★ Stability

- Recall our definition of Bounded Input-Bounded Output or BIBO stability. Given a system with input x and response y.

$$x \rightarrow \boxed{} \rightarrow y$$

$$|x| < \infty \;\; \forall t \longrightarrow |y| < \infty \;\; \forall t$$

if $|x| < A < \infty \;\; \forall t$ then x is Bounded

if $|y| < B < \infty \;\; \forall t$ as well then y is bounded and the system is BIBO stable.

- Now we know that $y_{zs}(t) = (h * x)(t)$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$|y(t)| = \left|\int h(\tau)x(t-\tau)\,d\tau\right| \le \int |h(\tau)|\,\underbrace{|x(t-\tau)|}_{A}\,d\tau$$

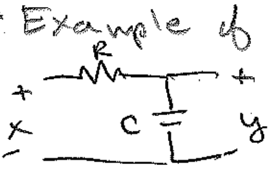
if $x(t) \le A < \infty$ then $|y(t)| \le A \int |h(\tau)|\,d\tau$

★ Thus if $\int |h(\tau)|\,d\tau \le B < \infty$ then $|y(t)| \le A \cdot B < \infty$ and the system is BIBO stable. [h is an Energy Signal]

- Stability is about the tendency of a system to reach equilibrium and return to it if perturbed.
  - Absolute Stability is a YES/NO ANSWER. (our focus)
  - Relative Stability is the degree to which it is stable.

- For a system to be practically useful then it should be stable.

- Example of BIBO Stability



$$h(t) = e^{-at}u(t)$$

$$a = \frac{1}{RC}$$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_0^{\infty} e^{-at}\,dt = -\frac{1}{a}e^{-at}\Big|_0^{\infty} = -0 + \frac{1}{a} = \frac{1}{a} < \infty$$

∴ BIBO Stable.

- Note, showing a result on an upper bound is sufficient to show BIBO stability.

Example: $h(t) = e^{-\alpha t}(A\cos\omega_0 t + B\sin\omega_0 t)u(t)$

Then $|h(t)|$ is bounded above by $(A+B)e^{-\alpha t}u(t)$

$$\int_{-\infty}^{\infty}|h(t)|dt \le \int_{0}^{\infty}(A+B)e^{-\alpha t}dt = \frac{A+B}{\alpha} \quad \underline{\text{if}} \ \alpha > 0$$

then BIBO Stable.

- Note in physical systems, cannot go to ∞, but may still be unstable.

★ A related notion of stability is concerned with zero-input. This is called many things: internal stability, asymptotic stability, Lyapunov stability, or zero-input stability.

- The zero-input response is the solution to

$$Q(D)y_0 = 0 \quad \text{given } D^{N-1}y(0^-), D^{N-2}y(0^-), \ldots y(0^-)$$

if, given <u>finite</u> initial conditions the zero-input response approaches 0 as $t \to \infty$ then the system is asymptotically stable.

- When is this the case? Recall the form of y_0

$$y_0(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \quad \text{where } \lambda_k \text{ are the roots of } Q(D).$$

if the initial conditions are finite, then c_k are.

So <u>if</u> $Re\{\lambda_k\} < 0$ for each root. then

$$|y_0(t)| < \infty \text{ and } y_0(t) \to 0 \text{ as } t \to 0.$$

as each exponential decays to zero.

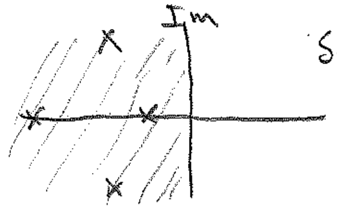
- Example: A pendulum with friction is a classic example.

- Some examples

• Suppose $Q(D) = D^4 + 9D^3 + 32D^2 + 56D + 32$

this has roots

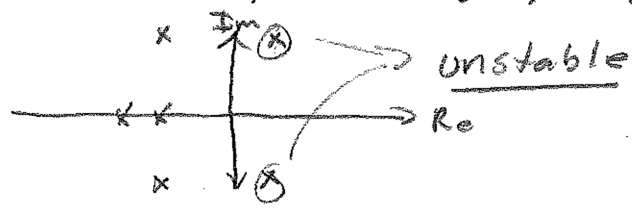
$-1, -4, -2 \pm j2$



Stable

- Another Example: $Q(D) = D^6 + 7D^5 + 21D^4 + 41D^3 + 90D^2 + 224D + 240$

  has roots: $-2, -3, -2 \pm j2, 1 \pm j2$



- What about the edge case where $Re\{\} = 0$ exactly? if any non repeated root is on the imaginary axis and all others are in LHP then the system is <u>marginally stable</u>

Summary:

- A system is BIBO stable if $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.
- A system is Asymptotic/internal/Lyapunov, zero-input stable or unstable according to
  1. if <u>any</u> root of $Q(D)$ is in RHP then <u>unstable</u>.
  2. if a repeated root is on the imaginary axis then <u>unstable</u>
  3. if one or more non-repeated roots are on imaginary axis then <u>marginally</u> stable.
  4. else it is stable.
- Note Asymptotic stability implies BIBO stability.

Examples:

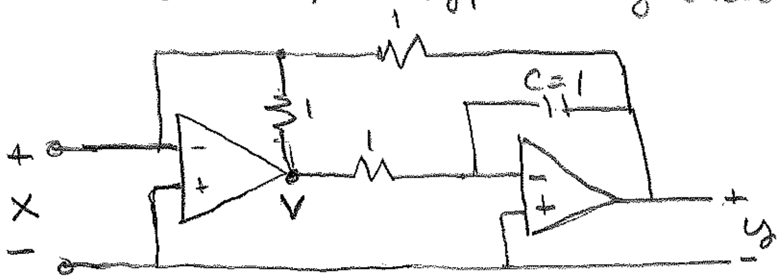
$y'' - y' - 2y = x$ — roots of $Q(D) = -1, 2$

$y'' + 3y' + 2y = x$ — $Q(D)$ roots $-1, -2$

$D^3 y + 4D^2 y = x$ — $Q(D)$ roots $-4, 0, 0$

$D^3 y + 2D^2 y + 4Dy + 8y = Dx$ — $Q(D)$ roots are $-2, \pm j2$

- Given the following circuit, determine if it is BIBO stable? asymptotically stable?



① KCL at input: $\frac{V-x}{1} + \frac{y-x}{1} = 0 \Rightarrow V + y = 2x$

② KCL at 2^{nd} input: $\frac{V}{1} + y' = 0 \Rightarrow V = -1y'$

Subst. ② → ① gives $-y' + y = 2x$ or $y' - y = -2x$.

- Now to find $h(t)$

$$Q(D)y_n = 0 \quad y_n(0^+) = 1 \quad \Rightarrow y_n = Ce^t$$

Q(D) has 1 root at +1. $\quad y_n(0^+) = C = 1$

$$h(t) = P(D)y_n u(t) = -2e^t u(t).$$

- Now compute: $\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} -2e^t dt = -2\left(e^t \Big|_0^{\infty}\right)$

$$= -2(\infty) + 0 = \infty$$

$\therefore$ BIBO <u>unstable</u>

- Note $Q(D) = D - 1$ with single root at +1

$\therefore$ asymptotically <u>unstable</u>

- What does it mean practically for this circuit to be unstable?

- If I apply $x = u(t)$ $\quad y(t) = -2e^t u(t) * u(t)$

gives $y(t) = \frac{1-e^t}{-1} u(t)$ $\quad y(t) \to -\infty$ as $t \to \infty$

?? can the voltage do this?