★ Properties of Laplace transform.

- The table of transforms and the properties are the tool of choice when working in Laplace analysis

- Linearity: if $x_1(t) \xrightarrow[ROC_1]{\mathcal{L}} X_1(s)$ and $x_2(t) \xrightarrow[ROC_2]{\mathcal{L}} X_2(s)$

then

$$a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_1(s)$$

$$a, b \in \mathbb{C} \qquad ROC = ROC_1 \cup ROC_2$$

Example: Suppose $x(t) = (3e^{-t} - 9e^{-4t})u(t)$

Let $x_1(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1} \quad Re(s) > -1$

$x_2(t) = e^{-4t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+4} \quad Re(s) > -4$

Then $\mathcal{L}\{3x_1 - 9x_2\} = \frac{3}{s+1} + \frac{-9}{s+4} \quad ROC: Re\{s\} > -4$

- The inverse Laplace transform uses this via partial Fraction expansions.

$$X(s) = \frac{P(s)}{Q(s)} \begin{array}{l} \leftarrow \text{polynomial in } s \\ \leftarrow \text{polynomial in } s. \end{array}$$

- if order of $P(s) > Q(s)$ first re-write as $X(s) = C + \frac{P'(s)}{Q'(s)}$ for constant C.

(this will be rare) ← then Focus on this term.

- Factor $Q(s)$, in general a mixture of real and imaginary roots. Factor into

$$Q(s) = \prod_{i=1}^{N} (s + r_i) \prod_{i=1}^{M} (s + \alpha_i + j\omega_i)(s + \alpha_i - j\omega_i)$$

N real roots, M complex pairs, where $N + M/2 \overset{!}{=}$ order of $Q(s)$

Then write as

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{N} \frac{A_i}{s + r_i} + \sum_{i=1}^{M} \frac{B_i s + C_i}{s^2 + 2\alpha_i s + \alpha_i^2 + \omega_i^2}$$

AND Solve for constants A_i, B_i, C_i.

– Example: $X(s) = \dfrac{2s^3 + 10s^2 + 31s + 13}{s^4 + 5s^3 + 17s^2 + 13s}$

Factor $Q(s) \rightarrow$ roots $0, -1, -2 \pm j3$

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{Bs + C}{s^2 + 4s + 13}$$

Solve for constants A_1, A_2, B, C

– equate coefficients 4 eq, 4 unknows

– "cover up" method (See section B.5)

We find $A_1 = 1$ $A_2 = 1$, $B = 1$, $C = 2$

$$X(s) = \frac{1}{s} + \frac{1}{s+1} + \frac{s+2}{s^2+4s+13}$$

$$= \frac{1}{s} + \frac{1}{(s+1)} + \frac{s+2}{(s+2)^2 + 9}$$

using the Table then.

$$x(t) = u(t) + e^{-t}u(t) + e^{-2t}\cos(3t)u(t)$$

$$= [1 + e^{-t} + e^{-2t}\cos(3t)]u(t)$$

– Another Property: time-shift.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$x(t-\tau) \overset{\mathcal{L}}{\longleftrightarrow} e^{-s\tau}X(s)$$

Example: $x(t) = u(t) - u(t-\tau)$ $\tau > 0$

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-s\tau}$$

– When using an inverse, isolate any terms with $e^{-s\tau}$ and invert seperately. then shift.

Example: $\dfrac{1}{s^2}e^{-2s} - \dfrac{1}{s^2}e^{-4s} \Rightarrow r(t-2) - r(t-4)$

- Most properties have "duals", eg. Frequeny shift.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$x(t)e^{at} \longleftrightarrow X(s-a) \quad a>0$$

- time differentiation

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$\frac{dx}{dt} = Dx \overset{\mathcal{L}}{\longleftrightarrow} sX(s) - \underbrace{x(0^-)}_{\text{initial value.}}$$

• For higher order.

$$D^2x \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - s\,x(0^-) - Dx(0^-)$$

$$D^3x \overset{\mathcal{L}}{\longleftrightarrow} s^3X(s) - s^2x(0^-) - s\,Dx(0^-) - D^2x(0^-)$$

$\vdots$ etc.

This property + linearity becomes the basis for solving causal LCCDE

$$y' + ay = x \implies sY(s) - y(0^-) + aY(s) = X(s)$$

an algebraic expression in s.

- The dual of time derivatives is Frequency derivative.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX}{ds}$$

- time integration:

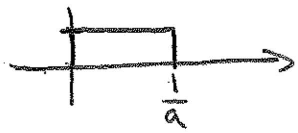
$$\int_{0^-}^{t} x(\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) + \frac{1}{s}\underbrace{\int_{-\infty}^{0^-} x(\tau)\,d\tau}_{\text{summary of past.}}$$

– time scaling: $x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$, $a > 0$

Example: rectangular pulse $x(t) = u(t) - u(t-1)$

$x(at)$



$\xleftrightarrow{\mathcal{L}}$ $\frac{1}{a} X\left(\frac{s}{a}\right)$

$\frac{1}{a}\left(\frac{a}{s} - \frac{a}{s} e^{-\tau s}\right)$

⟵ $a > 0$ compress in time

expansion in freq.

– A very important Property is convolution.

$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \qquad x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$

implies $(x_1 * x_2)(t) \xleftrightarrow{\mathcal{L}} X_1(s) X_2(s)$

convolution in time-domain is <u>multiplication</u> in the Laplace domain.

- It's dual is $x_1(t)\, x_2(t) \xleftrightarrow{\mathcal{L}} \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$

– Two useful properties are

– Initial Value

$$x(0^+) = \lim_{s \to \infty} s X(s)$$ if order of $P(s)$ < order of $Q(s)$

– Final value

$$x(\infty) = \lim_{s \to 0} s X(s)$$ if roots of denominator of $sX(s)$ are in the Left hand plane.

Example: $X(s) = \dfrac{s+3}{s^2 + 6s + 18}$

$$x(0^+) = \lim_{s \to \infty} \frac{s(s+3)}{s^2+6s+18} = \lim_{s \to \infty} 1 - \frac{3s + 24}{s^2+6s+18} \to 0 = \underline{1}$$

$$x(\infty) = \lim_{s \to 0} \frac{s(s+3)}{s^2+6s+18} = \frac{0}{18} = \underline{0}.$$