + \underbrace{y'(0^-)}_{168}$$

$$(s^2 + 7s + 6)Y(s) = 252 + 168 = 420$$

$$Y(s) = \frac{420}{s^2 + 7s + 6} = \frac{420}{(s+1)(s+6)} = \frac{A}{s+1} + \frac{B}{s+6}$$

- clearing fractions and equating coefficients
  $A = 84 \quad B = -84$

$$Y(s) = \frac{84}{s+1} - \frac{84}{s+6}$$

- Taking $\mathcal{L}^{-1}$ $\quad \underline{y(t) = 84(e^{-t} - e^{-6t})u(t)}$

- Yet another Example: Solve

$$y'' + 2y' + 5y = x' + x \qquad y(0^-) = y'(0^-) = 0 \quad \text{(at rest)}$$
$$x = \delta(t)$$

Taking the Laplace transform

$$s^2 Y(s) + 2s Y(s) + 5 Y(s) = s X(s) - x(0^-) + X(s)$$

$$X(s) = \mathcal{L}\{\delta(t)\} = 1 \qquad x(0^-) = 0$$

$$(s^2 + 2s + 5) Y(s) = s + 1$$

$$Y(s) = \frac{s+1}{s^2+2s+5}$$

- The denominator has complex roots $-1 \pm j2$

If we complete the square on the bottom.

$$Y(s) = \frac{s+1}{(s+1)^2 + 4} \qquad \text{Table Row 9a} \quad a = 1 \quad b = 2$$

$$\underline{y(t) = e^{-t} \cos(2t)\, u(t)}$$

- Another approach is to use the complex roots directly.

$$Y(s) = \frac{s+1}{(s+1-j2)(s+1+j2)} = \frac{A}{(s+1-j2)} + \frac{B}{(s+1+j2)}$$

$$A(s+1+j2) + B(s+1-j2) = s+1$$
$$As + A + j2A + Bs + B - j2B = s+1$$
$$\underline{(A+B)}s + \underline{A+B} + j2A - j2B = s+1$$

$$A + B = 1 \implies A = 1 - B \qquad 2A - 2B = 2 \implies B = \tfrac{1}{2}$$

$$Y(s) = \frac{1/2}{s+1-j2} + \frac{1/2}{s+1+j2}$$

Using Table

$$y(t) = \frac{1}{2} e^{(-1+j2)t} u(t) + \frac{1}{2} e^{(-1-j2)t} u(t)$$

$$= e^{-t}\left(\frac{1}{2} e^{j2t} + \frac{1}{2} e^{-j2t}\right) u(t)$$

$$= \underline{e^{-t} \cos(2t)\, u(t)}$$ the same result (but a bit more work)