★ Introduction to Integral transforms.

- Summary of time domain techniques.

$$x(t) \longrightarrow \boxed{Q(D)\,y = P(D)\,x} \longrightarrow y(t) = y_0 + y_{zs}(t)$$

$$y_{zs}(t) = (h * x)(t)$$

- Advantages
  - anlaysis is straightforward if cumbersome
  - applies to all LTI systems
  - time domain representations are intuitive.
- Disadvantages.
  - does not scale well to large systems, decomposition into blocks requires convolution.
  - design is difficult, how do I "design" $h(t)$?
  - implementation (inverse of modeling) from a given $h(t)$ to a circuit is difficult.

- We can borrow a technique from mathematics to overcome those disadvantages by transforming the <u>Domain</u> of the signals, and thus the system.

- This is done using an <u>Integral</u> <u>Transform</u>.

Consider an abstract function

$$F : U \to \mathbb{R} \qquad f(u) \qquad u \in U$$

The integral transform T

$$(Tf)(v) = \int \underbrace{k(u,v)}_{\text{kernel}} f(u)\,du$$

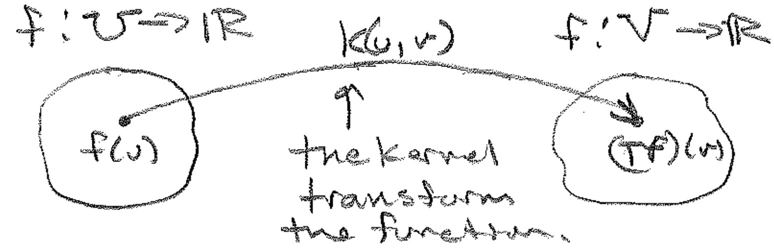
converts $f(u)$ to another function

$$(Tf) : V \to \mathbb{R} \qquad F(v) \qquad v \in V$$

with a different Domain.

From the "U" domain to the "V" domain.

- Graphically



- If the kernel is invertible with inverse $K^{-1}(u,v)$ then the inverse transformation

$$(Tf)u) = \int K^{-1}(u,v)F(v)\,dv$$

maps $F(v)$ <u>back</u> to $f(u)$.

- Note that if K is invertable then the two functions f and F represent the <u>same</u> function, just in different spaces.

- Convolution is an Integral transform.

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\,x_2(\tau)\,d\tau$$

  - the two domains, $t, \tau \in \mathbb{R}$ are the same.
  - the kernel $K(\tau, t) = x_1(t-\tau)$
  - Does it have an inverse?

  Lets look at an example:

$$x \longrightarrow \boxed{Dy + ay = x} \longrightarrow y \longrightarrow \boxed{x = Dy + ay} \longrightarrow x$$

$$h(t) = e^{-at}u(t) \qquad h^{-1}(t) = D\delta(t) + a\delta(t)$$

  So, technically, yes, however we will see later severe practical issues.

  - we can also rearrange the blocks to get the combined

$$Dy + ay = Dx + ax \quad \Rightarrow x = y$$

  whose overall impulse response is $h(t) = \delta(t)$

- Another simple example is the identity transform

$$(Tx)(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\, x(\tau)\, d\tau = x(t)$$

where the Kernel is $K(t,\tau) = \delta(t-\tau)$

- Two very important integral transforms for us will be

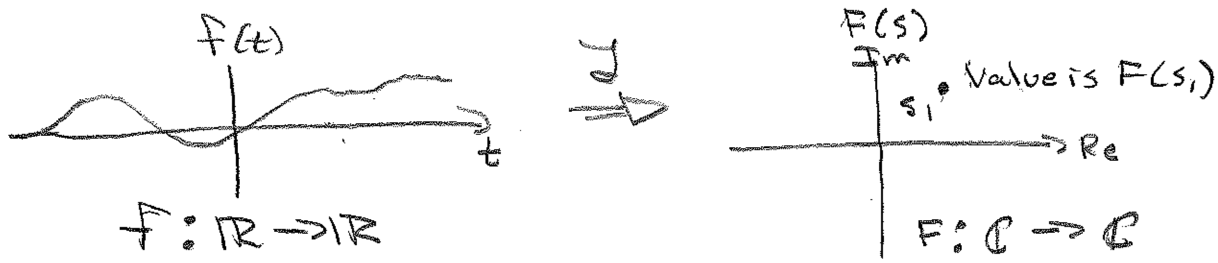
I. $K(s,t) = e^{-st}$ the complex exponential. $s \in \mathbb{C}$ and $t \in \mathbb{R}$.

$$F(s) \equiv (Tf)(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

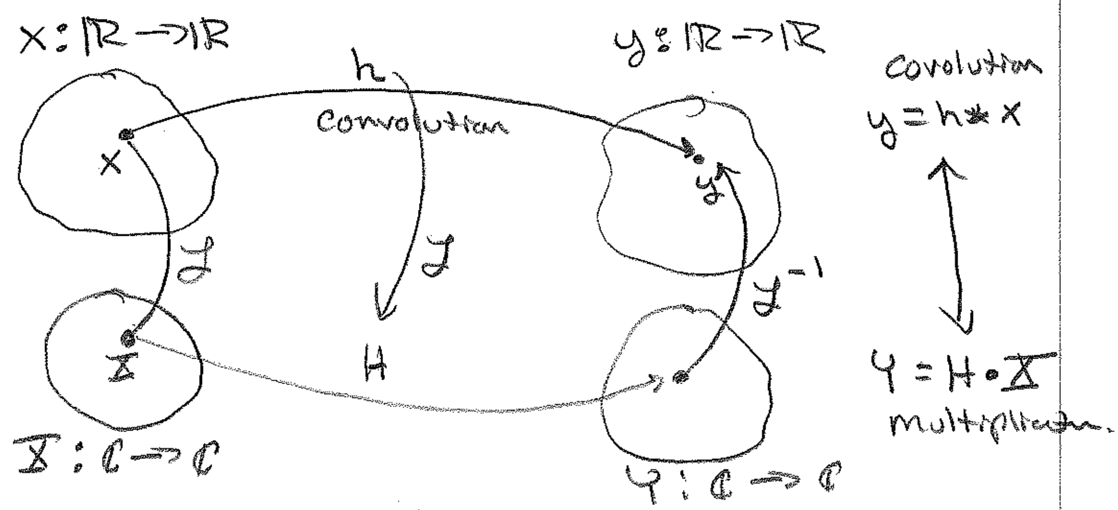
Maps functions of $\mathbb{R}$ to functions of $\mathbb{C}$

we will denot Tf in this case by $\mathcal{L}$.

becaus this is the (bilataral) Laplace Transform.



- while this might seem like a complication we will see this transformation turns LCCDE into algebraic equations in s.



- This gives us tools for design + implementation.

2. The second important transform will be when
$$K(\omega, t) = e^{-j\omega t} \qquad \omega, t \in \mathbb{R}$$
$$F(\omega) \equiv (Tf)(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, dt$$
which we denote $\mathcal{F} \equiv (Tf)(\omega)$

the <u>Fourier Transform</u>.

Note: the Fourier transform results in functions over the reals

$$F : \mathbb{R} \to \mathbb{C}$$

where $\omega \in \mathbb{R}$ has an interpretation as a frequency.

Because the Fourier transform is relatively simple compared to the Laplace, it is prefered in some cases, Those with:

- Stable System and non-causal signals.

- Some intuition about these transforms.

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt$$

- for a fixed $s = s_1$ the integral measures the similarity of two functions: $f(t)$ and $e^{-s_1 t}$

$$F(s_1) = \int_{-\infty}^{\infty} f(t) e^{-s_1 t}\, dt$$

in fact this is the continuous extension of the inner product in vector spaces.

$(v_1 \cdot v_2)$ is the component of v_1 along v_2.

This is used to change basis.

The idea here is identical.