★ response to complex exponential

- consider the zero-state response due to the input

  $x(t) = e^{st}, \; s = \sigma + j\omega$ a complex signal,

  given a system with impulse response $h(t)$.

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

$$= H(s)\, e^{st} = H(s)\, x(t)$$

- This shows if we apply e^{st} as input we get a weighted e^{st} as output
  - The weight, for fixed s is $H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

    the Laplace transform of $h(t)$.

  (much more about this later)

- Example: Suppose we have a first order system with

$$h(t) = e^{-5t} u(t)$$

  and input $x(t) = e^{(-1+j2\pi)t}$, i.e. $s = -1 + j2\pi$

  The output is $y(t) = H(-1+j2\pi)\, e^{(-1+j2\pi)t}$

  where $H(\underbrace{-1+j2\pi}_{s}) = \int_{-\infty}^{\infty} h(\tau) e^{-(-1+j2\pi)\tau} d\tau$

$$= \int_{0}^{\infty} e^{-5\tau} e^{(1-j2\pi)\tau}\, d\tau$$

$$= \int_{0}^{\infty} e^{-4\tau} e^{-j2\pi\tau}\, d\tau$$

we will see this $\int$ in detail later.

$$= \frac{1}{(-1+j\omega) + 5}$$

$$= \frac{1}{4 + j\omega}$$

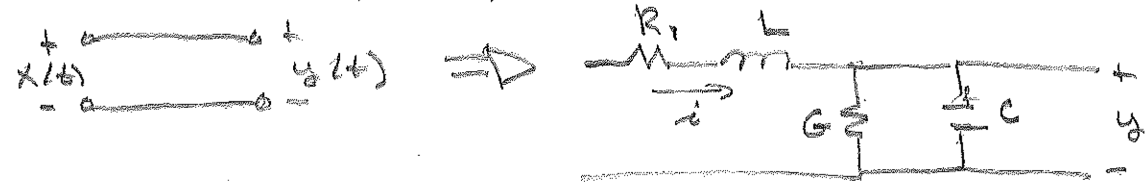
Thus

$$y(t) = \frac{1}{4+j2\pi} e^{(-1+j2\pi)t}$$

also a complex signal.

★ System time constant.

— Motivation: Consider a model of a short wire, or trace on a circuit board



where G is admittance = $\frac{1}{R_2}$

- Suppose we put a switch on a 5 volt source at x and throw it. How long does it take the voltage to reach 5 V on the other end (at y)?

— Circuit analysis

$$\text{KVL: } \quad x - y = Ri + Li' \qquad (1)$$

$$\text{KCL: } \quad i = Gy + Cy' \qquad (2)$$

$$i' = Gy' + Cy'' \qquad (3)$$

substitute 2,3 into 1 gives

$$y'' + \underbrace{\left(\frac{R}{L} + \frac{G}{C}\right)}_{a_1} y' + \underbrace{\left(\frac{1}{LC} + \frac{RG}{LC}\right)}_{a_2} y = \underbrace{\frac{1}{LC}}_{b} x$$

a second order system $y'' + a_1 y' + a_2 y = bx$.

where $x(t) = Vu(t)$ where $V = 5$ in our example.

- Impulse response.

$$y_n = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \text{where } \lambda_1, \lambda_2 \text{ are roots of } D^2 + a_1 D + a_2.$$

$$y_n'(0) = 1 \quad y_n(0) = 0$$

$$y_n' = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t}$$

$$y_n'(0) = \lambda_1 C_1 + \lambda_2 C_2 = 1 \quad \Rightarrow \quad C_1 = \frac{1}{\lambda_1 - \lambda_2}$$

$$y_n(0) = C_1 + C_2 = 0 \qquad C_2 = \frac{1}{\lambda_2 - \lambda_1}$$

Applying P(D) gives

$$h(t) = \left[\frac{b}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{b}{\lambda_2 - \lambda_1} e^{\lambda_2 t}\right] u(t)$$

– Transmission line example cont.

- To find the response to $V_u(t)$

$$y(t) = (h * x)(t) = \int_0^t h(\tau)\, d\tau$$

$$y(t) = \int_0^t \frac{b}{\lambda_1 - \lambda_2} e^{\lambda_1 \tau} d\tau + \int_0^t \frac{b}{\lambda_2 - \lambda_1} e^{\lambda_2 \tau} d\tau$$

$$= \left[\frac{b}{-\lambda_1(\lambda_1 - \lambda_2)} (1 - e^{\lambda_1 t}) + \frac{b}{-\lambda_2(\lambda_2 - \lambda_1)} (1 - e^{\lambda_2 t}) \right] u(t)$$

$$= \left[\frac{b(1 - e^{\lambda_1 t})}{-\lambda_1^2 + \lambda_1 \lambda_2} + \frac{b(1 - e^{\lambda_2 t})}{-\lambda_2^2 + \lambda_1 \lambda_2} \right] u(t)$$

- Recall λ_1, λ_2 depend on a_1, a_2 which depend on the model parameters R, L, G, C.

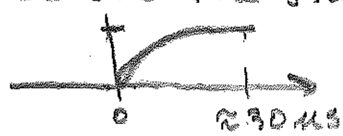
For $\approx$ 3 feet of CAT-5 cable.

$$R = 0.176\ \Omega \qquad L = 0.49 \times 10^{-6}\ H$$

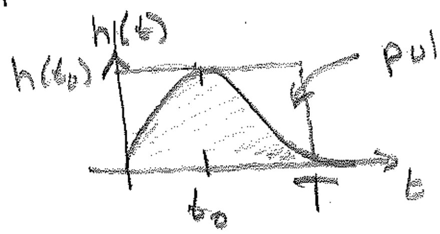
$$G = \frac{1}{10^9} \text{ Siemens } \left(\frac{1}{\Omega}\right) \qquad C = 0.049 \times 10^{-9}\ F$$

$\boxed{\text{DEMO}}$ Lets plot this.

- We see the step input gets attenuated and dispersed



This delay can be thought of as a square pulse approximation of the impulse response.



pulse with same area as $h(t)$

$$h(t_0)\, T = \int_0^\infty h(\tau)\, d\tau$$

☆ T is the system's time constant, how <u>fast</u> it responds and depends on the modes.

★ Resonance.

- Consider what happens when the input is close to modes of the system (roots of $Q(D)$)

example

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

$$\lambda = -\frac{1}{RC}$$

Suppose $x(t) = Ae^{-\alpha t} u(t)$ then.

From Table:
$$y(t) = (x * h)(t) = \frac{A}{RC} \frac{e^{-\alpha t} - e^{-\lambda t}}{\alpha - \lambda} u(t)$$

Note when $\alpha \approx \lambda$ $\frac{1}{\alpha - \lambda}$ is large

When $\alpha - \lambda$ is large $\frac{1}{\alpha - \lambda}$ is small

★ This will become the basis of filter design ★

- Taken to the extreme $\alpha = \lambda$

Let $\alpha = \lambda - \epsilon$ and $\lim_{\epsilon \to 0} y(t) = \frac{A}{RC} t e^{\lambda t} u(t)$

This is called resonance $y(t)$ grows as t until the $e^{\lambda t}$ term decreases.

DEMO

- Note resonance is usefull in electrical systems but can be destuctive in mechanical systems.