Introduction to Integral Transforms.

- Summary of time domain techniques.

\[ x(t) \xrightarrow{Q(P)} y(t) = y_0 + y_{es}(t) \]

\[ y_{es}(t) = (h * x)(t) \]

- Advantages
  - Analysis is straightforward if cumbersome
  - Applies to all LTI systems
  - Time domain representations are intuitive.

- Disadvantages.
  - Does not scale well to large systems, decomposition into blocks requires convolution.
  - Design is difficult, how do I "design" \( h(t) \)?
  - Implementation (inverse of modeling) from a given \( h(t) \) to a circuit is difficult.

- We can borrow a technique from mathematics to overcome those disadvantages by transforming the domain of the signals, and thus the system.

- This is done using an **Integral Transform**.

Consider an abstract function

\[ F : U \to \mathbb{R} \quad f(u) \quad u \in U \]

The integral transform \( T \)

\[(TF)(v) = \int k(u, v) f(u) \, du\]

converts \( f(u) \) to another function

\[(TF) : V \to \mathbb{R} \quad F(v) \quad v \in V\]

with a different domain.

From the "\( U \)" domain to the "\( V \)" domain.
- Graphically:

\[ f : U \rightarrow \mathbb{R} \quad \text{and} \quad K(u, v) \quad \text{and} \quad f : V \rightarrow \mathbb{R} \]

\[ f(u) \quad \overset{\text{the kernel}}{\uparrow} \quad \text{transform} \quad \text{the function.} \]

- If the kernel is invertible with inverse \( K^{-1}(u, v) \) then the inverse transformation

\[ (T f)(u) = \int K^{-1}(u, v) F(v) \, dv \]

maps \( F(v) \) back to \( f(u) \).

- Note that if \( K \) is invertible then the two functions \( f \) and \( F \) represent the same function, just in different spaces.

- Convolution is an integral transform,

\[ (x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(t-\tau) x_2(\tau) \, d\tau \]

- the two domains, \( t, \tau \in \mathbb{R} \) are the same.

- the kernel \( K(t, \tau) = x_1(t-\tau) \)

- Does it have an inverse?

Let's look at an example:

\[ x \rightarrow \boxed{Dy + ay + x} \rightarrow y \rightarrow \boxed{x = Dy + ay} \rightarrow x \]

\[ h(t) = e^{at} \delta(t) \quad h^{-1}(t) = D\delta(t) + a\delta(t) \]

So, technically, yes, however we will see later severe practical issues.

- We can also rearrange the blocks to get the combined

\[ Dy + ay = Dx + a \cdot x \implies x = y \]

whose overall impulse response is \( h(t) = \delta(t) \)
Another simple example is the identity transform:

\[
(TX)(t) = \int_{-\infty}^{\infty} S(t-\tau) x(\tau) d\tau = x(t)
\]

where the kernel is \( K(t, \tau) = S(t-\tau) \)

Two very important integral transforms for us will be:

1. \( K(s, t) = e^{-st} \) the complex exponential, \( s \in \mathbb{C} \) and \( t \in \mathbb{R} \),

\[
F(s) \equiv (Tf)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt
\]

Maps functions of \( \mathbb{R} \) to functions of \( \mathbb{C} \)
We will denote \( Tf \) in this case by \( \mathcal{L} \)

because this is the (bilateral) Laplace

\[
\begin{array}{c|c}
\text{f(t)} & \mathcal{L} \\
\hline
\cdot & \cdot \\
\end{array}
\]

\( f: \mathbb{R} \rightarrow \mathbb{R} \)

\( F: \mathbb{C} \rightarrow \mathbb{C} \)

\( \text{Value is } F(s) \)

While this might seem like a complication
we will see this transformation turns
LDE into algebraic equations in \( s \).

\[
\begin{array}{c|c}
\text{x: } \mathbb{R} \rightarrow \mathbb{R} & \text{convolution} \\
\hline
\text{y: } \mathbb{R} \rightarrow \mathbb{R} & \text{multiplication} \\
\end{array}
\]

\( y = h \ast x \)

\( y : \mathbb{C} \rightarrow \mathbb{C} \)

\( x : \mathbb{C} \rightarrow \mathbb{C} \)

This gives us tools for design and implementation.
2. The second important transform will be when

\[ K(\omega, t) = e^{-j\omega t}, \quad \omega, t \in \mathbb{R} \]

\[ F(\omega) \equiv (T_F)(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) \, dt \]

which we denote \( F \equiv (T_F)(\omega) \)

the \textit{Fourier Transform}.

Note: the Fourier transform results in functions over the reals

\[ F : \mathbb{R} \to \mathbb{C} \]

where \( \omega \in \mathbb{R} \)

has an interpretation as a frequency.

Because the Fourier transform is relatively simple compared to the Laplace, it is preferred in some cases, those with:

- Stable System and non-causal signals.

Some intuition about these transforms.

\[ F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt \]

For a fixed \( s=s_i \), the integral measures the similarity of two functions: \( F(t) \) and \( e^{-s_i t} \)

\[ F(s_i) = \int_{-\infty}^{\infty} f(t) e^{-s_i t} \, dt \]

In fact this is the continuous extension of the inner product in vector spaces.

\[ (v_1 \cdot v_2) \text{ is the component of } v_1 \text{ along } v_2 \]

This is used to change basis.

The idea here is identical.