★ One-sided Laplace transforms.

- Recall last time the Integral transform

$$X(s) = (\mathcal{L}x)(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad s \in \mathbb{C}, t \in \mathbb{R}$$

When the signal is causal ($x(t) = 0 \ \forall t < 0$) then this is the one-sided (unilateral) Laplace transform

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt. \qquad s = \alpha + j\omega$$

- Example: $x(t) = e^{\lambda t} u(t) \qquad \lambda \in \mathbb{R}$

$$X(s) = \int_0^\infty e^{\lambda t} e^{-st} dt = \int_0^\infty e^{-(s-\lambda)t} dt$$

$$= \frac{-1}{s-\lambda} e^{-(s-\lambda)t} \Big|_0^\infty$$

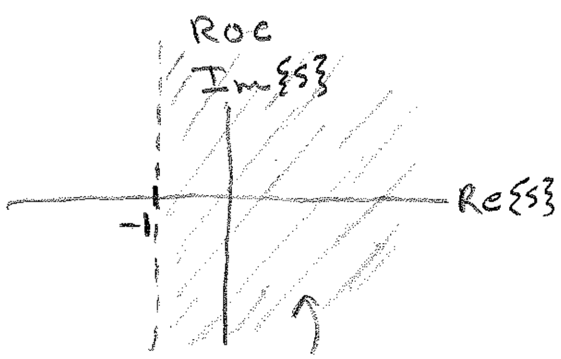
$$= \underbrace{\frac{-1}{s-\lambda} e^{-(s-\lambda)\infty}} + \underbrace{\frac{1}{s-\lambda} e^{-(s-\lambda)0}}_{\frac{1}{s-\lambda}}$$

Converges to 0 <u>if</u> $\operatorname{Re}\{s-\lambda\} > 0$

or $\operatorname{Re}\{s\} > \lambda$

Thus $\mathcal{L}\{e^{\lambda t} u(t)\} = \dfrac{1}{s-\lambda}$ with Region of Convergence (ROC) $\operatorname{Re}\{s\} > \lambda$

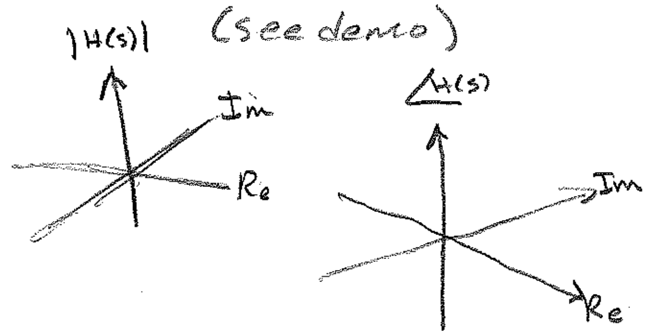
- Suppose $\lambda = -1$



$X(s)$ is <u>only</u> defined in this region

- A plot of $X(s)$ requires a surface plot of magnitude and phase

(see demo)



– More Examples

- In previous example $u(t) = e^{\lambda t} u(t) \Big|_{\lambda = 0}$

  AND $\mathcal{L}\{u(t)\} = \frac{1}{s}$ ROC: $Re\{s\} > 0$

- $x(t) = r(t) = t\,u(t)$

$$X(s) = \int_0^{\infty} t e^{-st} dt \qquad \text{using } \int \text{ by parts}$$

$$= uv - \int v\, du \qquad \begin{matrix} u = t & dv = e^{-st} \\ du = 1 & v = -\frac{1}{s} e^{-st} \end{matrix}$$

$$= -\frac{t}{s} e^{-st} + \int_0^{\infty} \frac{1}{s} e^{-st} dt$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \Big/_0^{\infty}$$

If $Re\{s\} > 0$ then

$$= (-0 - 0) - \left(-0 - \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2} \qquad Re\{s\} > 0$$

- $x(t) = \cos(\omega t)\, u(t) = \left(\frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}\right) u(t)$

$$X(s) = \frac{1}{2}\int_0^{\infty} e^{-(s-j\omega)t} dt + \frac{1}{2}\int_0^{\infty} e^{-(s+j\omega)t} dt$$

$$= \frac{1}{2} \frac{-1}{s-j\omega} \Big/_0^{\infty} + \frac{1}{2} \frac{-1}{s+j\omega} \Big/_0^{\infty} \qquad Re\{s\} > 0$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{1}{2}\left[\frac{s+\cancel{j\omega}+s-\cancel{j\omega}}{s^2+\omega^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2+\omega^2}\right] = \frac{s}{s^2+\omega^2} \qquad Re\{s\} > 0$$

– So, we see taking the Laplace transform is a straightforward application of integration.

— What about the inverse Laplace transform?

Given $x(t) \xrightarrow{\mathcal{L}} X(s)$, $\xleftarrow{\mathcal{L}^{-1}?}$

The inverse Kernel is $K^{-1}(s,t) = \frac{1}{2\pi j} e^{st}$

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} \, ds$$

↑ complex integration

Contour Integration over the Region of Convergence.

- Seeing how to do this (using method of Residues) is outside the course scope.

So, How will we take the Inverse transform? Tables
Assuming Unilateral Laplace we can ignore ROC.

— Table 4.1 Two columns $x(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\longleftrightarrow}} X(s)$

This combined with the properties (next time) gives us a flexible way to do forward and Inverse Laplace Transforms, using just Algebra.

- Example: $X(s) = \dfrac{2}{s^2 + 6s + 13}$

This is close to column 2, row 11

$x(t)$	$X(s)$
$e^{-at}\sin(bt)u(t)$	$\dfrac{b}{(s+a)^2+b^2}$

$$s^2 + \underset{a=3}{2as} + \underbrace{a^2+b^2}_{13}$$

$(3)^2 + b^2 = 13$
$b^2 = 4$
$b = 2$

thus

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+6s+13}\right\} = e^{-3t}\sin(2t)u(t)$$

- two basic properties are scaling and addition.
  - Scaling: if $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$
    then $a\,x(t) \longleftrightarrow a X(s)$
  - Addition: if $x_1(t) \longleftrightarrow X_1(s)$

    $x_2(t) \longleftrightarrow X_2(s)$

    then

    $x_1(t) + x_2(t) \longleftrightarrow X_1(s) + X_2(s)$

  These are easily proved using the integral properties.

- Now, lets use these properties to do transforms
  - Example: $x(t) = (e^{-t} - 4e^{-3t})u(t)$

    Let $x_1 = e^{-t}u(t)$ $\quad x_2 = e^{-3t}u(t)$

    and $x(t) = x_1(t) - 4\,x_2(t)$

    Thus $X(s) = \dfrac{1}{s+1} + (-4)\dfrac{1}{s+3}$

    ROC: $Re(s) > -1$ $\qquad$ ROC: $Re(s) > -3$

    $$X(s) = \frac{1}{s+1} + \frac{-4}{s+3}$$

    ROC: $Re\{s\} > -1$
    The intersection of the individual ROC's.

    $$= -\frac{3s+1}{s^2+4s+3}$$

  - Now, suppose we had been given $X(s)$ and asked to find $x(t)$. We need to <u>undo</u> those steps.
    - First, we could factor $s^2+4s+3 = (s+1)(s+3)$
    - Then we could rewrite as the <u>partial fraction expansion</u> $\dfrac{A}{s+1} + \dfrac{B}{s+3}$
    - Then find A, B (See section B.5 of text)

      $A(s+3) + B(s+1) = -(3s+1)$ $\qquad \underbrace{\begin{array}{l} A + B = -3 \\ 3A + B = -1 \end{array}}_{A = 1 \quad B = -4}$

    - To get $\dfrac{1}{s+1} + \dfrac{-4}{s+3}$
    - using table:

      $$\underline{x(t) = (e^{-t} - 4e^{-3t})u(t)}.$$